Chapter 1

Algebra in Mesopotamia

1.1. Introduction

Most of the mathematical cuneiform texts we have today date from about 1800 B.C. They are written on clay tablets in Akkadian, the language of the Semitic people who conquered the land of the Sumerians, in the modern Iraq, around 2000 B.C. These invaders adopted both the culture and the writing of the Sumerians. They also retained the knowledge of the Sumerian language, which would most likely have remained unintelligible to us without the Sumerian–Akkadian lexicons that they compiled and that have been preserved to the present day.[1]

It is to the Akkadians, too, that we owe the transmission of that outstanding feature of Sumerian culture, the development of mathematics. But this was not known until modern times: first came the deciphering of cuneiform writing, in the nineteenth century, then the study of cuneiform mathematical texts, in the first half of the twentieth century. Although these texts were written in Akkadian, they continued to use Sumerian words for technical terms. What was thus presumed to be a Sumerian heritage then revealed the existence of a quite particular form of algebra with pairs of linear equations as well as quadratic equations and systems being solved, with surprising ease, by applying a few elementary identities. But before going any further, we must first consider another particularity of these texts, which has to do with the number system in use at the time.

The earliest Sumerian clay tablets date from around 3200 B.C. From these we see that, even before the development of an alphabet, a primitive number system involving both a decimal base and a sexagesimal base was already in use. Thus D, ○, D, ◯ represented, respectively, 1, 10, 60, 3600; then, with the sign for 10 combined with D and ◯, signs for 600 and 36000

[1]The grammatical structure of Sumerian, which is an agglutinative language, appears to place it in the family of Ural-Altaic languages, along with Finnish, Hungarian, and Turkish. But this would have been of no help in understanding the language itself. With Akkadian, which is closely related to the other Semitic languages, such a problem did not arise.

were obtained. The other numbers were formed by repetition of these six symbols. That base 10 was used is not surprising: throughout the world, the ten human fingers had long provided a primitive method of counting. However, no such obvious reason can account for the choice of base 60. Modern speculations about the numerous divisors of the integer 60 appear to overlook the age of the earliest evidence for this system. On the tablets mentioned above, the symbol used for 60 appears alongside primitive drawings from before the invention of writing—that is, from a time when people could hardly have been considering the divisibility of numbers. A more reasonable explanation might be found in connection with the fundamental requirements of astronomy or the calendar. Perhaps, for example, the use of 60 had to do with the approximate number of the days and nights in a lunar month.

Whatever its origin may be, the above sexagesimal system, and thus the sexagesimal base, continued to be used after the development of written language. Around the middle of the third millenium B.C., a similar but improved numeral system came into being. Two symbols, one for unity (𒁹) and one for ten (𒌋), were used to compose the sexagesimal digits from 1 to 59 via repetition. Thus 𒐈, 𒌋𒁹, 𒌍𒐈 represent 3, 11, 33 (with the three unit symbols stacked), respectively. Note that, unlike the digits we use, these sexagesimal digits were of widely varying length. For purely practical reasons, the system evolved into a "place-value system" for expressing numbers above 59, that is, one in which the value of each symbol depends on its place within the number. The sexagesimal digits were simply juxtaposed and, when read from left to right, indicated the coefficients of decreasing (positive or negative) powers of 60. Thus 𒐈 𒌋𒁹 𒌍𒐈 represents the three-digit number $3'11'33$.

So far, this is just like our system except that we have 9 significant (i.e. nonzero) digits, whereas they had 59. Now such a place-value system must have some sign to indicate an empty space, that is, in the present case, the absence of a certain power of 60; but at the time, and for more than two millenia (till about 300 B.C.), there was no proper symbol for zero. An intermediate zero was marked simply by an empty space on the tablet, a gap in the writing. Now this could easily lead to confusion: first, no precise length can be attributed to such a gap since the lengths of the sexagesimal digits are not uniform—and thus two consecutive zeros could appear to be a single one; second, an empty space could be left merely to accommodate a rough or uneven surface of the clay tablet. With such an inherent weakness, this system can hardly have been created deliberately. But there is another, even more troublesome, weakness. Since an empty space is to be inserted between two sexagesimal digits, it is impossible to represent a number with initial or final zeros. Consequently, all numbers in cuneiform script, whether consisting of one digit or more, are defined only up to a factor equal to some (positive or negative) power of 60—which is as if our 132 were supposed to be preceded or followed by an indeterminate number of zeros. This should have been an obstacle to any development of mathematics. In actual fact, in mathematical problems this inconvenience was often avoided by mentioning at the outset what units of measurement were being used. This helped to determine the order of magnitude of the given values, whereby the second weakness of the system was overcome.

