

## Chapter III

### Isometries in the Plane: Classification and Structure

In the previous chapter we considered general properties of isometries in the plane, culminating in the *First Structure Theorem* (Theorem II.4.5), stating that any isometry of the plane can be expressed as the product of three or fewer reflections. In this chapter we consider isometries in finer detail. In particular, we will obtain a complete list — a *classification* — of all isometries of the plane.

The *First Structure Theorem* provides the organizing theme for our analysis: consider the products of 0, 1, 2, or 3 reflections. If we know what is produced by such products, then *we will know all the isometries of the plane!* The first two cases are completely trivial: “zero” reflections produce the **identity** transformation, while “one” reflection produces a **reflection** (we said these were trivial!).

In the first section we consider the products of *two* reflections. Such products result in all the **translations** when the lines are parallel and **rotations** (hence also **point inversions**) when the lines are distinct but intersecting. These facts depend on non-trivial foundational results concerning the *translation of rays* and *addition of directed angle measures*.

In subsequent sections we study the products of three reflections. (A question to consider: what do you think is produced by products of *three* reflections?) This will lead to the *classification* of all isometries in the plane as well as the formalization of the concept of **orientation** of an isometry. The chapter ends with introducing the fundamental concept of a **group of transformations** and the **rotation angle** of an orientation-preserving isometry. All the concepts introduced in this chapter are absolutely fundamental for what follows in subsequent chapters of this book.

#### §III.1 Two Reflections: Translations and Rotations

Our primary concern in this section is the analysis of *translations* and *rotations* of the plane as first introduced in Definitions II.1.10 and II.1.17. The collection of translations of the plane is of particular importance in a transformational understanding of Euclidean geometry.

**Parallel Lines and Translations.** Consider the composite of two reflections in parallel lines  $\ell$  and  $m$ . We suggest you experiment: draw two parallel lines and a triangle, then determine what happens as the triangle is reflected over the first line, then over the second (triangles drawn on transparencies are particularly useful in determining how they transform under a sequence of reflections). Hopefully you'll convince yourself that a composition of two reflections in parallel lines produces a *translation*.

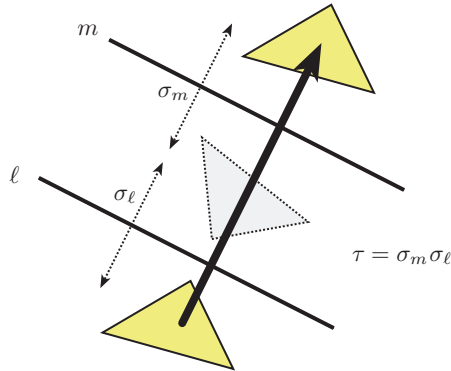


Figure 1.1. The translation  $\tau$  as a product of reflections.

What follows is the fundamental result we'll use to prove that *all* translations can be expressed in this manner.

**Proposition 1.2.**

Suppose  $\ell$  and  $m$  are two distinct parallel lines. Choose  $A \in \ell$  and define  $B = \sigma_m(A)$  so that  $m$  is the perpendicular bisector of  $\overline{AB}$ . Then  $\sigma_m \sigma_\ell = \tau_{AB}$ . The translational displacement of  $\tau_{AB}$  is twice the distance between  $\ell$  and  $m$ .

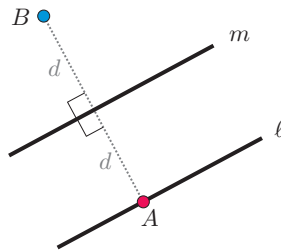


Figure 1.3. The standard translation configuration.

*Proof.* Proving  $\sigma_m \sigma_\ell = \tau_{AB}$  is equivalent to proving  $\phi = \sigma_m \tau_{AB}$  equals the reflection  $\sigma_\ell$ . However, by Theorem II.4.4, *Isometries and Fixed Points*, to verify  $\phi = \sigma_\ell$ , you have only to show that  $\phi \neq \iota$  and  $\phi$  fixes at least two points on the line  $\ell$ . If  $\phi = \iota$ , then  $\tau_{AB} = \sigma_m$ , which is impossible since  $\tau_{AB}$  has no fixed points (or fixes *all* points in the trivial case) while  $\sigma_m$  fixes one line of points. To show  $\phi$  fixes at least two point on  $\ell$ , one such point is  $A$ :

$$\phi(A) = \sigma_m \tau_{AB}(A) = \sigma_m(B) = A$$

since  $m$  is the perpendicular bisector of  $\overline{AB}$ . Then take any point  $x \in \ell$  other than  $A$  and define  $y = \sigma_m(x)$ . We claim  $\square AByx$  is a parallelogram, which will show  $\tau_{AB}(x) = y = \sigma_m(x)$  and  $\phi(x) = \sigma_m \tau_{AB}(x) = \sigma_m(y) = x$ , as desired:



To verify that  $\square AByx$  is a parallelogram, we verify the conditions of Theorem I.10.16, *Parallelogram Construction*:

- (1)  $\overrightarrow{AB}$  is perpendicular to  $m$  by definition of  $B$ , and  $\overrightarrow{xy}$  is perpendicular to  $m$  since  $y = \sigma_m(x)$ . Hence  $\overline{AB}$  and  $\overline{xy}$  are parallel by Proposition I.9.2.
- (2) Let  $d$  be the distance between the lines  $\ell$  and  $m$ . Then  $AB = 2d = xy$  since there is a well defined distance  $d$  between the parallel lines  $\ell$  and  $m$  by Exercise I.10.4.
- (3) Since  $\overline{AB}$  and  $\overline{xy}$  both contain points of  $m$  and all points of  $m$  lie on the same side of  $\ell$ , then  $B$  and  $y$  must also lie on the same side of  $\ell$  by Proposition I.5.5.

Thus  $\square AByx$  is a parallelogram, finishing the proof of Proposition 1.2.  $\square$

Proposition 1.2 can be extended in an extremely important fashion: all translations arise as products of two reflections in parallel lines. Of even more importance, *translating* the two parallel lines to two new parallel lines will not change the translation. This ability to “pairwise adjust” the lines of reflection is far more useful than you would likely imagine. These results are all contained in the following theorem.

**Theorem 1.4.**

- (a) An isometry  $\phi$  is a translation if and only if  $\phi = \sigma_m \sigma_\ell$  for two parallel lines  $m$  and  $\ell$ .
- (b) Let  $\ell_1$  and  $\ell_2$  be distinct parallel lines. Then  $\sigma_{\ell_2} \sigma_{\ell_1} = \sigma_{m_2} \sigma_{m_1}$  for two other lines  $m_1$  and  $m_2$  if and only if there exists a translation  $\tau$  such that  $\tau(\ell_1) = m_1$  and  $\tau(\ell_2) = m_2$ .

*Proof.* (a) This is essentially a rephrasing of Proposition 1.2. If  $\phi = \iota$ , then  $\phi = \sigma_\ell \sigma_\ell$  for any line  $\ell$ . In the non-trivial case  $\phi = \tau_{AB}$ ,  $A \neq B$ , let  $m$  be the perpendicular bisector of  $\overline{AB}$  and  $\ell$  the line perpendicular to  $\overline{AB}$  at  $A$ .

Then Proposition 1.2 gives  $\phi = \sigma_m \sigma_\ell$ , as desired. Conversely, if  $\phi = \sigma_m \sigma_\ell$  for two parallel lines  $m$  and  $\ell$ , choose  $A$  to be any point on  $\ell$  and define  $B = \sigma_m(A)$ . Then  $\phi = \tau_{AB}$ , again from Proposition 1.2.

(b) Suppose  $\ell_1$  and  $\ell_2$  are parallel lines and  $\tau$  is a translation mapping  $\ell_1$  onto the line  $m_1$  and  $\ell_2$  onto the line  $m_2$ . As shown in Figure 1.5 choose  $A \in \ell_1$  and  $B = \sigma_{\ell_2}(A)$  so that  $A, B, \ell_1, \ell_2$  are in the configuration of Proposition 1.2. Hence

$$\sigma_{\ell_2} \sigma_{\ell_1} = \tau_{AB}.$$

Now let  $C = \tau(A)$  and  $D = \tau(B)$ , also shown in Figure 1.5. Then, from Theorem II.2.11,  $\overline{AB}$  and  $\overline{CD}$  are directed line segments with the same length and direction; hence

$$\tau_{AB} = \tau_{CD}$$

by Theorem II.1.13. However, since  $\tau$  is an isometry, it preserves lengths, parallelism, and perpendicularity (Exercise II.2.7). Thus  $\overline{CD}$  is perpendicular to both  $m_1$  and  $m_2$ , with  $m_2$  bisecting  $\overline{CD}$ . Hence  $D = \sigma_{m_2}(C)$ , verifying that  $C, D, m_1, m_2$  are in the configuration of Proposition 1.2 and yielding

$$\tau_{CD} = \sigma_{m_2} \sigma_{m_1}.$$

Combining the last three equations gives  $\sigma_{\ell_2} \sigma_{\ell_1} = \sigma_{m_2} \sigma_{m_1}$ , as desired.

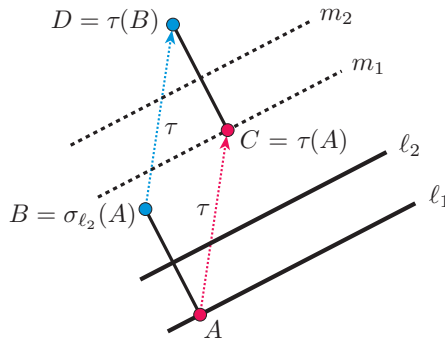


Figure 1.5. Assuming a translation  $\tau$  mapping  $\ell_1, \ell_2$  onto  $m_1, m_2$ .

Conversely, suppose  $\sigma_{\ell_2} \sigma_{\ell_1} = \sigma_{m_2} \sigma_{m_1}$ ; we need to construct a translation  $\tau$  that maps  $\ell_1$  and  $\ell_2$  onto  $m_1$  and  $m_2$ . First observe that  $m_1$  and  $m_2$  must be parallel lines since otherwise the isometry  $\sigma_{m_2} \sigma_{m_1}$  (which is a translation since it equals  $\sigma_{\ell_2} \sigma_{\ell_1}$ ) would have a unique fixed point, an impossibility for a translation. We can thus choose points  $A, B, C, D$  so that both  $A, B, \ell_1, \ell_2$ , and  $C, D, m_1, m_2$  are in Proposition 1.2 configuration, as shown in Figure 1.6.

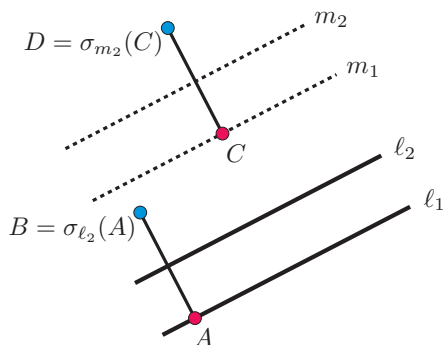


Figure 1.6. Assuming  $\sigma_{l_2}\sigma_{l_1} = \sigma_{m_2}\sigma_{m_1}$ , determine the translation  $\tau$ .

Thus  $\tau_{AB} = \sigma_{l_2}\sigma_{l_1} = \sigma_{m_2}\sigma_{m_1} = \tau_{CD}$ , so Theorem II.1.13 gives that  $\overline{AB}$  and  $\overline{CD}$  have the same length and direction, and Theorem II.2.11 then implies there exists a translation  $\tau$  mapping  $A$  to  $C$  and  $B$  to  $D$ . Thus  $\tau(l_1)$  will be a line containing  $\tau(A) = C$  that is perpendicular to  $\tau(\overline{AB})$ . Since  $m_1$  has these same properties and such a line is unique,  $\tau(l_1) = m_1$ . Similarly  $\tau(l_2) = m_2$ , as desired.  $\square$

**Intersecting Lines and Rotations.** We have seen that a translation is merely the composite of two reflections in parallel lines. We now consider the composite of two reflections in intersecting lines  $\ell$  and  $m$ . Experiment as you did in the parallel lines case: draw two intersecting lines and a triangle, then determine what happens as the triangle is reflected over the first line, then over the second. The result should look like that in Figure 1.7; it should convince you that a composite of reflections in intersecting lines produces a *rotation* (Definition II.1.17) about the point of intersection.

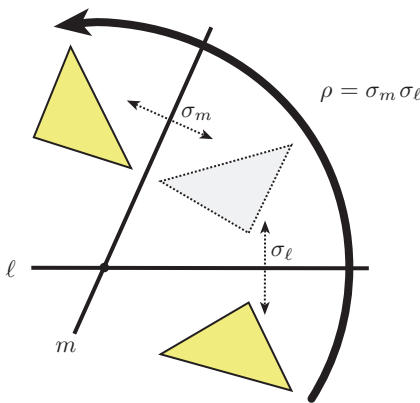


Figure 1.7. A rotation  $\rho$  as a product of reflections.

In fact, *all* rotations can be expressed as products of two reflections in intersecting lines. Establishing this fact will depend on the following result.

**Proposition 1.8.**

Suppose  $m$  and  $\ell$  are two distinct lines intersecting at  $p$ . Pick points  $A \in \ell$ ,  $A \neq p$ , and  $C \in m$ ,  $C \neq p$ , and define  $\theta = m\angle ApC$ . Then  $\sigma_m\sigma_\ell = \rho_{p,2\theta}$ , the rotation about  $p$  through the directed angle  $2\theta$ .

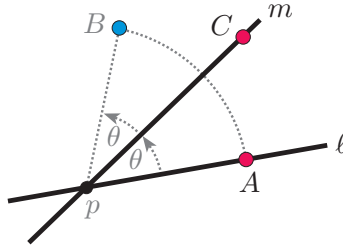


Figure 1.9. The standard rotation configuration.

*Remark.* Given intersecting lines  $\ell$  and  $m$ , there are two possibilities for the directed angle between them, and thus two choices for the directed angle measure  $\theta$ . These are shown in Figure 1.10. To understand how these two values relate to each other, suppose  $\overrightarrow{pC'}$  is the ray opposite  $\overrightarrow{pC}$ .

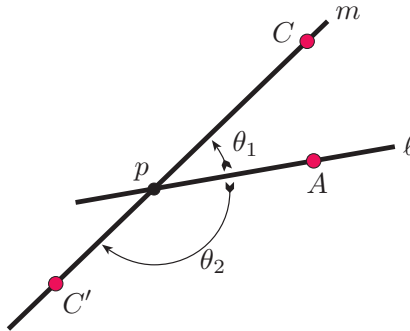


Figure 1.10. The two directed angles for intersecting lines.

The properties of directed angle measure (Definition I.6.15) give

$$\begin{aligned}
 \pi &= m\angle C'pC \pmod{2\pi} \\
 &= m\angle C'pA + m\angle ApC \pmod{2\pi} \\
 &= -m\angle ApC' + m\angle ApC \pmod{2\pi} \\
 &= -\theta_2 + \theta_1 \pmod{2\pi},
 \end{aligned}$$

where  $\theta_1 = m\angle ApC$  and  $\theta_2 = m\angle ApC'$  are the two possible values for  $\theta$ . However, from the above computation we have  $\theta_1 = \theta_2 + \pi \pmod{2\pi}$ , and thus doubling the angles gives  $2\theta_1 = 2\theta_2 + 2\pi = 2\theta_2 \pmod{2\pi}$ . This proves, no matter which of the angles  $\theta_1$  or  $\theta_2$  is given to  $\theta$  in Proposition 1.8, that the rotation  $\rho_{p,2\theta}$  will be the same. Thus, choosing  $\overrightarrow{pC}$  or  $\overrightarrow{pC'}$  will change nothing in the statement of Proposition 1.8 except for  $\theta = \theta_1$  or  $\theta_2$ . Hence,

without loss of generality, we may assume the choice of ray  $\overrightarrow{pC}$  for which  $\angle ApC$  is an acute or a right angle, i.e.,  $-\pi/2 < \theta \leq \pi/2 \pmod{2\pi}$ .

*Proof.* As just observed, we can assume  $-\pi/2 < \theta \leq \pi/2 \pmod{2\pi}$ . For convenience we select  $\theta$  to be its specific value in this range — hence we can dispense with  $\pmod{2\pi}$ .

We desire to show  $\rho_{p,2\theta} = \sigma_m\sigma_\ell$ , which is equivalent to proving that the isometry  $\phi = \sigma_m\rho_{p,2\theta}$  equals the reflection  $\sigma_\ell$ . However, from Theorem II.4.4 (*Isometries and Fixed Points*) to prove  $\phi = \sigma_\ell$ , it is sufficient to show  $\phi \neq \iota$  and that  $\phi$  fixes two distinct points on the line  $\ell$ . We know  $\phi \neq \iota$  since otherwise  $\rho_{p,2\theta} = \sigma_m$  which is impossible since a rotation and a reflection have different fixed point sets. As for fixed points for  $\phi$  on  $\ell$ , one is  $p$ :

$$\phi(p) = \sigma_m(\rho_{p,2\theta}(p)) = \sigma_m(p) = p.$$

To verify that  $\phi$  fixes other points on  $\ell$ , we need to first consider the generic case  $-\pi/2 < \theta < \pi/2$ , i.e.,  $\rho_{p,2\theta}$  is not equal to the point inversion  $\nu_p$ . Choose any  $x \in \overrightarrow{pA}$  other than  $p$  and define  $y = \rho_{p,2\theta}(x)$ . Since  $-\pi < 2\theta < \pi$ , the ray  $\overrightarrow{pC}$  is in the interior of the angle  $\angle xpy$  and thus intersects the line segment  $\overline{xy}$  by Theorem I.5.9 (*Crossbar Theorem*). Denote this point of intersection as  $c$ . This is shown in Figure 1.11.

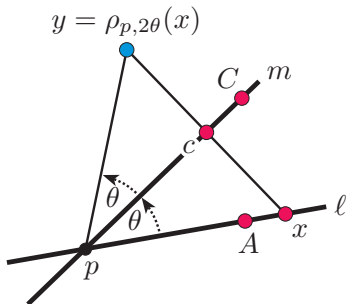


Figure 1.11. The two directed angles for intersecting lines.

By *SAS* it is easily seen that  $\triangle xpc \cong \triangle ypc$ , which implies  $yc = cx$  and  $\angle pcy \cong \angle pcx$ . Since these are supplementary angles, both must be right angles. Hence  $m = \overleftrightarrow{pc}$  is the perpendicular bisector of  $\overline{xy}$ , which gives  $x = \sigma_m(y)$ . Thus

$$\phi(x) = \sigma_m(\rho_{p,2\theta}(x)) = \sigma_m(y) = x,$$

proving  $x$  is a fixed point of  $\phi$ . This finishes the proof that  $\phi = \sigma_\ell$ , hence  $\rho_{p,2\theta} = \sigma_m\sigma_\ell$ , except for the case where  $\rho_{p,2\theta} = \nu_p$ , which is Exercise 1.1.  $\square$

We are now able to show that *all* rotations arise as composites of two reflections in intersecting lines. Moreover, *rotating* the two intersecting lines about their point of intersection  $p$  to two new lines intersecting at  $p$  will not

change the rotation given by the two reflections. This ability to “pairwise adjust” the lines of reflection matches the same property we found for translations in Theorem 1.4b. As you will see, *pairwise adjustment* of reflections is an extremely important tool in our transformational view of geometry.

**Theorem 1.12.**

- (a) *An isometry  $\phi$  is a rotation about  $p$  if and only if  $\phi = \sigma_m\sigma_\ell$  for two lines  $m$  and  $\ell$  intersecting at  $p$ .*
- (b) *Let  $\ell_1, \ell_2$  be distinct lines intersecting at  $p$ . Then  $\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{m_2}\sigma_{m_1}$  for two other lines  $m_1, m_2$  if and only if there exists a rotation  $\rho_{p,\psi}$  about  $p$  such that  $\rho_{p,\psi}(\ell_1) = m_1$  and  $\rho_{p,\psi}(\ell_2) = m_2$ .*

*Proof.* (a) Suppose  $\phi = \rho_{p,2\theta}$ . In the trivial case where  $2\theta = 0 \pmod{2\pi}$ ,  $\phi = \iota = \sigma_\ell\sigma_\ell$  for any line  $\ell$ . In the non-trivial case,  $\phi = \rho_{p,2\theta}$  will have only one fixed point (the center of rotation  $p$ ) and thus Theorem II.4.4 (*Isometries and Fixed Points*) applies to show  $\phi$  equals  $\sigma_m\sigma_\ell$  for two distinct lines  $\ell, m$  intersecting at  $p$ .

Conversely, suppose  $\phi = \sigma_m\sigma_\ell$  for lines  $\ell, m$  intersecting at  $p$ . The simple case is where  $\ell = m$ , which gives  $\phi = \iota$ , the trivial rotation. If  $\ell$  and  $m$  are distinct lines, choose points  $A \in \ell, C \in m$  such that neither equals  $p$ , and let  $\theta = m\angle ApC$ . Then Proposition 1.8 gives  $\rho_{p,2\theta} = \sigma_m\sigma_\ell = \phi$ , as desired.

(b) Suppose  $\ell_1, \ell_2$  are distinct lines intersecting at  $p$ , and assume  $\rho = \rho_{p,\psi}$  is a rotation about  $p$  that maps  $\ell_1$  onto  $m_1$  and  $\ell_2$  onto  $m_2$ , i.e.,  $\rho(\ell_1) = m_1$  and  $\rho(\ell_2) = m_2$ . We need to prove  $\sigma_{m_2}\sigma_{m_1} = \sigma_{\ell_2}\sigma_{\ell_1}$ .

As shown in Figure 1.13 choose points  $A \in \ell_1, C \in \ell_2$  such that neither equals  $p$ , and let  $\theta = m\angle ApC$ . By Proposition 1.8 we see  $\sigma_{\ell_2}\sigma_{\ell_1} = \rho_{p,2\theta}$ .

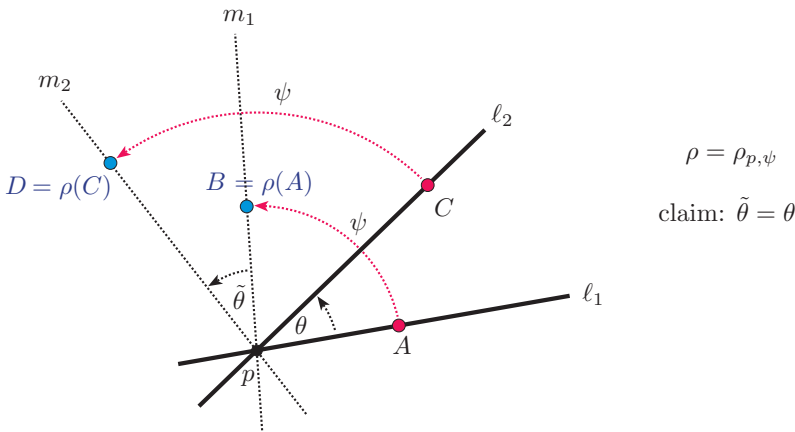


Figure 1.13. Rotating  $\ell_1, \ell_2$  to  $m_1, m_2$  will not change  $\sigma_{\ell_2}\sigma_{\ell_1} = \rho_{p,2\theta}$ .

Now define  $B = \rho(A) \in m_1$ ,  $D = \rho(C) \in m_2$ , and  $\tilde{\theta} = m\angle BpD$ . Then Proposition 1.8 again applies (since  $m_1$  and  $m_2$  must intersect at  $p$  because  $\rho$  fixes  $p$ ) to show  $\sigma_{m_2}\sigma_{m_1} = \rho_{p,2\tilde{\theta}}$ . We will thus obtain our desired equality of  $\sigma_{\ell_2}\sigma_{\ell_1}$  and  $\sigma_{m_2}\sigma_{m_1}$  by simply showing  $\theta = \tilde{\theta}$ .

We prove  $\theta = \tilde{\theta}$  by using the properties of directed angle measure as given in Definition I.6.15. “Split” the directed angle  $\angle ApD$  in two different ways: via the ray  $\overrightarrow{pC}$  and also by the ray  $\overrightarrow{pB}$ . Since directed angle measure is *additive*, the measure of the directed angle  $\angle ApD$  will thus be expressed in two ways as the sum of the measures of two other directed angles. Equating these two expressions for  $m\angle ApD$  will yield the desired equality of  $\theta$  and  $\tilde{\theta}$ . Here are the computations:

$$\begin{aligned} m\angle ApD &= m\angle ApC + m\angle CpD = \theta + \psi, \\ m\angle ApD &= m\angle ApB + m\angle BpD = \psi + \tilde{\theta}. \end{aligned}$$

We have, of course, used  $m\angle ApB = m\angle CpD = \psi$  since  $B$  and  $D$  are the rotations of the points  $A$  and  $C$  about  $p$  with directed angle measure  $\psi$ . Thus  $\theta + \psi = \psi + \tilde{\theta}$ , which gives  $\theta = \tilde{\theta}$ , as desired.

Conversely, suppose  $\ell_1, \ell_2$  are distinct lines intersecting at  $p$  and such that  $\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{m_2}\sigma_{m_1}$  for two other lines  $m_1, m_2$ . We need to construct a rotation  $\rho = \rho_{p,\psi}$  such that  $\rho(\ell_1) = m_1$  and  $\rho(\ell_2) = m_2$ . Our procedure will be to define  $\rho$  as any rotation about  $p$  such that  $\rho(\ell_1) = m_1$ . We will then prove it must also map  $\ell_2$  onto  $m_2$ , as desired.

First choose  $A \in \ell_1$  and  $C \in \ell_2$ , neither equal to  $p$ , and define  $\theta = m\angle ApC$ . Then  $\sigma_{\ell_2}\sigma_{\ell_1} = \rho_{p,2\theta}$  from Proposition 1.8, and hence  $\sigma_{m_2}\sigma_{m_1} = \rho_{p,2\theta}$ . From (a) we thus have that the directed angle from  $m_1$  to  $m_2$  has the same measure  $\theta$  as the directed angle from  $\ell_1$  to  $\ell_2$ . See Figure 1.14.

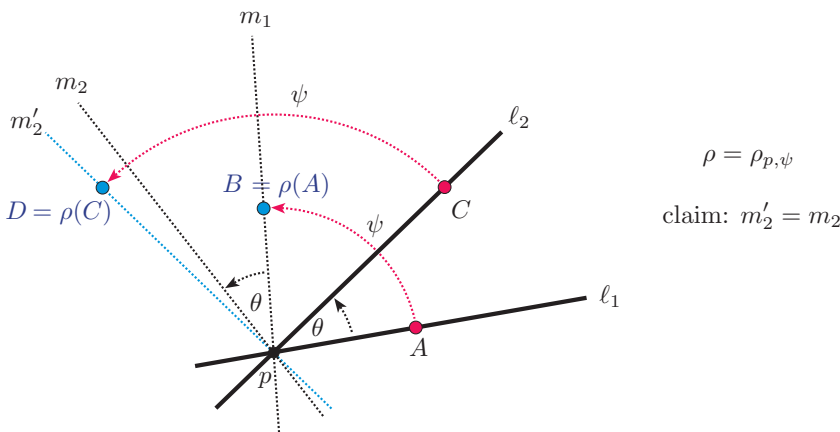


Figure 1.14. If  $\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{m_2}\sigma_{m_1}$ , then  $\ell_1, \ell_2$  rotate to  $m_1, m_2$ .