Learning those 59 digits was no more difficult than learning our nine is; with only two symbols to be repeated as necessary, it was in fact less so. It was thus not in writing numbers but in calculating that the use of a sexagesimal system led to difficulty. Performing calculations with our base 10 requires knowing 36 products of one digit by another, not counting products where one of the factors is 0 or 1 or those made superfluous by commutativity. On the other hand, after eliminating banal and repetitive products in the same way, base 60 requires knowing 1711 products. As it would hardly have been reasonable to expect people to learn them all by heart, written multiplication tables were needed for performing calculations. The abundance of such tables in museums today attests both to the disadvantages of the sexagesimal system and to its widespread use in calculating.

Still, even with the help of tables, much mental agility was required of Mesopotamian mathematicians. The results of the multiplication of two numbers with several sexagesimal digits were given directly in the solutions of problems, which means that after extracting all the necessary products of two sexagesimal digits from the tables, Mesopotamians were able to mentally add the digits occupying the same sexagesimal places or, at least, to verify the result. Thus, in one of the examples we shall see below (page 12), $14'30$ by $14'30$ is directly given as $3'30'15$.[2] On the other hand, division was reduced to multiplication by the inverse of the divisor; this required first consultation of tables indicating, for a given integer less than 60, its inverse and, second, multiplication tables indicating the products by the inverse in question[3]. Finally, note that there were also tables giving exact roots of squares (as well as some approximate roots of non-square integers). This is why, in one of our examples (page 10), it is just said that $14'30'15$ is the square of $29'30$.[4] In short, mathematics in Mesopotamia was clearly inextricably linked to the existence of auxiliary tables.

As inconvenient as it was, the sexagesimal system was used in Mesopotamia for two millenia, and astronomers in particular continued to use it to record their observations. Ptolemy (A.D. 150) mentions in his fundamental text on astronomy, which was to be influential throughout the Middle Ages, that since the beginning of the reign of Nabonassar (747 B.C.), "the ancient (Mesopotamian) observations are, on the whole, preserved down to

[2] In our writing: $14 + \frac{30}{60}$ by $14 + \frac{30}{60}$ is $3 \cdot 60 + 30 + \frac{15}{60}$, that is, $(14 + \frac{1}{2})^2 = 210 + \frac{1}{4}$.

[3] Assuming that the inverse had a finite sexagesimal expansion, rather than an infinite periodic one. In base 60, this is the case for all numbers of the form $2^\alpha 3^\beta 5^\gamma$, just as the inverses of numbers of the form $2^\alpha 5^\beta$ have a finite expansion in base 10. In general, the necessary and sufficient condition for a number to have an inverse with finite expansion in a given base is that the representation of the number as a product of primes contains only prime factors of that base.

[4] In our writing: $14 \cdot 60 + 30 + \frac{15}{60}$ $(= 870 + \frac{1}{4})$ is the square of $29 + \frac{30}{60}$ $(= 29 + \frac{1}{2})$.

our own time."[5] Several centuries of continuous observations provided an invaluable body of data for the computation of planetary periods, and Greek scientists knew how to best make use of this information. As it would have been an overwhelming task to convert all these data into the decimal system they used, the Greeks maintained the sexagesimal system for astronomical measurements, including measurements of time and angles. This was also adopted by the Indians as early as antiquity, probably via the Greeks. Then, together with Indian and Greek astronomy, it reached the Muslims, who in turn transmitted this notation to medieval Christian Europe. The sexagesimal division of degrees and hours still used today is thus a living witness to the sexagesimal base once used by the Sumerians already in prehistoric times.

1.2. Linear Systems

Consider a linear system of one of the two forms

$$\begin{cases} ax + by = k \\ x \pm y = l, \end{cases}$$

where a, b, k, and l are given quantities with k and l positive and at least one of a and b positive.

To solve this system, one could either

(1) express one of the unknown quantities in terms of the other by using the second equation, introduce this into the first equation, and solve the new equation $ax \pm b(l - x) = k$,

or

(2) take as new unknown $z = \frac{x-y}{2}$ (or $\frac{x+y}{2}$), and substitute $\frac{x+y}{2} \pm \frac{x-y}{2} = \frac{l}{2} \pm z$ (respectively $z \pm \frac{l}{2}$) for x and y in the first equation.

Using the second method to solve this system may well seem odd to modern readers. However, the Mesopotamians resorted to the second as much as to the first, as the following two examples will illustrate.