It should be intuitively clear that the directed angle from  $\ell_1$  to  $m_1$  must have the same measure  $\psi$  as the directed angle from  $\ell_2$  to  $m_2$ . The rotation we desire with then be  $\rho = \rho_{p,\psi}$ . We need only verify our intuitive insights.

Choose  $B \in m_1$  such that the length  $pB$  equals  $pA$ , define the directed angle measure  $\psi$  as  $m\angle ApB$ , and let  $\rho$  denote the rotation  $\rho_{p,\psi}$ . Then  $B = \rho(A)$  from the definition of  $B$ , proving  $\rho(\ell_1) = m_1$  since  $\rho$  maps both points  $p, A \in \ell_1$  to  $p, B \in m_1$ .

We are left with verifying  $\rho(\ell_2) = m_2$ . To do so, define  $D = \rho(C)$  and let  $m'_2$  be the line containing  $p$  and  $D$ . Since  $\rho(\ell_2) = m'_2$ , we must prove  $m'_2 = m_2$ . We know  $m_2$  is that unique line through  $p$  such that the directed angle measure from  $m_1$  is  $\theta$ . We now show  $m'_2$  satisfies this same condition, which will complete the proof that  $\rho(\ell_2) = m_2$ . We need only use the properties of directed angle measure as given in Definition I.6.15. As done earlier, we “split” the directed angle  $\angle ApD$  in two different ways:

$$\begin{aligned} m\angle ApD &= m\angle ApC + m\angle CpD = \theta + \psi, \\ m\angle ApD &= m\angle ApB + m\angle BpD = \psi + m\angle BpD. \end{aligned}$$

Thus  $m\angle BpD = \theta$ , proving  $\sigma_{m'_2}\sigma_{m_1} = \rho_{p,2\theta}$  from Proposition 1.8. This gives  $\sigma_{m'_2}\sigma_{m_1} = \sigma_{m_2}\sigma_{m_1}$ , and hence  $\sigma_{m'_2} = \sigma_{m_2}$ , or  $m'_2 = m_2$ . We have therefore established  $\rho_{p,\psi}(\ell_2) = m_2$ , as desired!  $\square$

**Pairwise Adjustment of Reflections.** As shown in Theorems 1.4 and 1.12 there is a useful redundancy in the product of two reflections  $\sigma_{\ell_2}\sigma_{\ell_1}$ : we do not need to know the *absolute* positions of each line  $\ell_1$  and  $\ell_2$ ; we only need to know their positions *relative to each other*. In particular, given a product of two reflections  $\sigma_{\ell_2}\sigma_{\ell_1}$ , we may “adjust” the lines  $\ell_1$  and  $\ell_2$  by any translation (if  $\ell_1$  and  $\ell_2$  are parallel) or by any rotation about their intersection point (if  $\ell_1$  and  $\ell_2$  intersect) without changing the isometry that results from the product  $\sigma_{\ell_2}\sigma_{\ell_1}$ . We refer to the movement of reflection lines allowed in this way as *pairwise adjustment of reflections*. This provides us with a powerful tool for simplifying complicated transformational problems. This will become quite obvious in our subsequent material.

We summarize this important technique in the next theorem. This is merely a rephrasing of the relevant portions of Theorems 1.4b and 1.12b.

**Theorem 1.15. *Pairwise Adjustment of Reflections.***

*Suppose  $\ell_1$  and  $\ell_2$  are two lines in the plane.*

- (a) *Suppose the lines  $\ell_1$  and  $\ell_2$  are parallel. Translate the lines by any fixed translation  $\tau$  to obtain two new lines  $m_1$  and  $m_2$ . Then the product of the reflections  $\sigma_{\ell_2}\sigma_{\ell_1}$  remains unchanged, i.e.,*

$$\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{m_2}\sigma_{m_1}.$$

- (b) Suppose the lines  $\ell_1$  and  $\ell_2$  intersect at a point  $p$ . Rotate the lines about  $p$  by any fixed rotation  $\rho$  to obtain two new lines  $m_1$  and  $m_2$  through  $p$ . Then the product of the reflections  $\sigma_{\ell_2}\sigma_{\ell_1}$  remains unchanged, i.e.,

$$\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{m_2}\sigma_{m_1}.$$

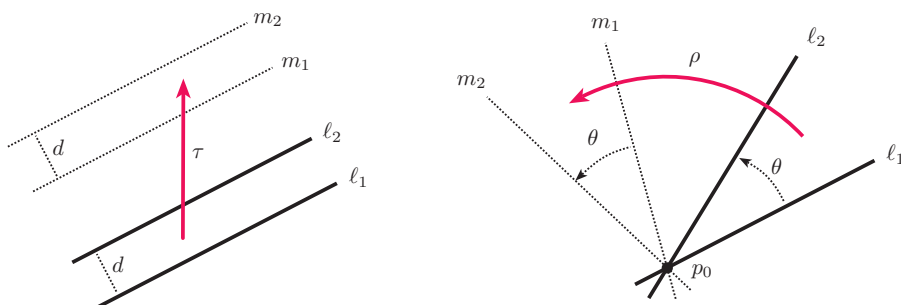


Figure 1.16. Pairwise adjustment of reflections.

**Translations and Point Inversions.** From Theorem 1.4a we know that a transformation  $\tau$  is a translation if and only if it is the product of two reflections in parallel lines. As a first application of *pairwise adjustment of reflections* we will show that a transformation  $\tau$  is a translation if and only if it is the product of two point inversions. This characterization of translations in terms of point inversions will prove useful numerous times in our subsequent work.

**Proposition 1.17.**

- (a) The product of two point inversions  $\nu_b\nu_A$  about the points  $A$  and  $b$  is the translation  $\tau_{AB}$ , where  $b$  is the midpoint of  $\overline{AB}$ .
- (b) Fix a point  $A$ . Any translation  $\tau$  can be expressed as a product of point inversions, the first of which is centered on  $A$ :

$$\tau = \nu_b\nu_A \text{ for some point } b.$$

*Proof.* (a) From Proposition 1.8 we can express our point inversions in the forms  $\nu_A = \sigma_{\ell_2}\sigma_{\ell_1}$  and  $\nu_b = \sigma_{m_2}\sigma_{m_1}$ , where  $\ell_1$  and  $\ell_2$  are perpendicular lines intersecting at  $A$  and  $m_1$  and  $m_2$  are perpendicular lines intersecting at  $b$ . Then  $B$ , the image of  $A$  under  $\nu_b$ , will appear as in Figure 1.18.

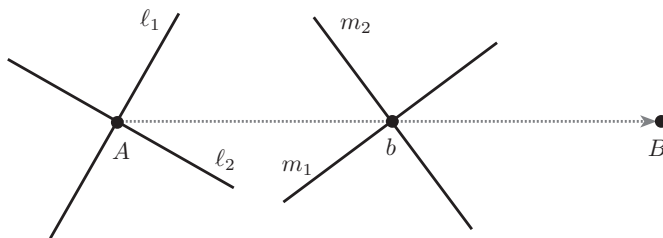


Figure 1.18. The product  $\nu_b\nu_A$  should equal  $\tau_{AB}$ .

However, from Theorem 1.15, *Pairwise Adjustment of Reflections*, we can rotate the lines  $\ell_1, \ell_2, m_1,$  and  $m_2$  to a more convenient configurations. In particular, pick  $m_1$  to contain the point  $A$  (in addition to the point  $b$ ) and  $\ell_2$  to contain  $b$  (in addition to  $A$ ). Hence  $m_1 = \ell_2$  and Figure 1.18 is replaced by Figure 1.19.

Hence  $\nu_b\nu_A = (\sigma_{m_2}\sigma_{m_1})(\sigma_{\ell_2}\sigma_{\ell_1}) = \sigma_{m_2}\sigma_{\ell_1}$  since  $\sigma_{m_1}\sigma_{\ell_2} = \sigma_{m_1}^2 = \iota$ . But  $\ell_1$  and  $m_2$  are both perpendicular to  $m_1$ , hence parallel to each other. Thus  $\sigma_{m_2}\sigma_{\ell_1}$  is a translation  $\tau$  by Theorem 1.4a. Let  $B$  be the image of  $A$  under this translation. Then  $\nu_b\nu_A = \tau = \tau_{AB}$ . We claim  $b$  is the midpoint of  $\overline{AB}$ . To see this, observe that  $B = \tau(A) = \nu_b\nu_A(A) = \nu_b(A)$  since point inversion about  $A$  leaves  $A$  unchanged. But by definition of point inversion,  $B = \nu_b(A)$  means  $b$  is the midpoint of  $\overline{AB}$ , as desired. This proves (a).

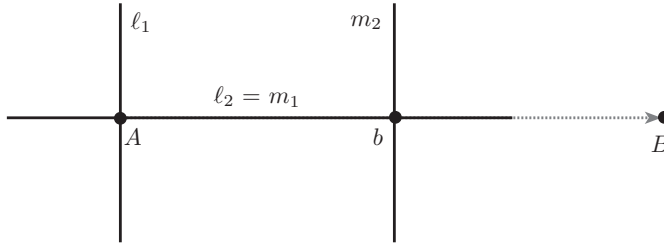


Figure 1.19. Better choices for the point inversion reflection lines.

(b) Fix a point  $A$  and let  $\tau$  be any translation. Then by Theorem 1.4a we can express  $\tau$  in the form  $\tau = \sigma_{m_2}\sigma_{\ell_1}$  for two parallel lines  $\ell_1$  and  $m_2$ . But from Theorem 1.15, *Pairwise Adjustment of Reflections*, we can adjust the pair  $\ell_1, m_2$  so that  $\ell_1$  contains the point  $A$ . Then let  $\ell_2$  and  $m_1$  both denote the line which passes through  $A$  and is perpendicular to  $\ell_1$  and  $m_2$ . Let  $b$  denote the intersection of  $\ell_2 = m_1$  with  $m_2$ . We thus have exactly the same configuration of lines as was shown above in Figure 1.19. Thus

$$\begin{aligned} \tau &= \sigma_{m_2}\sigma_{\ell_1} = \sigma_{m_2}(\sigma_{m_1}\sigma_{\ell_2})\sigma_{\ell_1} \text{ since } \sigma_{m_1}\sigma_{\ell_2} = \iota, \\ &= (\sigma_{m_2}\sigma_{m_1})(\sigma_{\ell_2}\sigma_{\ell_1}) \\ &= \nu_b\nu_A \text{ by Proposition 1.8 since } m_1 \perp m_2 \text{ and } \ell_1 \perp \ell_2. \end{aligned}$$

This finishes the proof of Proposition 1.17. □

**Theorem 1.20.**

*The product of two translations in the plane is another translation. In particular, if  $\tau_1$  takes the point  $A$  to the point  $B$  and  $\tau_2$  takes the point  $B$  to the point  $C$ , then*

$$\tau_2\tau_1 = \tau_{BC}\tau_{AB} = \tau_{AC}.$$

*Proof.* This will follow from our characterization of translations as products of point inversions. For given the point  $A$  and the translation  $\tau_1 = \tau_{AB}$ , it

follows that  $\tau_1 = \nu_b\nu_A$  from Proposition 1.17 where  $b$  is the midpoint of  $\overline{AB}$ . Moreover, applying the same proposition to the point  $b$  and the translation  $\tau_2$  gives the existence of a point  $c$  such that  $\tau_2 = \nu_c\nu_b$ . Combining our formulas for  $\tau_1$  and  $\tau_2$  yields  $\tau_2\tau_1 = (\nu_c\nu_b)(\nu_b\nu_A) = \nu_c\nu_A$  since  $\nu_b^2 = \iota$ . Hence  $\tau_2\tau_1$  is seen to be a translation  $\tau$  by Proposition 1.17.

Since  $\tau_2\tau_1$  is a translation  $\tau$ , the equation in Theorem 1.20 follows from Theorem II.1.13. For let  $B = \tau_1(A)$  and  $C = \tau_2(B)$ . Then

$$\tau(A) = \tau_2\tau_1(A) = C.$$

Hence  $\tau_1 = \tau_{AB}$ ,  $\tau_2 = \tau_{BC}$ , and  $\tau = \tau_{AC}$ . Thus  $\tau_2\tau_1 = \tau$  gives the desired equality  $\tau_{BC}\tau_{AB} = \tau_{AC}$ .  $\square$

————— Exercises III.1 —————

**Exercise 1.1.**

Finish the proof of Proposition 1.8 by considering  $\theta = \pm\pi/2 \pmod{2\pi}$ , i.e., the case when  $\ell$  and  $m$  are perpendicular as shown in Figure 1.21. You must prove:

*Suppose  $\ell$  and  $m$  are perpendicular lines intersecting at a point  $p$ . Then the composite  $\sigma_m\sigma_\ell$  equals the point inversion  $\nu_p$  through  $p$ .*

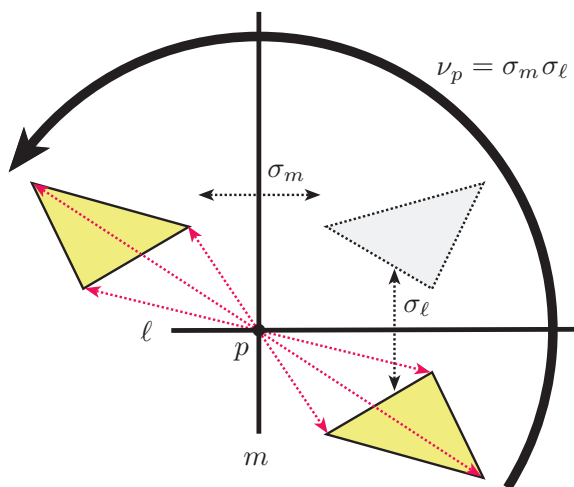


Figure 1.21. Reflections in perpendicular lines yield point inversions

**Exercise 1.2.**

Describe *as precisely as possible* what results from the product  $\nu_A\nu_B\nu_C$  of three point inversions in three distinct points  $A, B, C$ . When the points are non-collinear your description of  $\nu_A\nu_B\nu_C$  should include a parallelogram.