These two examples follow one another on the same tablet, and their close relationship is visible both in their form and in their similar givens and solutions. We will first present the original text in translation, then a summary of the computations involved, and finally explain the solution. Note that the sexagesimal numbers have been converted to decimal numbers in the translations; see Appendix A for the same texts with sexagesimal

[5]See Ptolemy's *Almagest*, III, 7. Of the Greek text published by J. Heiberg, *Syntaxis mathematica* (2 vol.), Leipzig 1898–1903, we have a German translation by K. Manitius (*Des Claudius Ptolemäus Handbuch der Astronomie* (2 vol.), Leipzig 1912–13; reprint: Leipzig 1963) and an English translation with commentary by G. Toomer (*Ptolemy's Almagest*, New York 1984). A French translation with the Greek text, in a less critical edition, exists by the Abbé N. Halma, *Composition mathématique de Claude Ptolémée* (2 vol.), Paris 1813–16 (reprint: Paris 1988).

numbers and, in bold face, the initial and final zeros that, as mentioned above, were not indicated in Mesopotamian texts[6].

In these two problems, we are asked to find the areas of two fields of grain knowing their respective yields per unit area, the difference in the total yields, as well as the sum of their areas (first problem) or the difference of their areas (second problem). The units employed are the *bur*, which equals 1800 *sar* (1 *sar* $\cong$ 36 m^2), for surface area; and the *kur*, which equals 300 *sila* (1 *sila* $\cong$ 1 liter), for volume. In both problems the respective yields per unit area of the two fields are the same: 4 *kur* = 1200 *sila* per *bur* for the first field and 3 *kur* = 900 *sila* per *bur* for the second. The difference in their total yields, 500 *sila*, is the same as well. The sum of the areas is given to be 1800 *sar* in the first problem, while the difference is given to be 600 *sar* in the second. Finally, note that in each text the statement of the problem is followed by a change of units (a welcome step in the absence of final zeros; see page 2) and that the answer is verified after it has been calculated.

Example 1. Tablet 8389, Museum of the Ancient Near East, Berlin, Problem 1 [Appendix A.1][7]

(α) *Per bur, I obtained* 4 *kur of grain. Per second bur, I obtained* 3 *kur of grain. One grain exceeds the other by* 500. *I added my fields;* (*it gives*) 1800. *What are my fields?*

Put 1800, *the bur. Put* 1200, *the grain he obtained. Put* 1800, *the second bur. Put* 900, *the grain he obtained.*[8] *Put* 500, *that by which one grain exceeds the other. And put* 1800, *the sum of the areas of the fields.*

Next, divide in two 1800, *the sum of the areas of the fields*[9]*;* (*it gives*) 900. *Put* 900 *and* 900, *twice. Take the inverse of* 1800, *the bur;* (*it gives*) $\frac{1}{1800}$.[10] *Multiply* $\frac{1}{1800}$ *by* 1200, *the grain he obtained;* (*it gives*) $\frac{2}{3}$, *the false grain. Multiply* (*it*) *by* 900, *which you have put twice*[11]*;* (*it gives*) 600. *Let your head hold* (*it*)[12]. *Take the inverse of* 1800, *the second bur;* (*it gives*) $\frac{1}{1800}$. *Multiply* $\frac{1}{1800}$ *by* 900, *the grain he obtained*[13]*;* (*it gives*) $\frac{1}{2}$, *the false*

[6]The translation of these texts follows that of F. Thureau-Dangin, *Textes mathématiques babyloniens*, Leiden 1938. For a (more literal) English translation of these texts, see J. Høyrup, *Lengths, Widths, Surfaces: a Portrait of Old Babylonian Algebra and Its Kin*, New York 2002. The particular algebraic way of solving equations in Mesopotamian mathematics was first noted by Kurt Vogel, "Zur Berechnung der quadratischen Gleichungen bei den Babyloniern", *Unterrichtsblätter für Mathematik und Naturwissenschaften*, 39 (1933), pp. 76–81 (reprinted pp. 265–273 in J. Christianidis (ed.), *Classics in the History of Greek Mathematics*, Dordrecht 2004).

[7]In all translations of texts, the words in parentheses have been added by us for clarity.

[8]Conversion of *bur* to *sar* and *kur* to *sila*.

[9]This specification is welcome, as there is also another 1800 (the *bur* converted into *sar*).

[10]Note that this way of writing fractions is anachronistic, as all fractions were expressed sexagesimally.

[11]and not the 900 that is the yield of the second field.

[12]This means that an intermediate result has been obtained that will be used later. A similar phrase is found in medieval mathematics (see note 65).