**Exercise 1.3.**

Use *pairwise adjustment of reflections* (Theorem 1.15) to strengthen Theorems 1.4a and 1.12a as follows:

- Fix a point  $q$  in the plane. Show that any rotation may be expressed as a composite  $\sigma_m\sigma_\ell$ , where the line  $\ell$  contains  $q$ . Obtain a similar result for translations.
- Fix a line  $\ell'$ . Show that any rotation may be expressed as a product  $\sigma_m\sigma_\ell$ , where the line  $\ell$  is parallel to  $\ell'$ . Is there a corresponding result for translations?

**Exercise 1.4.**

Prove that the composition of a non-trivial rotation and a translation (in either order) is another rotation with the same angle as the original rotation.

*Hint:* Use pairwise adjustment of reflections.

**Exercise 1.5.**

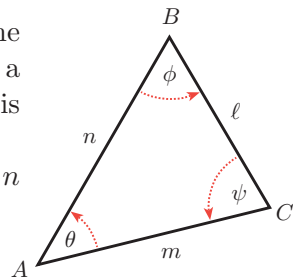
Showing that the product of two translations  $\tau_1$  and  $\tau_2$  is indeed another translation,  $\tau = \tau_2\tau_1$  in Theorem 1.20 involved writing  $\tau_1$  and  $\tau_2$  each as a product of half-turns. Here is another approach. Write  $\tau_1 = \sigma_{\ell_2}\sigma_{\ell_1}$  and  $\tau_2 = \sigma_{m_2}\sigma_{m_1}$  as reflections in pairs of parallel lines by Theorem 1.4a. There are two cases to consider.

- If all four lines  $\ell_1, \ell_2, m_1, m_2$  are parallel, then one judicious shift of two of these lines will show  $\tau = \tau_2\tau_1$  is a translation.
- If  $\ell_1, \ell_2$  are not parallel to  $m_1, m_2$ , let  $p$  be the point of intersection of  $\ell_2$  and  $m_1$ . A twist about  $p$  will prove  $\tau = \tau_2\tau_1$  is a translation.

**Exercise 1.6.**

- Consider the triangle  $\triangle ABC$  as shown to the right. Show that the product  $\rho_{B,2\phi}\rho_{A,2\theta}$  is a rotation about the third vertex  $C$ . What is the angle of this rotation?

*Hint:* Use reflections in the lines  $\ell, m$ , and  $n$  comprising the sides of  $\triangle ABC$ .



- Show, for *any* three lines  $\ell, m$ , and  $n$ ,

$$\sigma_m\sigma_\ell\sigma_\ell\sigma_n\sigma_n\sigma_m = \iota.$$

- Group the product in (b) in pairs and, in the case when  $\ell, m$ , and  $n$  form a triangle  $\triangle ABC$  as shown, interpret the resulting equation in terms of  $\triangle ABC$ .

**Exercise 1.7.**

Given isometries  $\phi$  and  $\psi$ , the **conjugate** of  $\psi$  by  $\phi$  is the isometry

$$\psi^\phi = \phi\psi\phi^{-1}.$$

Though it may look odd when first encountered, conjugation will prove to be one of our most important geometric tools. In this exercise you will have your first look at conjugation, using it to prove that the product of translations is *commutative*.

- (a) Prove that the conjugate of a reflection  $\sigma_\ell$  by  $\phi$  equals  $\sigma_{\phi(\ell)}$ , i.e.,

$$\phi\sigma_\ell\phi^{-1} = \sigma_{\phi(\ell)}.$$

*Hint:* The *Isometries and Fixed Points Theorem*. From this result you have only to show that (1)  $\phi\sigma_\ell\phi^{-1}$  fixes every point of the line  $\phi(\ell)$ , i.e.,  $\phi\sigma_\ell\phi^{-1}$  fixes every point of the form  $\phi(x)$  for  $x \in \ell$  and (2)  $\phi\sigma_\ell\phi^{-1} \neq \iota$  (if this were true, then a little composition algebra shows  $\sigma_\ell = \iota$ , which is false).

- (b) Prove the conjugate of a translation  $\tau_{AB}$  by  $\phi$  equals  $\tau_{\phi(A)\phi(B)}$ , i.e.,

$$\phi\tau_{AB}\phi^{-1} = \tau_{\phi(A)\phi(B)}.$$

*Hint:* Theorem 1.4a. From this result you have  $\tau_{AB} = \sigma_m\sigma_\ell$  for two parallel lines  $\ell$  and  $m$ . Thus

$$\phi\tau_{AB}\phi^{-1} = \phi\sigma_m\sigma_\ell\phi^{-1} = \phi\sigma_m\phi^{-1}\phi\sigma_\ell\phi^{-1},$$

setting you up to apply (a).

- (c) Use (b) to prove that a product of translations is commutative, i.e., if  $\tau_1$  and  $\tau_2$  are both translations, then  $\tau_2\tau_1 = \tau_1\tau_2$ .

*Hint:* The equality  $\tau_2\tau_1 = \tau_1\tau_2$  is equivalent to  $\tau_2\tau_1\tau_2^{-1} = \tau_1$ . So let  $\tau_1 = \tau_{AB}$ , and then apply (b) followed by Theorem II.2.11. (Part of this result was established using different methods in Exercise II.3.8d.)

- (d) What is the conjugate of a point inversion  $\nu_p$  by  $\phi$ ? Prove it!
- (e) Prove that a point inversion  $\nu_p$  commutes with an isometry  $\phi$  if and only if  $\phi$  fixes  $p$ , i.e.,  $\nu_p\phi = \phi\nu_p$  if and only if  $\phi(p) = p$ .

*Hint:* The equality  $\nu_p\phi = \phi\nu_p$  is equivalent to  $\phi\nu_p\phi^{-1} = \nu_p$ .

**Exercise 1.8.**

In Exercise 1.7 the *conjugate* of  $\psi$  by  $\phi$  was defined to be

$$\psi^\phi = \phi\psi\phi^{-1}.$$

As was shown in that exercise, the conjugates of any reflection  $\sigma_\ell$  or translation  $\tau_{AB}$  are

$$\phi\sigma_\ell\phi^{-1} = \sigma_{\phi(\ell)} \quad \text{and} \quad \phi\tau_{AB}\phi^{-1} = \tau_{\phi(A)\phi(B)}.$$

In this exercise you determine the conjugates of any rotation  $\rho_{p,\theta}$ . It is not hard to show that the conjugate of a rotation is another rotation.

However, determining the sign of the resulting rotation angle requires some care.

- (a) Prove  $\phi\rho_{p,\theta}\phi^{-1} = \rho_{\phi(p),\pm\theta}$  for any isometry  $\phi$ . The plus/minus sign indicates that for some isometries  $\phi$  the resulting rotation angle will be  $\theta$  while for others the angle will be  $-\theta$ .

*Hint:* Theorem 1.12b and Proposition II.2.9f.

- (b) Prove  $\tau\rho_{p,\theta}\tau^{-1} = \rho_{\tau(p),\theta}$  for any translation  $\tau$ .

*Hint:* Remember that directed angle measure is *translationally invariant*, as defined in Definition I.11.14.

- (c) Prove  $\sigma_m\rho_{p,\theta}\sigma_m^{-1} = \rho_{p,-\theta}$  for any reflection  $\sigma_m$  in a line  $m$  that contains the point  $p$ .

*Hint:* Express  $\rho_{p,\theta}$  as a product of reflections, one of which is  $\sigma_m$ .

- (d) Prove  $\sigma_\ell\rho_{p,\theta}\sigma_\ell^{-1} = \rho_{\sigma_\ell(p),-\theta}$  for any reflection  $\sigma_\ell$ .

*Hint:* Let  $m$  be the line through  $p$  which is parallel to  $\ell$ . Then use (c) to express  $\rho_{p,\theta}$  as  $\sigma_m\rho_{p,-\theta}\sigma_m^{-1}$ .

- (e) Strengthen (a) as follows. An isometry  $\phi$  is said to have **even parity** if it can be expressed as the product of an even number of reflections and **odd parity** if it can be expressed as the product of an odd number of reflections. Prove that

$$\phi\rho_{p,\theta}\phi^{-1} = \begin{cases} \rho_{\phi(p),\theta} & \text{if } \phi \text{ has even parity,} \\ \rho_{\phi(p),-\theta} & \text{if } \phi \text{ has odd parity.} \end{cases}$$

- (f) Use (e) to show that an isometry has either even parity or odd parity, *but not both!* (This topic will be revisited in §4.)

### Exercise 1.9.

Suppose  $m$  is a translationally invariant directed angle measure. For any isometry  $\phi$  define

$$m^\phi(\angle A) = m(\phi(\angle A)) \quad \text{for all directed angles } \angle A.$$

- (a) Prove  $m^\phi$  is a translationally invariant directed angle measure.

*Hint:* Use Exercises 1.7b and II.2.9b.

- (b) Show that  $m^\phi$  must equal  $m$  or  $-m$  for all directed angles, i.e.,

$$m^\phi(\angle A) = m(\angle A) \quad \text{for all directed angles } \angle A \text{ or}$$

$$m^\phi(\angle A) = -m(\angle A) \quad \text{for all directed angles } \angle A.$$

$\phi$  is called *rotation preserving* if  $m^\phi = m$  or *reversing* if  $m^\phi = -m$ .

- (c) Are the following isometries rotation preserving or reversing:

$$\sigma_\ell, \quad \tau_{AB}, \quad \nu_p, \quad \rho_{p,\theta}?$$

- (d) Show that an isometry  $\phi$  of the plane is *rotation preserving* if and only if it has *even parity* as defined in Exercise 1.8e.

### §III.2 Glide Reflections

From the previous section we know the isometries that result from a product of two reflections: translations and rotations. However, from the *First Structure Theorem*, Theorem II.4.5, knowing the isometries that result from a product of *three* reflections is the key to finishing our classification of all isometries of the plane. So in this section we consider products of the form

$$\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$$

for lines  $\ell_1$ ,  $\ell_2$ , and  $\ell_3$ . There are two special cases we dispose of quickly: when the three lines intersect in one point or when the lines are parallel.

#### Proposition 2.1.

- (a) *Suppose three lines intersect at a common point  $p$ . Then the triple product of reflections in these lines is again a reflection. The line of reflection for this product contains  $p$ .*
- (b) *Suppose three lines are mutually parallel. Then the triple product of reflections in these lines is again a reflection. The line of reflection for this product is parallel to the original three lines.*

*Proof.* (a) Suppose the three lines  $\ell_1$ ,  $\ell_2$ , and  $\ell_3$  intersect at a single point  $p$  as shown in the first panel of Figure 2.2. We consider the product  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$ . According to Theorem 1.15b we can rotate the pair of lines  $\ell_1$ ,  $\ell_2$  about the point  $p$  until  $\ell_2$  moves onto  $\ell_3$  and the isometry given by  $\sigma_{\ell_2}\sigma_{\ell_1}$  will not change in the process! Let  $m_1$  and  $m_2 = \ell_3$  be the adjusted lines (as in the center panel of Figure 2.2), so that  $\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{m_2}\sigma_{m_1}$ .

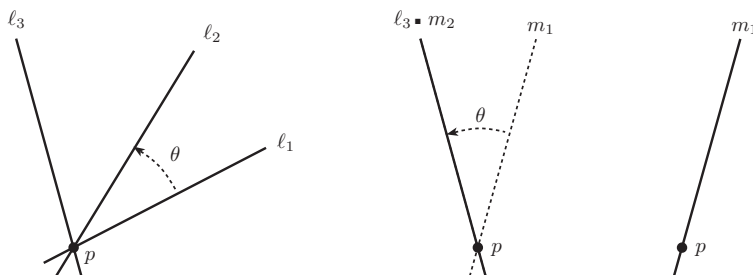


Figure 2.2. Simplifying a product of three reflections in intersecting lines.

Our original triple product then becomes

$$\begin{aligned} \sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1} &= \sigma_{\ell_3}\sigma_{m_2}\sigma_{m_1} \\ &= \sigma_{\ell_3}\sigma_{\ell_3}\sigma_{m_1} \quad \text{since } m_2 \text{ equals } \ell_3, \\ &= \sigma_{\ell_3}^2\sigma_{m_1} \\ &= \sigma_{m_1} \quad \text{since } \sigma^2 = \iota \text{ for any reflection } \sigma. \end{aligned}$$

Thus the triple product  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  is merely the reflection  $\sigma_{m_1}$ , as shown in the final panel of Figure 2.2. This proves statement (a).

(b) Suppose the three lines  $\ell_1$ ,  $\ell_2$ , and  $\ell_3$  are mutually parallel. Then from Theorem 1.15a we can translate the pair of lines  $\ell_1, \ell_2$  until  $\ell_2$  moves onto  $\ell_3$  and the isometry given by  $\sigma_{\ell_2}\sigma_{\ell_1}$  will not change in the process! The argument now proceeds as in the first case, resulting in our product  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  reducing to a single reflection, as desired.  $\square$

**Glide Reflections.** The products  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  of Proposition 2.1 did not produce new isometries — they yielded reflections. However, we now consider a special configuration of lines  $\ell_1, \ell_2, \ell_3$  that *does* produce something new.

The configuration we desire is shown in Figure 2.3: the reflection axes  $\ell_1$  and  $\ell_3$  are distinct, parallel lines, separated by a distance  $d > 0$ , while the remaining axis  $\ell_2$  is perpendicular to both  $\ell_1$  and  $\ell_3$ .

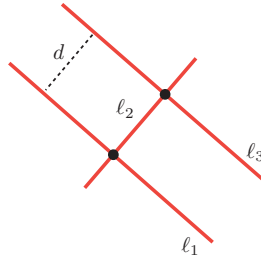


Figure 2.3. The “special” configuration for three reflection axes.