[13]and not the 900 of the half-area.

grain. Multiply (it) by 900*, which you have put twice; (it gives)* 450*. By what does* 600*, which your head holds, exceed* 450*? It exceeds (it) by* 150*. Subtract* 150*, that by which it exceeds, from* 500*, that by which one grain exceeds the other; you leave* 350*. Let your head hold* 350*, which you left. Add the coefficient* $\frac{2}{3}$ *and the coefficient* $\frac{1}{2}$*; (it gives)* $\frac{7}{6}$*. I do not know its inverse*[14]*. What must I put to* $\frac{7}{6}$ *to have* 350*, that your head holds? Put* 300*. Multiply* 300 *by* $\frac{7}{6}$*; this gives you* 350*. From (one)* 900*, which you have put twice, subtract, and to the other add,* 300*, which you have put; the first is* 1200*, the second* 600*. The area of the first field is* 1200*, the area of the second field* 600*.*

If the area of the first field is 1200 *and the area of the second field is* 600*, what is their grain? Take the inverse of* 1800*, the bur; (it gives)* $\frac{1}{1800}$*. Multiply* $\frac{1}{1800}$ *by* 1200*, the grain he obtained*[15]*; (it gives)* $\frac{2}{3}$*. Multiply (it) by* 1200*, the area of the first field; (it gives)* 800*, (which is) the grain of* 1200*, the area of the first field. Take the inverse of* 1800*, the second bur; (it gives)* $\frac{1}{1800}$*. Multiply* $\frac{1}{1800}$ *by* 900*, the grain he obtained; (it gives)* $\frac{1}{2}$*. Multiply* $\frac{1}{2}$ *by* 600*, the area of the second field; (it gives)* 300*, (which is) the grain of* 600*, the area of the second field. By what does* 800*, the grain of the first field, exceed* 300*, the grain of the second field? It exceeds (it) by* 500*.*

(β) The numerical calculations used to solve the problem and to verify the answer found are successively (with the conversions immediately following the statement omitted here):

$$1800/2 = 900$$
$$900$$
$$900$$
$$\begin{cases} 1800^{-1} = \frac{1}{1800} \\ 1200 \cdot \frac{1}{1800} = \frac{2}{3} \quad \text{“false grain”} \\ 900 \cdot \frac{2}{3} = 600 \quad \text{“hold”} \end{cases}$$
$$\begin{cases} 1800^{-1} = \frac{1}{1800} \\ 900 \cdot \frac{1}{1800} = \frac{1}{2} \quad \text{“false grain”} \\ 900 \cdot \frac{1}{2} = 450 \end{cases}$$
$$600 - 450 = 150$$
$$500 - 150 = 350 \quad \text{“hold”}$$
$$\frac{2}{3} + \frac{1}{2} = \frac{7}{6}$$

[14]The inverse of 7 is not found in the usual tables for inverses (nor in the multiplication tables), as it does not have a finite sexagesimal expansion (see note 3).

[15]and not the area of the first field.

$\frac{6}{7}$ "not known"

Now $\frac{7}{6} \cdot z = 350$ for $z = 300$

(The text verifies this.)

$900 + 300 = 1200$ "area of the first field"

$900 - 300 = 600$ "area of the second field"

$$\begin{cases} 1200 \cdot \frac{1200}{1800} = 800 & \text{"grain of the first field"} \\ 600 \cdot \frac{900}{1800} = 300 & \text{"grain of the second field"} \\ \text{the difference is, indeed, 500.} \end{cases}$$

(γ) Interpretation:

First, $\frac{1200}{1800} = \frac{2}{3}$ and $\frac{900}{1800} = \frac{1}{2}$ are the yields (in *sila*, not in *kur*) of the first and second fields per unit area (*sar*, not *bur*), respectively, which the tablet refers to as the "false grain." The "true grain" would then be the total production of each field, thus $\frac{2}{3}x$ and $\frac{1}{2}y$ if x and y represent the areas of the two fields. Therefore, in modern mathematical notation, we are asked to solve the system

$$\begin{cases} x + y = 1800 \\ \frac{2}{3}x - \frac{1}{2}y = 500. \end{cases}$$

We already know that $\frac{x+y}{2} = 900$. We now take, to replace x and y, the auxiliary unknown $z = \frac{x-y}{2}$. By adding and subtracting these, we obtain $x = 900 + z$ and $y = 900 - z$, and the second equation becomes

$$\frac{2}{3}(900 + z) - \frac{1}{2}(900 - z) = 500.$$

Looking back at the computations given in the text, we see that they correspond to calculating

$$\frac{2}{3}900 - \frac{1}{2}900 + \frac{2}{3}z + \frac{1}{2}z = 500,$$

$$600 - 450 + \frac{7}{6}z = 500,$$

$$150 + \frac{7}{6}z = 500,$$

$$\frac{7}{6}z = 500 - 150 = 350.$$

With $z = 300$, we then find, as in the text, that $x = 900 + z = 1200$ and $y = 900 - z = 600$.