Let  $q_1$  and  $q_3$  be the intersection points of  $\ell_2$  with  $\ell_1$  and  $\ell_3$ , respectively. Since  $\ell_2$  is perpendicular to both  $\ell_1$  and  $\ell_3$ , then  $\sigma_{\ell_1}\sigma_{\ell_2} = \nu_{q_1} = \sigma_{\ell_2}\sigma_{\ell_1}$  and  $\sigma_{\ell_2}\sigma_{\ell_3} = \nu_{q_3} = \sigma_{\ell_3}\sigma_{\ell_2}$ , i.e., the reflection  $\sigma_{\ell_2}$  in  $\ell_2$  commutes with both  $\sigma_{\ell_1}$  and  $\sigma_{\ell_3}$ . Hence, when considering the product  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$ , it does not matter where we insert the reflection  $\sigma_{\ell_2}$ , i.e.,

$$\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{\ell_2}\sigma_{\ell_3}\sigma_{\ell_1} = \sigma_{\ell_3}\sigma_{\ell_1}\sigma_{\ell_2}. \quad (2.4)$$

The advantage of these alternate forms for  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  is the appearance of the isometry  $\sigma_{\ell_3}\sigma_{\ell_1}$ . Since  $\ell_1$  and  $\ell_3$  are parallel lines, by Proposition 1.2 we know that the product  $\sigma_{\ell_3}\sigma_{\ell_1}$  is a translation  $\tau$  parallel to  $\ell_2$  and by a distance  $2d$ , twice the distance between  $\ell_1$  and  $\ell_3$ , as shown in Figure 2.5.

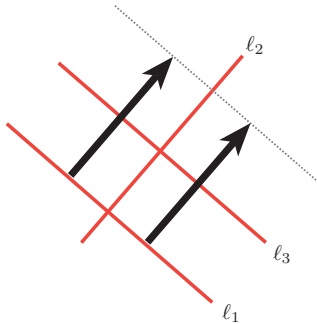


Figure 2.5. The translation  $\tau = \sigma_{\ell_3}\sigma_{\ell_1}$ .

Expressed using the translation  $\tau$ , equation (2.4) becomes

$$\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{\ell_2}\tau = \tau\sigma_{\ell_2}.$$

From this we can now explain the geometric action of the isometry  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$ : it takes a point  $x$  and *translates* (or *glides*) it parallel to  $\ell_2$  by a distance  $2d$ , twice the distance from  $\ell_1$  to  $\ell_3$ , and then *reflects* the resulting point across  $\ell_2$ . This action is shown in Figure 2.7 and leads to the following definition.

**Definition 2.6.** *Glide reflection  $\gamma_{\ell,s}$  with axis  $\ell$ , parameter  $s$ .*

Fix a directed line  $\ell$  and a number  $s \geq 0$ . The **glide reflection**  $\gamma_{\ell,s}$  with **glide axis**  $\ell$  and **glide length** (or **glide parameter**)  $s \geq 0$  is defined by

$$\gamma_{\ell,s} = \sigma_{\ell}\tau_s$$

where  $\tau_s$  is translation parallel to  $\ell$  with translational displacement  $s$  in the positive direction of  $\ell$ .

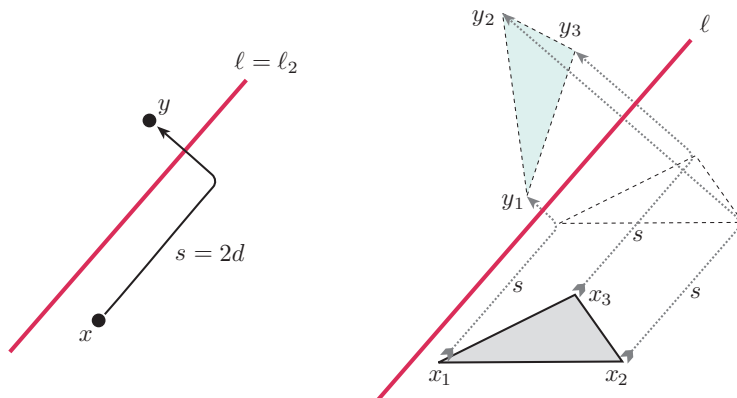


Figure 2.7. The glide reflection  $\gamma_{\ell,s}$  with axis  $\ell$  and parameter  $s$ .

It is convenient to allow for a glide in the negative direction of  $\ell$ . To do so, for any  $s < 0$  let  $\gamma_{\ell,s}$  denote the glide reflection parallel to  $\ell$  with glide length  $-s$  in the negative direction of  $\ell$ . The **glide parameter** of  $\gamma_{\ell,s}$  is still defined to be  $s$ . When  $s = 0$ , the glide reflection is merely reflection in the line  $\ell$ ; hence  $\gamma_{\ell,s}$  is considered *non-trivial* when  $s \neq 0$ . Summarizing,

given a directed line  $\ell$  and  $s \in \mathbb{R}$ , define  $\gamma_{\ell,s} = \sigma_{\ell}\tau_s$  where

$$\tau_s = \begin{cases} \text{translation with displacement } s \\ \text{in the positive direction of } \ell \text{ if } s \geq 0, \\ \text{translation with displacement } -s \\ \text{in the negative direction of } \ell \text{ if } s < 0. \end{cases}$$

From our various descriptions of the glide reflection  $\gamma_{\ell,s}$  you should see that the transformation preserves the line  $\ell$  while interchanging the two half

planes on either side of  $\ell$ . On the line  $\ell$  itself  $\gamma_{\ell,s}$  is merely a translation by the distance  $|s|$ . It should also be clear that a non-trivial  $\gamma_{\ell,s}$  has no fixed points, i.e., there are no points that  $\gamma_{\ell,s}$  maps to themselves. These facts lead to the next proposition, whose proof is left to Exercise 2.3.

**Proposition 2.8.**

Two glide reflections  $\gamma_{\ell,s}$  and  $\gamma_{m,r}$  are equal if and only if either

- $\ell = m$  and  $s = r$  when  $\ell$  and  $m$  have the same direction or
- $\ell = m$  and  $s = -r$  when  $\ell$  and  $m$  have opposite directions.

A helpful illustration for a glide reflection  $\gamma_{\ell,s}$  is the “footprints” picture of Figure 2.9. The image of the first footprint is the second; the image of the second is the third; etc.

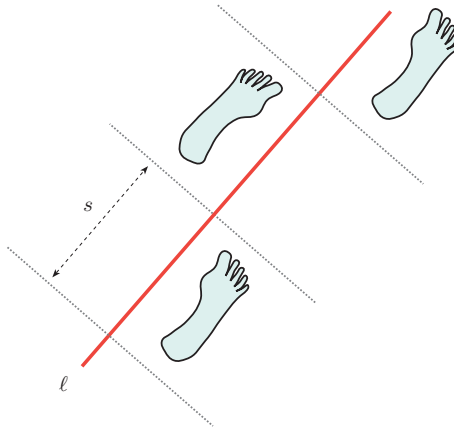


Figure 2.9. Glide reflections generate “footprints.”

**General Three Reflection Products.** If three lines are arranged in the special way illustrated in Figure 2.3, then the associated reflections produce an isometry which, while subtler than those studied earlier, can still be readily visualized. What happens when the three lines are placed in an arbitrary arrangement is less easy to picture directly. Remarkably, it turns out that we still get only glide reflections! This is the next result.

**Theorem 2.10.**

Suppose three lines  $\ell_1, \ell_2, \ell_3$  do not have a common intersection point nor are they all parallel to each other. Then the triple product  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  of the reflections defined by the lines is a non-trivial glide reflection.

*Remark.* It is important and interesting to explicitly identify the glide axis  $\ell$  and glide length  $s$  produced by this product. These will emerge during the proof of the theorem and will be further investigated in Exercise 2.9.

*Proof.* Our conditions on the lines  $\ell_1, \ell_2,$  and  $\ell_3$  imply that at least one of the lines is not parallel to either of the others and that this non-parallel line

intersects the remaining two in *distinct* points. So, in particular, we know that  $\ell_1$  intersects  $\ell_2$  and/or  $\ell_2$  intersects  $\ell_3$ . We will assume the former case, i.e., that  $\ell_1$  intersects  $\ell_2$  at a point  $p_0$ . (It is easy to show that the argument we give in the first case will apply with minor changes to the second.)

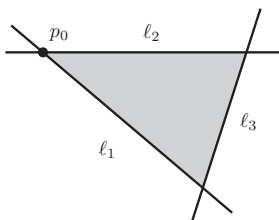


Figure 2.11. The general case for a product of three reflections.

According to Theorem 1.15b we may rotate the lines  $\ell_1$  and  $\ell_2$  about  $p_0$  without changing the isometry  $\sigma_{\ell_2}\sigma_{\ell_1}$ . Hence we rotate to two new lines —  $\ell'_1$  and  $\ell'_2$  — where  $\ell'_2$  is now perpendicular to  $\ell_3$ . Designate the point of intersection as  $q$  — see Figure 2.12. If the angle between  $\ell'_1$  and  $\ell'_2$  is  $\pi/2$ , then  $\ell'_1$  and  $\ell_3$  are parallel and we are in the situation of Figure 2.3. Hence  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{\ell_3}\sigma_{\ell'_2}\sigma_{\ell'_1}$  is a glide reflection with glide axis  $l = \ell'_2$ , as we desired to show.

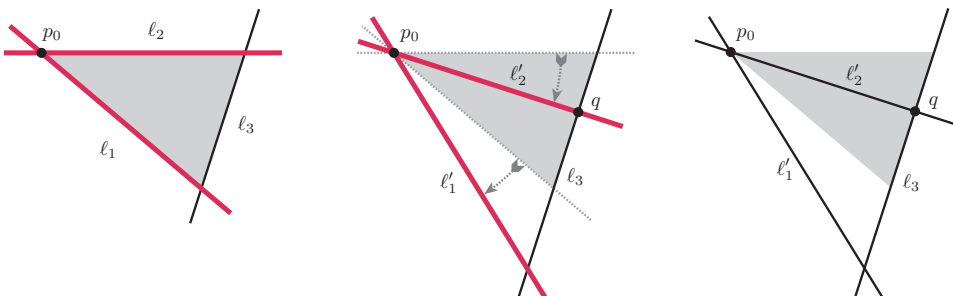


Figure 2.12. Rotating  $\ell_1$  and  $\ell_2$  about  $p_0$ .

So assume the angle between  $\ell'_1$  and  $\ell'_2$  is not  $\pi/2$ . Then rotate the lines  $\ell'_2$  and  $\ell_3$  about  $q$  to new lines  $\ell''_2$  and  $\ell'_3$ , where  $\ell''_2$  is perpendicular to  $\ell'_1$  — see Figure 2.13.

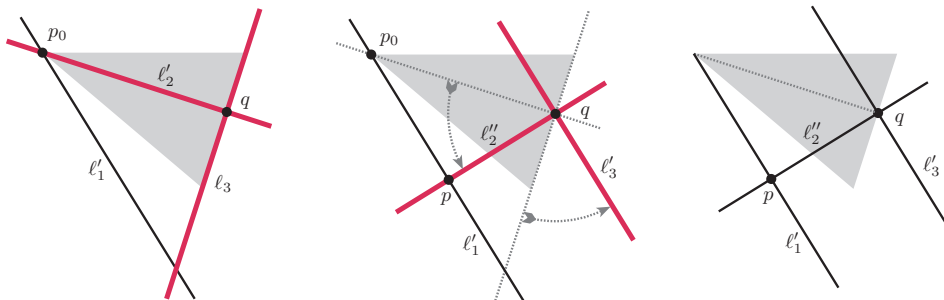


Figure 2.13. Rotating  $\ell'_2$  and  $\ell_3$  about  $q$ .

The end result of this process is that our original product of three reflections is still expressed as a product of three reflections,

$$\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{m_3}\sigma_{m_2}\sigma_{m_1},$$

but now the lines are quite nicely related:  $m_1 = \ell'_1$  and  $m_3 = \ell_3$  are parallel and  $m_2 = \ell''_2$  is perpendicular to both — see Figure 2.14 for this final arrangement. This, however, shows that  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  is indeed a glide reflection. That the glide reflection is not-trivial is verified in Exercise 2.10.  $\square$

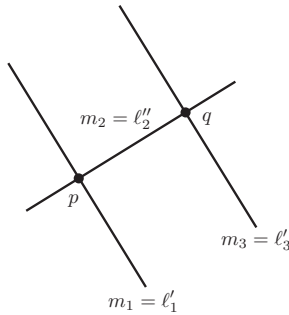


Figure 2.14. The final arrangement:  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  equals  $\sigma_{m_3}\sigma_{m_2}\sigma_{m_1}$ .

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### Exercises III.2

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#### Exercise 2.1.

Show that the inverse of any glide reflection  $\gamma_{\ell,s}$  is also a glide reflection. What is its glide axis and what is its glide parameter?

#### Exercise 2.2.

Show that a non-trivial glide reflection  $\gamma_{\ell,s}$  preserves no line other than the glide axis  $\ell$ , i.e., if  $m$  is any line other than  $\ell$ , then there is a point  $p$  on  $m$  for which  $\gamma_{\ell,s}(p)$  is not on  $m$ .

#### Exercise 2.3.

Prove Proposition 2.8:

*Two glide reflections  $\gamma_{\ell,s}$  and  $\gamma_{m,r}$  are equal if and only if either*

- $\ell = m$  and  $s = r$  when  $\ell$  and  $m$  have the same direction or
- $\ell = m$  and  $s = -r$  when  $\ell$  and  $m$  have opposite directions.

#### Exercise 2.4.

Suppose  $\gamma_{\ell,s}$  is any glide reflection and  $y = \gamma_{\ell,s}(x)$  for  $x \in \mathcal{E}$ . What can you say about the location of  $p$ , the midpoint of  $\overline{xy}$ ? Prove your claim.

#### Exercise 2.5.

Show that any glide reflection  $\gamma_{\ell,s}$  can be expressed as the product of a reflection and a point inversion, i.e.,  $\gamma_{\ell,s} = \sigma_m\nu_{p_0}$ . How does the reflection line  $m$  relate to the glide axis  $\ell$ ? How is the glide length  $|s|$  related to  $m$  and  $p_0$ ? Do  $\sigma_m$  and  $\nu_{p_0}$  commute?

**Exercise 2.6.**

Suppose  $\gamma_{\ell,s}$  is a glide reflection with glide parameter  $s$ . Prove that the distance between any point  $x$  and its image  $y$  under  $\gamma_{\ell,s}$  is at least as large as  $|s|$ , i.e.,  $|xy| \geq |s|$  for any  $x$  and  $y = \gamma_{\ell,s}(x)$ . What can you say about  $x$  if  $|xy| = |s|$ ?

**Exercise 2.7.**

Show that the square of a non-trivial glide reflection  $\gamma_{\ell,s}$  is a non-trivial translation  $\tau$ . How is the translational displacement of  $\tau$  related to  $\ell$  and  $s$ ? Is the converse true, i.e., given any non-trivial translation  $\tau$ , does there exist a glide reflection  $\gamma_{\ell,s}$  such that  $\tau = \gamma_{\ell,s}^2$ ?

**Exercise 2.8.**

The conjugate of one isometry by another,  $\psi^\phi = \phi\psi\phi^{-1}$ , was considered in Exercises 1.7 and 1.8. Suppose  $\gamma_{\ell,s}$  is a glide reflection and  $\phi$  is any isometry. Determine the conjugate of  $\gamma_{\ell,s}$  by  $\phi$ , i.e.,  $\gamma_{\ell,s}^\phi = \phi\gamma_{\ell,s}\phi^{-1}$ .

**Exercise 2.9.**

Let  $\ell_1$ ,  $\ell_2$ , and  $\ell_3$  be lines forming a triangle  $\triangle ABC$  as in Figure 2.15.

- Show that the axis of the glide reflection  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  passes through the foot of the altitude from  $C$  to  $\ell_3$ . *Hint:* Study the proof of Theorem 2.10.
- Similarly, show that the axis also passes through the foot of the altitude from  $A$  to  $\ell_1$ . Hence the axis of  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  may be described as the line passing through the two feet of the altitudes from  $A$  and  $C$  of the triangle  $\triangle ABC$ .
- Clearly the glide axis of  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  cannot also contain the foot of the altitude from  $B$  to  $\ell_2$ . Why do the arguments used in (a) and (b) for vertices  $A$  and  $C$  fail for vertex  $B$ ?

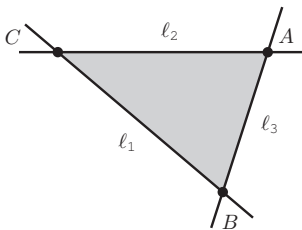


Figure 2.15. The triangle for Exercise 2.9.

The feet of the altitudes of a triangle form another triangle, known as the **orthic triangle** of the original triangle. Exercise 2.9 is evidence that the orthic triangle is closely related to the multiplication of reflections. In Chapter VI we will show that this connection implies several interesting geometric properties of the orthic triangle.

**Exercise 2.10.**

Suppose  $\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  equals a reflection  $\sigma_m$ . Show that either

- (i)  $\ell_1, \ell_2, \ell_3, m$  all intersect at a common point  $p$  or
- (ii)  $\ell_1, \ell_2, \ell_3, m$  are all parallel.

This proves that the glide reflection in Theorem 2.10 is non-trivial, as claimed. *Hint:* Notice that  $\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{\ell_3}\sigma_m$ .

**Exercise 2.11.**

Determine the product of a rotation and a reflection, i.e.,  $\phi = \sigma_\ell\rho_{p,\theta}$ . Be as explicit as possible, e.g., if you claim a translation, then specify the translational displacement in terms of  $\ell, p$ , and  $\theta$ .

*Hint:* Consider the case where  $p \in \ell$  and the case where  $p \notin \ell$ .

**§III.3 The Classification Theorem**

From the *First Structure Theorem*, Theorem II.4.5, we know that any isometry can be built by the product of three or fewer reflections. A “product of zero reflections” is just the **identity** transformation  $\iota$ . A “product of one reflection” is merely the **reflection** itself. A product of two reflections will either degenerate into the identity or yield a non-trivial **rotation** or **translation** (Propositions 1.2 and 1.8). Finally, a product of three reflections will either degenerate into a single reflection (Proposition 2.1) or yield a non-trivial **glide reflection** (Theorem 2.10). This covers all possibilities, producing the following theorem.<sup>1</sup>

**Theorem 3.1. Classification of Plane Isometries.**

*Any isometry of the plane is of exactly one of the following types:*

- (0) **identity** transformation  $\iota$ ,
- (1) **reflection**  $\sigma_\ell$  in a line  $\ell$ ,
- (2a) **rotation**  $\rho_{p,\theta}$  of angle  $0 < \theta < 2\pi$  about a point  $p$ ,
- (2b) **translation**  $\tau_{AB}$  parallel to a non-trivial line segment  $\overline{AB}$ ,
- (3) **glide reflection**  $\gamma_{\ell,s}$  with glide axis  $\ell$  and glide length  $|s| \neq 0$ .

*The isometry types are denoted by the minimum number of reflections (0, 1, 2, or 3) needed in a product to produce the stated type.*

**Fixed Points and Fixed Lines.** The claim in Theorem 3.1 that every isometry is of **exactly** one of the listed types has not yet been verified. We need to be sure that the sets of isometries listed in the various parts of the theorem are indeed *disjoint* from one another. One way to do this is to consider fixed points and fixed lines for these classes of isometries.

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<sup>1</sup>Recall that a *rotation* or *translation* is non-trivial if it is not the identity transformation, and a *glide reflection* is non-trivial if it is not a pure reflection, i.e., if its glide length is non-zero. All the rotations, translations, and glide reflections listed in Theorem 3.1 are non-trivial.

Recall that given a transformation  $\Phi$  and a point  $p$ , we say that  $p$  is a **fixed point** for  $\Phi$  if  $\Phi(p) = p$ , i.e., if  $\Phi$  does not move  $p$ . Similarly, given a line  $\ell$ , we say  $\ell$  is a **fixed line** for  $\Phi$  if  $\Phi(\ell) = \ell$ , i.e., if the image of  $\ell$  under  $\Phi$  is  $\ell$  itself. Note that this does *not* mean that  $\Phi$  fixes every individual point of  $\ell$ ! It only means that every point of the line  $\ell$  is again mapped to a point of  $\ell$ . For example, a non-trivial translation  $\tau$  has no fixed points at all but it fixes every line parallel to the line segment defining the translation.

In the *Isometries and Fixed Points Theorem* (Theorem II.4.4) we noted that the number of fixed points for an isometry gives much information about the minimum number of reflections necessary to produce the isometry. However, given the *Classification Theorem*, we can now greatly strengthen Theorem II.4.4, especially if we include fixed lines. The result is Theorem 3.2, essentially a table describing the fixed points and fixed lines of each type of isometry. (We distinguish between half-turns, i.e., point inversions, and other rotations since half-turns have more fixed lines than non-half-turn rotations.) The straightforward verification of this result is Exercise 3.1.

**Theorem 3.2. Fixed Points and Fixed Lines.**

*The plane isometry types listed in the Classification Theorem are distinct, i.e., every plane isometry belongs to one and only one of the listed types. More specifically, each isometry type has a different configuration of fixed points and fixed lines, as summarized in the following table.*

**Fixed Points and Fixed Lines for Plane Isometries.**

|   | <i>necessary</i>   | <i>fixed</i>                       | <i>fixed</i>   |
|---|--------------------|------------------------------------|--|
|   | <i>reflections</i> | <i>points</i>                      | <i>lines</i>   |
| <i>identity</i> $\iota$                                 | 0                  | <i>all points</i>                  | <i>all lines</i>   |
| <i>reflections</i> $\sigma_\ell$                        | 1                  | <i>points on <math>\ell</math></i> | <i><math>\ell</math> and lines <math>\perp</math> to <math>\ell</math></i> |
| <i>point inversions</i> $\nu_p$                         | 2                  | <i>only <math>p</math></i>         | <i>lines through <math>p</math></i>  |
| <i>rotations</i> $\rho_{p,\theta}$ , $\theta \neq n\pi$ | 2                  | <i>only <math>p</math></i>         | <i>none</i>  |
| <i>translations</i> $\tau_{AB}$ , $A \neq B$            | 2                  | <i>none</i>                        | <i>lines parallel to <math>\overline{AB}</math></i>                        |
| <i>glide reflections</i> $\gamma_{\ell,s}$ , $s \neq 0$ | 3                  | <i>none</i>                        | <i>only <math>\ell</math></i>  |

————— **Exercises III.3** —————

**Exercise 3.1.**

Prove Theorem 3.2, *Fixed Points and Fixed Lines*.

**Exercise 3.2.**

Given points  $p \neq q$ , determine all lines  $\ell$  such that  $\nu_q \sigma_\ell \nu_p \sigma_\ell = \iota$ .

*Hint:* Rewrite the equation in the form  $\nu_q \sigma_\ell = \dots$ , and then consider what each side of this equality does to the point  $p$ .

**Exercise 3.3. Isometries of the Line.**

An *isometry of the line*  $\ell$  is a transformation  $\phi$  mapping  $\ell$  onto  $\ell$  which preserves distance between points. For simplicity we fix a coordinate system on  $\ell$ , allowing us to identify  $\ell$  with  $\mathbb{R}$ , i.e., we will identify each point of  $\ell$  with its coordinate in  $\mathbb{R}$ .

- (a) Identify as many types of isometries of the line as possible, assigning appropriate notation in all cases. In particular, give a formula for *reflection in a point*  $p \in \mathbb{R}$ ,  $\sigma_p(x) = \dots$ . *Hint:* For example,  $\tau_{ab}(x) = x + (b - a)$ . The formulas you obtain in this exercise are *very useful* when working with isometries of the line.
- (b) The results of §II.3 on composition and inversion for isometries of the plane all hold without change for isometries of the line. However, the *Bisector/Fixed Point Relation* (Proposition II.4.2), Proposition II.4.3, the *Isometries and Fixed Points Theorem* (Theorem II.4.4), and the *First Structure Theorem* (Theorem II.4.5) all need modification. State and prove versions of these results for isometries of the line. *Hint:* For Proposition II.4.2 think midpoints.
- (c) State and prove versions of *Pairwise Adjustment of Reflections* (Theorem 1.15), the *Triangle Isometry Theorem* (Theorem II.5.1), the *Classification Theorem* (Theorem 3.1), and the *Fixed Points and Fixed Lines Theorem* (Theorem 3.2) for isometries of the line.

**Exercise 3.4. Orthogonal Extension.**

Suppose  $\phi$  is an isometry of the line  $\ell_0$ . Define  $\phi^e$ , the **orthogonal extension** of  $\phi$  to the plane, in the following manner. To each point  $p_0$  in  $\ell_0$ , define  $m_{p_0}$  to be that line through  $p_0$  which is perpendicular to  $\ell_0$ . To every  $p$  in the plane, let  $\ell_p$  be that line through  $p$  which is parallel to  $\ell_0$  (if  $p$  is on  $\ell_0$ , let  $\ell_p$  equal  $\ell_0$ ), and let  $p'$  be that point such that  $p$  is on  $m_{p_0}$ . Define  $\phi^e(p) = p'$ , where  $p'$  is the intersection of the perpendicular lines  $m_{\phi(p_0)}$  and  $\ell_p$ . Thus  $\square pp_0\phi(p_0)p'$  is a rectangle. The goal of this exercise is to show that  $\phi^e$  is an *isometry of the plane* and that it inherits certain properties from its generator  $\phi$ .

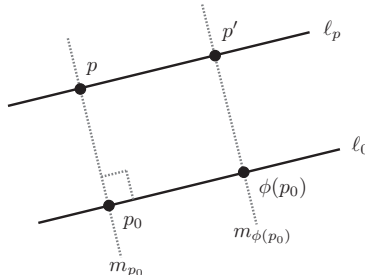


Figure 3.3. The orthogonal extension is  $\phi^e(p) = p'$ .

- (a) Show  $\phi^e$  fixes every line  $\ell$  which is parallel to  $\ell_0$ .

- (b) Show  $\phi^e$  maps line  $m_{p_0}$  onto line  $m_{\phi(p_0)}$  for any point  $p_0$  on  $\ell_0$ .
- (c) If  $p_0$  is a fixed point for  $\phi$  on the line  $\ell_0$ , show that all points on  $m_{p_0}$  are fixed points for  $\phi^e$ . More generally, if any point  $p$  on  $m_{p_0}$  is a fixed point for  $\phi^e$ , show *every* point on  $m_{p_0}$  is fixed by  $\phi^e$ .
- (d) Show  $\phi^e$  preserves distance between points  $p, q$  which lie on a line  $\ell$  parallel to  $\ell_0$ . *Hint:* Show  $pq = p_0q_0$  and  $p'q' = \phi(p_0)\phi(q_0)$ .
- (e) Show  $\phi^e$  is an isometry of the plane. *Hint:* Suppose  $p$  and  $q$  are points in the plane. Then let  $q_p$  be the intersection of  $m_{q_0}$  and  $\ell_p$ . Using (d), prove that  $\triangle pq_pq$  and  $\triangle p'q'_p q'$  are congruent triangles.