In summary, what the text does is replace the two unknowns by a single unknown and then solve the resulting equation. But all this is done without the slightest explanation. The only comments given in the text are to specify the concrete meaning of the quantities involved, particularly in order to differentiate between two equal numerical values attributed to different quantities, and to point out some problematic step in the computations (see the footnotes).

Example 2. Tablet 8389, Museum of the Ancient Near East, Berlin, Problem 2 [Appendix A.2]

Per bur, I obtained 4 *kur of grain. Per second bur, I obtained* 3 *kur of grain. Now* (*there are*) *two fields. One field exceeds the other by* 600*. One grain exceeds the other by* 500*. What are my fields?*

Put 1800*, the bur. Put* 1200*, the grain he obtained. Put* 1800*, the second bur. Put* 900*, the grain he obtained. Put* 600*, that by which one field exceeds the other. And put* 500*, that by which one grain exceeds the other.*

Take the inverse of 1800*, the bur;* (*it gives*) $\frac{1}{1800}$*. Multiply* (*it*) *by* 1200*, the grain he obtained;* (*it gives*) $\frac{2}{3}$*, the false grain. Multiply* $\frac{2}{3}$*, the false grain, by* 600*, that by which one field exceeds the other;* (*it gives*) 400*. Subtract* (*it*) *from* 500*, that by which one grain exceeds the other; you leave* 100*. Let your head hold* 100*, that you left. Take the inverse of* 1800*, the second bur;* (*it gives*) $\frac{1}{1800}$*. Multiply* $\frac{1}{1800}$ *by* 900*, the grain he obtained;* (*it gives*) $\frac{1}{2}$*, the false grain. By what does* $\frac{2}{3}$*, the false grain, exceed* $\frac{1}{2}$*, the false grain? It exceeds* (*it*) *by* $\frac{1}{6}$*. Take the inverse of* $\frac{1}{6}$*, that by which it exceeds;* (*it gives*) 6*. Multiply* 6 *by* 100*, which your head holds;* (*it gives*) 600*, the area of the first field. To* 600*, the area of the field, add* 600*, that by which one field exceeds the other;* (*it gives*) 1200*. The area of the second field is* 1200*.*

If the area of the first field[16] *is* 1200 *and the area of the second field* (*is*) 600*, what is their grain? Take the inverse of* 1800*, the bur;* (*it gives*) $\frac{1}{1800}$*. Multiply* $\frac{1}{1800}$ *by* 1200*, the grain he obtained;* (*it gives*) $\frac{2}{3}$*. Multiply* (*it*) *by* 1200*, the area of the* (*first*) *field;* (*it gives*) 800*,* (*which is*) *the grain of* 1200*, the area of the* (*first*) *field. Take the inverse of* 1800*, the second bur;* (*it gives*) $\frac{1}{1800}$*. Multiply* $\frac{1}{1800}$ *by* 900*, the grain he obtained;* (*it gives*) $\frac{1}{2}$*. Multiply* (*it*) *by* 600*, the area of the second field;* (*it gives*) 300*,* (*which is*) *the grain of* 600*, the area of the* (*second*) *field. By what does* 1200*, the area of the* (*first*) *field, exceed* 600*, the area of the second field? It exceeds* (*it*) *by* 600*. By what does* 800*, the* (*first*) *grain, exceed* 300*, the second grain? It exceeds* (*it*) *by* 500*.*

With the same notation as before, the system of equations involved in the second problem is

$$\begin{cases} x - y = 600 \\ \frac{2}{3}x - \frac{1}{2}y = 500. \end{cases}$$

We could set $z = \frac{x+y}{2}$ as new unknown and solve the problem as before. The text, however, proceeds as we would, by substituting $x = y + 600$ from the first equation into the second. This yields

$$\frac{2}{3}(y + 600) - \frac{1}{2}y = 500.$$

[16] The "second field" above, the "first field" in the verification of the previous problem.

The subsequent steps in the text correspond to simplifying this equation and solving for y; indeed, the author computes

$$\frac{2}{3} \cdot 600 = 400,$$

then

$$500 - 400 = 100,$$

and finally

$$y = \frac{100}{\frac{2}{3} - \frac{1}{2}} = 6 \cdot 100 = 600.$$

1.3. Quadratic Equations and Systems

From Mesopotamian times to the Renaissance, quadratic equations usually appeared in one of three standard forms, differing according to the place of each of the three terms, all positive, on either side of the equality. The equation $ax^2+bx+c=0$ with $a, b, c > 0$ did not appear, and could not appear, as only equations with positive (and rational) solutions were considered. Thus the three possible cases are

(1) $ax^2 + bx = c$, with solution

$$x = \frac{-\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + ac}}{a};$$

or, equivalently, in reduced form

$$x^2 + px = q,$$

with solution

$$x = -\frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + q}\,.$$

(2) $ax^2 = bx + c$, with solution

$$x = \frac{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + ac}}{a};$$

or (in reduced form)

$$x^2 = px + q,$$

with solution

$$x = \frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + q}\,.$$

(3) $ax^2 + c = bx$, with the two possible solutions

$$x = \frac{\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - ac}}{a};$$

or (in reduced form)

$$x^2 + q = px,$$

with solutions

$$x = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}\,.$$

Only the last of these cases has two positive solutions—as long as the discriminant is positive.

Some of the preserved Mesopotamian tablets clearly demonstrate a knowledge of the first two formulas. This is illustrated by the next two examples, where, as before, we present the text followed by its mathematical interpretation.

Example 1. Tablet 13901, British Museum, Problem 1 [Appendix A.3]

I added the area and the side of my square; (it gives) $\frac{3}{4}$.

You put 1, *the unit. You divide in two* 1; *(it gives)* $\frac{1}{2}$. *You multiply* $(\frac{1}{2})$ *by* $\frac{1}{2}$; *(it gives)* $\frac{1}{4}$. *You add* $\frac{1}{4}$ *to* $\frac{3}{4}$; *(it gives)* 1. *It is the square of* 1. *You subtract* $\frac{1}{2}$, *which you multiplied, from* 1; *(it gives)* $\frac{1}{2}$, *the side of the square.*

The equation here is $x^2 + x = \frac{3}{4}$, which is of the form $x^2 + px = q$. The calculations performed (along with their symbolic equivalents) are

$$\begin{array}{ll}
1\ (= p) & p \\
\downarrow /2 & \\
\frac{1}{2} & \frac{p}{2} \\
\downarrow \text{square} & \\
\frac{1}{4} & \left(\frac{p}{2}\right)^2 \\
\downarrow +\frac{3}{4} & \\
1 & \left(\frac{p}{2}\right)^2 + q \\
\downarrow \text{square root} & \\
1 & \sqrt{\left(\frac{p}{2}\right)^2 + q} \\
\downarrow -\frac{1}{2} & \\
\frac{1}{2} & \sqrt{\left(\frac{p}{2}\right)^2 + q} - \frac{p}{2} = x.
\end{array}$$

The reader may not be fully convinced by this solution, as the quantity 1 we started with is both the coefficient of x^2 and that of x. But later we will see an example of the solution of an equation of this type where the coefficient of x^2 is not 1.

Example 2. Same tablet, Problem 2 [Appendix A.4]

I subtracted from the area the side of my square; (it gives) 870.

You put 1, *the unit. You divide in two* 1; *(it gives)* $\frac{1}{2}$. *You multiply* $(\frac{1}{2})$ *by* $\frac{1}{2}$; *(it gives)* $\frac{1}{4}$. *You add (it) to* 870; *(it gives)* $870 + \frac{1}{4}$. *It is the square of* $29 + \frac{1}{2}$. *You add* $\frac{1}{2}$, *which you multiplied, to* $29 + \frac{1}{2}$; *(it gives)* 30, *the side of the square.*

The equation, $x^2 - x = 870$, is of the form $x^2 = px + q$. The text performs the following calculations:

$$\begin{array}{ll}
1 \ (= p) & p \\
\downarrow /2 & \\
\frac{1}{2} & \frac{p}{2} \\
\downarrow \text{square} & \\
\frac{1}{4} & \left(\frac{p}{2}\right)^2 \\
\downarrow +870 & \\
870 + \frac{1}{4} & \left(\frac{p}{2}\right)^2 + q \\
\downarrow \text{square root} & \\
29 + \frac{1}{2} & \sqrt{\left(\frac{p}{2}\right)^2 + q} \\
\downarrow +\frac{1}{2} & \\
30 & \sqrt{\left(\frac{p}{2}\right)^2 + q} + \frac{p}{2} = x.
\end{array}$$

Few such examples of single quadratic equations have been preserved. Less rare are those of quadratic systems, at least among the texts we still have today. These systems often take one of the two following forms, either initially or after a series of transformations:

$$\begin{cases} x + y = p \\ x \cdot y = q \end{cases}$$

$$\begin{cases} x - y = p \\ x \cdot y = q. \end{cases}$$

Expressed in modern terms, the solution proceeds as follows. In the identity

$$\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 = xy,$$

two terms are known, and we can therefore calculate the term on the left side that is still unknown. Then, after taking its square root, we are able to determine the two unknown quantities x and y by using the identity previously encountered in solving linear systems:

$$x,\ y = \frac{x+y}{2} \pm \frac{x-y}{2}.$$

These are not the only identities known to the Mesopotamians, and other identities allow different systems to be solved. Thus,

$$\left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 = \frac{x^2+y^2}{2}$$

is used to solve the system

$$\begin{cases} x^2 + y^2 = p \\ x \pm y = q, \end{cases}$$

and

$$\frac{x^2+y^2}{4} \pm \frac{x \cdot y}{2} = \left(\frac{x \pm y}{2}\right)^2$$

is used to solve

$$\begin{cases} x^2+y^2 = p \\ x \cdot y = q. \end{cases}$$

Example 3. Tablet 8862, Louvre Museum, Problem 1 [Appendix A.5]

Here the answer obtained is also verified at the end, but this time it is given immediately after the statement of the problem as well.