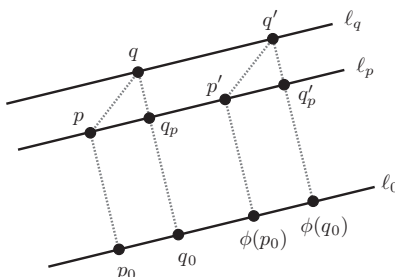


Figure 3.4. The orthogonal extension is  $\phi^e(p) = p'$ .

- (f) Determine those types of isometries of the plane which are extensions via orthogonal extension of isometries of the line. Then determine those which are not. Justify your claims.
- (g) Suppose  $\ell_0$  and  $\ell_1$  are distinct but intersecting lines. Are any isometries of the plane orthogonal extensions from *both*  $\ell_0$  and  $\ell_1$ ?
- (h) Suppose two plane isometries  $\phi^e$  and  $\psi^e$  are orthogonal extensions of isometries  $\phi$  and  $\psi$  of the line  $\ell_0$ . Prove that the product  $\psi^e\phi^e$  is the orthogonal extension of the isometry  $\psi\phi$  of the line  $\ell_0$ , i.e.,  $(\psi\phi)^e = \psi^e\phi^e$ . Further show  $(\phi^{-1})^e = (\phi^e)^{-1}$ .

### §III.4 Orientation

You may have noticed a qualitative difference between the isometries which are made from an even number of reflections (rotations, translations) and those which are made from an odd number (reflections and glide reflections). Inspection of Figures II.1.8 and 2.7 reveals that reflections and glide reflections change “handedness,” i.e., they convert left hands into right hands, and vice versa. Rotations and translations, however, preserve handedness. For this reason, reflections and glide reflections could never be realized as actual motions of physical objects (if the objects are forced to remain in a plane) — appropriately enough we experience them only when we look in a mirror. By contrast, we frequently experience rotations and translations of objects (even when constrained to remain in a fixed plane).

The mathematical term for handedness is **orientation**. Orientation comes in two states, such as *left* and *right*, or *clockwise* and *counterclockwise* (as we studied carefully in §§I.6 and I.11). Neither of these states has an intrinsic meaning by itself: each is defined as the opposite of the other! A reflection interchanges them. Thus an even number of reflections will preserve orientation, while an odd number of reflections reverses orientation.

In this section we justify the above observations from our understanding of isometries as products of reflections. Although any isometry is a composite of reflections, there are many ways to build a given isometry from reflections. Even the *number* of reflections used to make a given isometry can vary. For example, we can compose together any number of reflections, but the *First Structure Theorem* (Theorem II.4.5) says we can always obtain the same result by composing *at most three* reflections. Even products of just two or three reflections have such redundancy — see Theorems 1.15 and 2.10.

However, although the exact reflections, or even the number of them, that produce a given isometry are not uniquely determined, there is an important feature of factorization into reflections that will be the same for all factorizations: the **parity** of the number of factors. If one factorization of an isometry into reflections uses an *even* number of reflections, then all factorizations will use an *even* number. Likewise, if one factorization uses an *odd* number of reflections, then all factorizations will use an *odd* number. One proof of this result was outlined in Exercise 1.8 using conjugation. We now develop an alternate proof, based on the following geometric principle:

**Proposition 4.1.**

*A product of four reflections is expressible as a product of two reflections.*

*Proof.* Given  $\sigma_{\ell_4}\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$ , we perform the following *pairwise adjustments of reflections*. Pick any point  $p$  on the line  $\ell_1$ . Then pairwise adjust the reflection product  $\sigma_{\ell_3}\sigma_{\ell_2}$  to equal  $\sigma_{\ell'_3}\sigma_{\ell'_2}$ , where  $\ell'_2$  now also contains the point  $p$  — this is merely Theorem 1.15 in the form of Exercise 1.3a. Thus

$$\sigma_{\ell_4}\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{\ell_4}\sigma_{\ell'_3}\sigma_{\ell'_2}\sigma_{\ell_1}. \quad (4.2)$$

Now pairwise adjust the reflection product  $\sigma_{\ell_4}\sigma_{\ell'_3}$  to equal  $\sigma_{\ell''_4}\sigma_{\ell''_3}$ , where  $\ell''_3$  is a third line containing  $p$ . This gives

$$\sigma_{\ell_4}\sigma_{\ell'_3}\sigma_{\ell'_2}\sigma_{\ell_1} = \sigma_{\ell''_4}\sigma_{\ell''_3}\sigma_{\ell'_2}\sigma_{\ell_1}. \quad (4.3)$$

However, since the product  $\sigma_{\ell''_3}\sigma_{\ell'_2}\sigma_{\ell_1}$  consists of three reflections in lines *with a common point*  $p$ , a final pairwise adjustment of the product  $\sigma_{\ell''_2}\sigma_{\ell_1}$  to  $\sigma_{\ell''_3}\sigma_{\ell'_1}$ , i.e., a rotation of the two intersecting lines until  $\ell'_2$  matches  $\ell''_3$  (as in Proposition 2.1a), eliminates the middle two reflections since  $\sigma_{\ell''_3}\sigma_{\ell'_3} = \iota$ :

$$\sigma_{\ell''_4}\sigma_{\ell''_3}\sigma_{\ell'_2}\sigma_{\ell_1} = \sigma_{\ell''_4}\sigma_{\ell''_3}\sigma_{\ell''_3}\sigma_{\ell'_1} = \sigma_{\ell''_4}\sigma_{\ell'_1}. \quad (4.4)$$

Hence  $\sigma_{\ell_4}\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  has been collapsed to a product of two reflections.  $\square$

We illustrate the steps of the proof of Proposition 4.1 with a specific example. Let the lines  $\ell_1, \ell_2, \ell_3,$  and  $\ell_4$  be as shown in the first panel of Figure 4.5. Select the point  $p$  on  $\ell_1$  as shown. Since the lines  $\ell_2$  and  $\ell_3$  intersect, the first pairwise adjustment (4.2) is a rotation of these two lines into  $\ell'_2$  and  $\ell'_3$ , where  $\ell'_2$  contains  $p$ . This is illustrated in Figure 4.5.

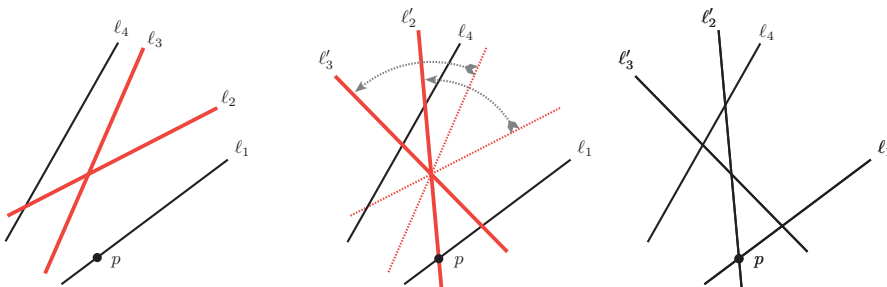


Figure 4.5. Adjustment (4.2) of  $\ell_2$  and  $\ell_3$ , results in  $\ell'_2$  containing  $p$ .

Since  $\ell'_3$  and  $\ell_4$  intersect, the second pairwise adjustment (4.3) is a rotation of these two lines into  $\ell''_3$  and  $\ell'_4$ , where  $\ell''_3$  contains  $p$ . This is Figure 4.6.

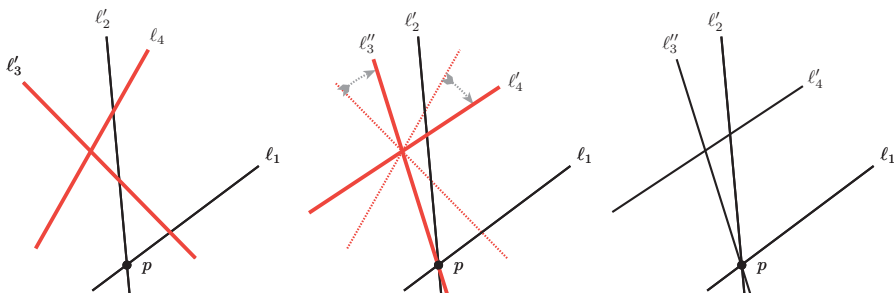


Figure 4.6. Adjustment (4.3) of  $\ell'_3$  and  $\ell_4$  results in  $\ell''_3$  containing  $p$ .

The lines  $\ell_1, \ell'_2,$  and  $\ell''_3$  all pass through  $p$ . Hence our third and final pairwise adjustment (4.4) is a rotation of  $\ell_1$  and  $\ell'_2$  about  $p$ , sending  $\ell'_2$  to  $\ell''_2 = \ell''_3$ . This eliminates the second and third reflections, as shown in Figure 4.7.

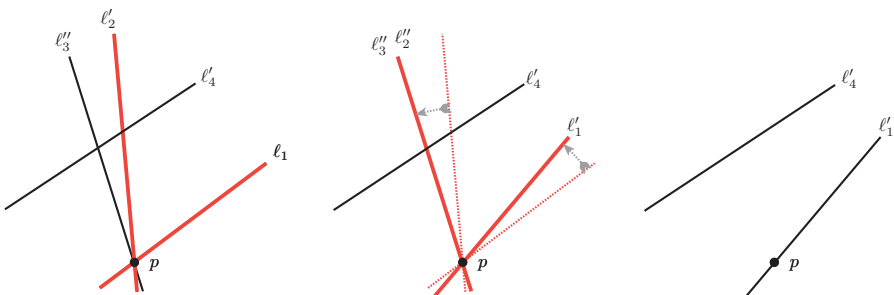


Figure 4.7. Adjustment (4.4) of  $\ell_1$  and  $\ell'_2$  eliminates  $\ell'_2$  and  $\ell'_3$ .

Thus  $\sigma_{\ell_4}\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  has been collapsed to a product of just two reflections,  $\sigma_{\ell'_4}\sigma_{\ell'_1}$ , as desired. You will be asked in Exercise 4.1 to illustrate this process starting with three other configurations of four lines.

Proposition 4.1 will now show that *orientation* of an isometry is well-defined.

**Corollary 4.8.**

*No isometry can be expressed as both the product of an even number of reflections and the product of an odd number of reflections.*

*Proof.* We can use Proposition 4.1 repeatedly to reduce any product of an odd number of reflections down to a product of three reflections and any product of an even number of reflections down to a product of two reflections. Hence we have only to rule out the following situation:

$$\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{m_2}\sigma_{m_1}.$$

Applying the reflection  $\sigma_{m_2}$  to both sides of this equation gives

$$\sigma_{m_2}\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{m_1}.$$

However, from Proposition 4.1 we know that the product of four reflections on the left can be rewritten as a product of two reflections, say  $\sigma_{n_2}\sigma_{n_1}$ . Hence this product of two reflections would equal the single reflection  $\sigma_{m_1}$ .

However, from our work in §1 we know that the product of two reflections is either a rotation or a translation. Thus the set of fixed points for a product of two reflections is either empty, a single point, or (in the trivial case) the whole plane. Since the set of fixed points for a reflection is a line, this shows a single reflection can never equal a product of two reflections.  $\square$

Corollary 4.8 guarantees that the following definition makes sense.

**Definition 4.9.**

An isometry of the plane is **orientation-preserving** (or **parity zero**) if it is the product of an *even* number of reflections in lines. It is **orientation-reversing** (or **parity one**) if it is the product of an *odd* number of reflections in lines.

An orientation-preserving isometry is also called *proper*, *direct*, or *even parity*, and an orientation-reversing isometry is called *improper*, *opposite*, or *odd parity*.

The notion of orientation allows us to formulate precisely what is special about the rotations and translations among the full class of isometries. This is worth stating formally.

**Theorem 4.10.**

- (a) Every orientation-preserving isometry of the plane is either the identity, a rotation, or a translation.
- (b) The product of two orientation-preserving isometries is orientation-preserving. The inverse of an orientation-preserving isometry is orientation-preserving.
- (c) The product of two orientation-reversing isometries is orientation-preserving. The inverse of an orientation-reversing isometry is orientation-reversing.
- (d) The product of an orientation-preserving isometry and an orientation-reversing isometry is orientation-reversing.

*Proof.* Since an orientation-preserving isometry must be the product of an even number of reflections and since we need at most three by the *First Structure Theorem* (Theorem II.4.5), we need either zero or two. Statement (a) follows from this and the *Classification Theorem* (Theorem 3.1). Statements (b) and (c) follow easily from the definition of orientation together with Proposition II.3.10. Statement (d) follows similarly.  $\square$

————— Exercises III.4 —————

**Exercise 4.1.**

Show how the process illustrated in (4.2), (4.3), and (4.4) proceeds for the three starting configurations shown below in Figure 4.11.

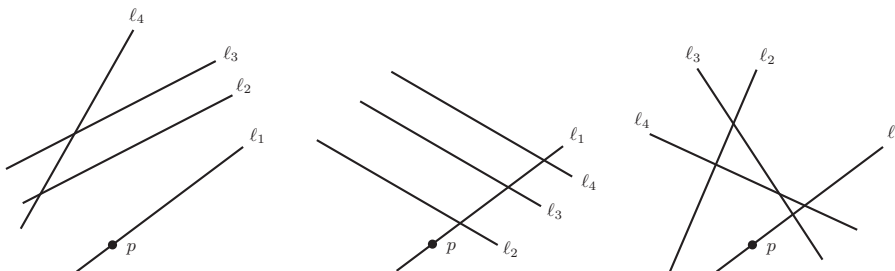


Figure 4.11. Starting configurations for Exercise 4.1.

**Exercise 4.2.**

Our proof of Proposition 4.1, that a product of four reflections always reduces to a product of two reflections, has the virtue that it applies to *all* possible situations, i.e., no special cases need be considered. However, considering particular cases allows for proofs in which the pairwise adjustment of reflections has more specific geometric meaning. As an example, consider the situation shown in Figure 4.12 where  $l_1$  and  $l_2$  intersect at  $p$  and  $l_3$  and  $l_4$  intersect at  $q$ . Let  $\alpha$  be the directed angle measure from  $l_1$  to  $l_2$  and  $\beta$  the directed angle measure from  $l_3$  to  $l_4$ .