(α) *A rectangle. I multiplied the length by the width, I thus constructed an area. Then I added to the area that by which the length exceeds the width;* (*it gives*) 183. *Then I added the length to the width;* (*it gives*) 27. *What are the length, the width, and the area?*

27	183	*the sums*	
15	*the length*		
		180	*the area*
12	*the width.*		

You proceed thus. Add 27, *the sum of the length and the width, to* 183*;* (*it gives*) 210. *Add* 2 *to* 27*;* (*it gives*) 29. *Divide in two* 29*;* (*it gives*) $14+\frac{1}{2}$*; by* $14+\frac{1}{2}$, (*it gives*) $210+\frac{1}{4}$. *From* $210+\frac{1}{4}$ *you subtract* 210*;* (*it gives*) $\frac{1}{4}$, *the remainder.* $\frac{1}{4}$ *is the square of* $\frac{1}{2}$. *Add* $\frac{1}{2}$ *to the first* $14+\frac{1}{2}$*;* (*it gives*) 15, *the length. You subtract* $\frac{1}{2}$ *from the second* $14+\frac{1}{2}$*;* (*it gives*) 14, *the width. You subtract* 2, *which you added to* 27, *from* 14, *the width;* (*it gives*) 12, *the true width.*

I multiplied 15, *the length, by* 12, *the width.* 15 *by* 12 (*gives*) 180, *the area. By what does* 15, *the length, exceed* 12, *the width? It exceeds* (*it*) *by* 3. *Add* 3 *to* 180, *the area;* (*it gives*) 183, (*the sum of the excess of the length over the width and*) *the area.*

(β) We must solve the following system:

$$\begin{cases} xy + (x-y) = 183 \\ x + y = 27. \end{cases}$$

The calculations performed in the text are

$$\begin{aligned}
&27 + 183 = 210\\
&27 + 2 = 29\\
&29/2 = 14 + \frac{1}{2}\\
&\left(14 + \frac{1}{2}\right)^2 = 210 + \frac{1}{4}\\
&210 + \frac{1}{4} - 210 = \frac{1}{4}\\
&\sqrt{\frac{1}{4}} = \frac{1}{2}\\
&\left(14 + \frac{1}{2}\right) + \frac{1}{2} = 15 \qquad \text{“length”}\\
&\left(14 + \frac{1}{2}\right) - \frac{1}{2} = 14 \qquad \text{“width”}\\
&14 - 2 = 12 \qquad\qquad\quad \text{“true width.”}
\end{aligned}$$

The verification follows these calculations.
(γ) Interpretation: Again, we consider the system

$$\begin{cases} xy + (x - y) = 183\\ x + y = 27. \end{cases}$$

Adding the two equations yields

$$xy + (x - y) + (x + y) = x(y + 2) = 210.$$

Now add 2 to both sides of the second equation to obtain

$$x + (y + 2) = 29.$$

Setting $y' = y + 2$ then yields the standard system already seen (page 11):

$$\begin{cases} x + y' = 29\\ x \cdot y' = 210. \end{cases}$$

Indeed, the text calculates

$$\left(\frac{x + y'}{2}\right)^2 - xy' = \left(\frac{x - y'}{2}\right)^2$$

and then

$$\frac{x + y'}{2} \pm \frac{x - y'}{2} = x,\ y'.$$

The true width, as opposed to the “width” y', is then $y = y' - 2$. The text has thus performed what we would call a change of variable in order to reduce the proposed system to the standard form. Although nothing is explicitly said about this, the computations performed do not allow of any other interpretation.

Example 4. Tablet 13901, British Museum, Problem 18 [Appendix A.6]
(α) *I added the area of my three squares;* (*it gives*) 1400*. The side of one exceeds the side of the other by* 10*.*

You multiply by 1 *the* 10 *that exceeds;* (*it gives*) 10*. You multiply* (*it*) *by* 2*;* (*it gives*) 20*. You multiply* 20 *by* 20*;* (*it gives*) 400*. You multiply* 10 *by* 10*;* (*it gives*) 100*. You add* (*it*) *to* 400*;* (*it gives*) 500*. You subtract* (*it*) *from* 1400*;* (*it gives*) 900*. You multiply* (*it*) *by* 3*, the squares*[17]*; you write* 2700*. You add* 10 *and* 20*;* (*it gives*) 30*. You multiply* 30 *by* 30*;* (*it gives*) 900*. You add* (*it*) *to* 2700*;* (*it gives*) 3600*. It is the square of* 60*. You subtract* (*from it*) 30*, which you multiplied; you write* 30*. You multiply by* 30 *the inverse of* 3*, the squares,* (*which is*) $\frac{1}{3}$*;* (*it gives*) 10*, the side of the* (*first*) *square. You add* 10 *to* 10*;* (*it gives*) 20*, the side of the second square. You add* 10 *to* 20*;* (*it gives*) 30*, the side of the third square.*
(β) The calculations performed are

$$1 \cdot 10 = 10$$
$$2 \cdot 10 = 20$$
$$20 \cdot 20 = 400$$
$$10 \cdot 10 = 100$$
$$400 + 100 = 500$$
$$1400 - 500 = 900$$
$$3 \cdot 900 = 2700$$
$$10 + 20 = 30$$
$$30 \cdot 30 = 900$$
$$900 + 2700 = 3600$$
$$\sqrt{3600} = 60$$
$$60 - 30 = 30$$
$$30 \cdot \frac{1}{3} = 10, \text{ yielding the other values 20 and 30.}$$

(γ) Let us now examine the problem in our way. We are to solve

$$\begin{cases} x^2 + y^2 + z^2 = 1400 \\ x - y = 10 \\ y - z = 10. \end{cases}$$

Taking $y = z + 10$ and $x = y + 10 = z + 20$, the first equation becomes

$$(z + 20)^2 + (z + 10)^2 + z^2 = 1400.$$

Equivalently,

$$3z^2 + 2(20 + 10)z + (20^2 + 10^2) = 1400,$$

[17]That is, the *number* of squares (according to the statement, and also the coefficient of x^2 in the equation). This way of expressing a coefficient is common and not specifically Mesopotamian; it is found until the Renaissance (see pages 129–130, 135, 138, note 175).

or

$$3z^2 + 2 \cdot 30z + 500 = 1400.$$

Thus

$$3z^2 + 2 \cdot 30z = 900,$$

so

$$z = \frac{\sqrt{30^2 + 2700} - 30}{3} = \frac{60 - 30}{3} = 10.$$

Note that all numerical values occurring above are calculated successively in the text. Once again, however, there is no hint of the method followed.

Example 5. Tablet 13901, British Museum, Problem 9 [Appendix A.7]
(α) *I added the area of my two squares;* (*it gives*) 1300*. The side of one exceeds the side of the other by* 10*.*

You divide in two 1300*; you write* 650*. You divide in two* 10*;* (*it gives*) 5*. You multiply* (*it*) *by* 5*;* (*it gives*) 25*. You subtract* (*it*) *from* 650*;* (*it gives*) 625*. It is the square of* 25*. You write* 25 *twice. You add* 5*, which you multiplied, to the first* 25*;* (*it gives*) 30*, the side of the* (*first*) *square. You subtract* 5 *from the second* 25*;* (*it gives*) 20*, the side of the second square.*
(β) The calculations are

$$\frac{1300}{2} = 650$$
$$\frac{10}{2} = 5$$
$$5 \cdot 5 = 25$$
$$650 - 25 = 625$$
$$\sqrt{625} = 25$$
$$25 + 5 = 30$$
$$25 - 5 = 20.$$

(γ) The corresponding system is

$$\begin{cases} x^2 + y^2 = 1300 \\ x - y = 10. \end{cases}$$

We can therefore apply the identities seen above (page 11) to find x and y:

$$\sqrt{\frac{x^2 + y^2}{2} - \left(\frac{x - y}{2}\right)^2} \pm \frac{x - y}{2} = \frac{x + y}{2} \pm \frac{x - y}{2} = x, \ y.$$

This is indeed how the text proceeds.

Now, consider all these examples together. Had we not followed each text by its interpretation, we might have imagined that Mesopotamian algebra consisted of no more than a series of blind computations arriving at the correct answer by trial and error. After considering the interpretation, however, another picture emerges. The Mesopotamian student needed to memorize a small number of identities, and the art of solving these problems then consisted in transforming each problem into a standard form and calculating the solution, often using the half-sum or half-difference of the two unknowns at the end. Although this is certainly not the only solution method in use at the time, it is the most characteristic, which is why we have focused on it here. Moreover, as we will soon see, this method did not disappear with Mesopotamian mathematics.