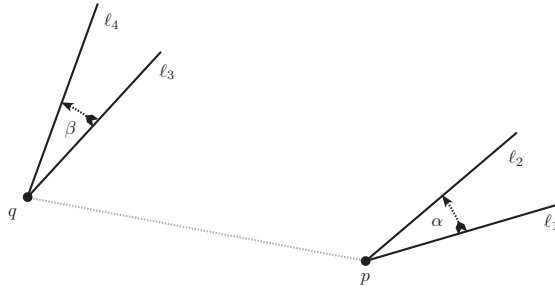


Figure 4.12. The configuration for Exercise 4.2.

- (a) Basing your proof on the procedure illustrated in Figure 4.13, show why  $\sigma_{\ell_4}\sigma_{\ell_3}\sigma_{\ell_2}\sigma_{\ell_1}$  reduces to a product of two reflections and identify the isometry so obtained. *Hint:* There are two cases: when  $\alpha + \beta$  is an integral multiple of  $\pi$  and when it is not.

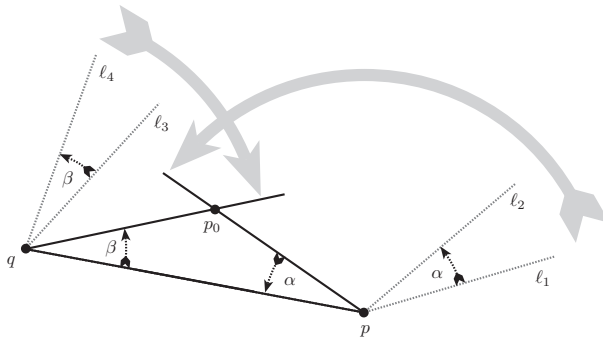


Figure 4.13. Exercise 4.2: A double rotation.

- (b) Determine all the possible types of isometries which result from the product of two rotations  $\rho_{q,\theta_2}\rho_{p,\theta_1}$  about two points  $p$  and  $q$ . How does this isometry type depend on the angles  $\theta_1$  and  $\theta_2$ ?
- (c) Suppose  $\phi'$  is the product in (b) taken in the opposite order, i.e.,  $\phi' = \rho_{p,\theta_1}\rho_{q,\theta_2}$ . Relate  $\phi'$  to  $\phi$  as precisely as possible.
- (d) Suppose  $\phi$  is a product of two rotations such that (1) the sum of the rotation angles equals  $2\pi$  and (2)  $\phi$  has a fixed point. Prove that  $\phi$  equals the identity.

**Exercise 4.3.**

- (a) If  $\phi$  and  $\psi$  are isometries with the same orientation whose values agree at *two* distinct points, prove  $\phi = \psi$ . *Hint:* Exercise II.4.1
- (b) Given two congruent directed line segments  $\overline{x_1x_2}$  and  $\overline{y_1y_2}$ , prove there exists a *unique* orientation-preserving isometry  $\phi$  mapping  $\overline{x_1x_2}$  onto  $\overline{y_1y_2}$  that preserves direction, i.e.,  $\phi$  maps  $x_1$  to  $y_1$  and  $x_2$  to  $y_2$ . *Hint:* Exercise II.4.2.
- (c) Repeat (b) for orientation-reversing isometries  $\psi$ .

**Exercise 4.4.**

Suppose  $x_1 \neq x_2$  and  $y_1 \neq y_2$  are points such that  $\overline{x_1x_2} \cong \overline{y_1y_2}$ . From Theorem II.2.11 we know there exists a translation  $\tau$  mapping  $x_1$  to  $y_1$  and  $x_2$  to  $y_2$  if and only if the directed congruent line segments  $\overline{x_1x_2}$  and  $\overline{y_1y_2}$  have the same direction (Definition II.1.12).

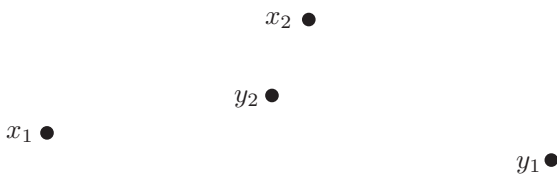
(a) Prove there exists a non-trivial rotation  $\rho$  mapping  $x_1$  to  $y_1$  and  $x_2$  to  $y_2$  if and only if the directed congruent line segments  $\overline{x_1x_2}$  and  $\overline{y_1y_2}$  do *not* have the same direction. *Hint:* Exercise 4.3.

(b) Suppose  $\rho_{p,\theta}$  is a rotation mapping  $x_1$  to  $y_1$  and  $x_2$  to  $y_2$ . Describe a geometric procedure for determining the center of rotation  $p$  and the angle of rotation  $\theta$  from the points  $x_1, x_2, y_1, y_2$ .

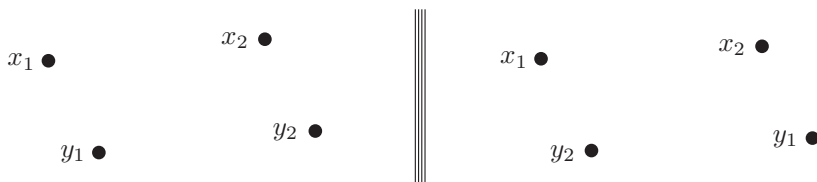
*Hint:* Let  $\ell_1$  and  $\ell_2$  be the perpendicular bisectors of  $\overline{x_1y_1}$  and  $\overline{x_2y_2}$ , respectively. Then consider each of the possible configurations:

- $\ell_1$  and  $\ell_2$  intersect at only one point,
- $\ell_1$  and  $\ell_2$  are equal (consider  $m_x = \overrightarrow{x_1x_2}$  and  $m_y = \overleftarrow{y_1y_2}$ ),
- $\ell_1$  and  $\ell_2$  are parallel (can this happen?).

(c) Determine if a rotation exists for the following set of points  $x_1, x_2, y_1, y_2$ . If so, locate  $p$ , the center of rotation, and compute  $\theta$ , the angle of rotation. If no rotation exists, state why.



(d) Repeat (c) for the following two sets of points:



**Exercise 4.5.**

Suppose  $\overline{x_1x_2}$  and  $\overline{y_1y_2}$  are two congruent line segments for which the midpoints of the “diagonal” line segments  $\overline{x_1y_1}$  and  $\overline{x_2y_2}$  coincide. Let  $p$  be this common midpoint.

(a) Prove that the line segments  $\overline{x_1x_2}$  and  $\overline{y_1y_2}$  are parallel.

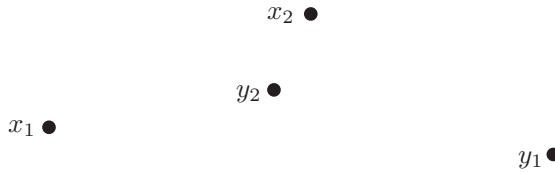
(b) Let  $\ell$  be the line through  $p$  perpendicular to  $\overleftarrow{x_1x_2}$  and  $\overleftarrow{y_1y_2}$ . Show there exists a glide reflection  $\gamma_{\ell,s}$  with axis  $\ell$  such that  $y_1 = \gamma_{\ell,s}(x_1)$  and  $y_2 = \gamma_{\ell,s}(x_2)$ . What is the length  $|s|$ ? When does  $|s| = 0$ ?

- (c) Prove that  $\gamma_{\ell,s}$  in (b) is unique, i.e., there is no other glide reflection  $\gamma$  (trivial or non-trivial) such that  $\gamma(x_1) = y_1$  and  $\gamma(x_2) = y_2$ .  
*Hint:* Exercise 4.3.

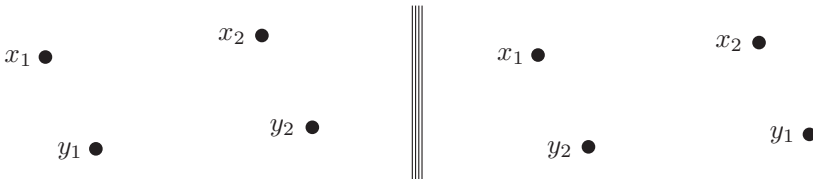
**Exercise 4.6.**

Suppose  $\overline{x_1x_2}$  and  $\overline{y_1y_2}$  are congruent line segments.

- (a) If  $y_1 = \gamma_{\ell,s}(x_1)$  and  $y_2 = \gamma_{\ell,s}(x_2)$  for a glide reflection  $\gamma_{\ell,s}$ , describe a procedure for determining the glide axis  $\ell$  and the glide length  $|s|$  by a geometric analysis of the points  $x_1, x_2, y_1,$  and  $y_2$ . Be sure to indicate why your  $\ell$  and  $|s|$  are unique. When will  $s$  equal zero?  
*Hint:* There are two cases to consider, and Exercise 4.5 is one of them! Also consult Exercise 2.4.
- (b) Prove there exists *exactly one* glide reflection  $\gamma_{\ell,s}$  (possibly trivial, i.e., just a reflection) such that  $\gamma_{\ell,s}(x_1) = y_1$  and  $\gamma_{\ell,s}(x_2) = y_2$ .  
*Hint:* Exercise 4.3.
- (c) Use your procedure in (a) to determine the glide reflection  $\gamma_{\ell,s}$  for the following set of points  $x_1, x_2, y_1, y_2$ . Draw the glide axis  $\ell$  and determine the glide length  $|s|$ . Is your glide reflection *non-trivial*?



- (d) Repeat (c) for the following two sets of points:



**Exercise 4.7.**

Given any two congruent directed line segments  $\overline{x_1x_2}$  and  $\overline{y_1y_2}$ , it was shown in Exercise 4.3 that there exist two unique isometries,  $\phi$  (orientation-preserving) and  $\psi$  (orientation-reversing), that map  $\overline{x_1x_2}$  onto  $\overline{y_1y_2}$  and preserve direction, i.e.,  $x_1$  is mapped to  $y_1$  and  $x_2$  is mapped to  $y_2$ .

- (a) Determine conditions on the configuration of points  $x_1, x_2, y_1, y_2$  that specify the isometry class of  $\phi$ , i.e., identity, non-trivial translation, or non-trivial rotation. *Hint:* Exercise 4.4.
- (b) Determine conditions on the configuration of points  $x_1, x_2, y_1, y_2$  that specify the isometry class of  $\psi$ , i.e., reflection or non-trivial

glide reflection. *Hint:* If  $x = y$ , call *any* line containing  $x = y$  a *perpendicular bisector* of the trivial “line segment”  $\overline{xy}$ . Then consider perpendicular bisectors of  $\overline{x_1y_1}$  and  $\overline{x_2y_2}$ .

**Exercise 4.8.**

Isometries of the line were defined and studied in Exercise 3.3. Develop a theory of *orientation* for such transformations. In particular, state and prove versions of Proposition 4.1, Corollary 4.8, and Theorem 4.10.

**Exercise 4.9.**

Suppose  $\angle ABC$  has directed angle measure  $\theta = m\angle ABC$ . Let  $\phi$  be any isometry with parity  $\epsilon$  (which equals 0 or 1) and define

$$\angle A'B'C' = \phi(\angle ABC)$$

where  $A' = \phi(A)$ ,  $B' = \phi(B)$ , and  $C' = \phi(C)$ . Prove that

$$m\angle A'B'C' = (-1)^\epsilon m\angle ABC.$$

*Hint:* Let  $\theta' = (-1)^\epsilon \theta$ . You need to show  $m\angle A'B'C' = \theta'$ , which is equivalent to showing  $\rho_{B',\theta'}(\overrightarrow{B'A'}) = \overrightarrow{B'C'}$ . To verify this latter equality, use Exercise 1.8e to express  $\rho_{B',\theta'}$  in terms of  $\rho_{B,\theta}$ .

## §III.5 Groups of Transformations

In our study of isometries we have relied heavily on the ability to compose and invert transformations. In applying these operations, it was crucial to know that the set of all isometries of the plane is *closed under composition and inversion*, i.e., the composite of two isometries is also an isometry and the inverse of an isometry is also an isometry (Propositions II.3.4 and II.3.9). This situation — a set of transformations closed under composition and inversion — arises over and over in mathematics. We formally identify this concept as the mathematical notion of a *group of transformations*.

**Definition 5.1.**

A **group of transformations** is a non-empty collection  $\mathcal{G}$  of transformations that is closed under composition and inversion, i.e., if  $\Phi$  and  $\Psi$  are in  $\mathcal{G}$ , then  $\Phi\Psi$  and  $\Phi^{-1}$  are also in  $\mathcal{G}$ .

**Proposition 5.2.**

*The collection  $\mathcal{A}_2$  of all transformations of the plane is a group.<sup>2</sup>*

*The collection  $\mathcal{E}_2$  of all isometries of the plane is a group.*

*Proof.* Propositions II.3.4 and II.3.8 show that the collection  $\mathcal{A}_2$  of all transformations is a group. Propositions II.3.4 and II.3.9 yield the same result for the collection  $\mathcal{E}_2$  of isometries.  $\square$

<sup>2</sup>When dealing with a “generic” group, we will use the symbols  $\mathcal{G}$ ,  $\mathcal{H}$ , etc., while important specific groups will have their own notation such as  $\mathcal{E}_2$  and  $\mathcal{A}_2$ .

Given a group  $\mathcal{G}$  of transformations, we are often interested in some subset  $\mathcal{H}$  of  $\mathcal{G}$  consisting of all transformations with some distinguishing properties. If this subset is also a group in its own right — i.e., the product of two members of  $\mathcal{H}$  is also in  $\mathcal{H}$  and the inverse of any element in  $\mathcal{H}$  is also in  $\mathcal{H}$  — then we call  $\mathcal{H}$  a *subgroup* of  $\mathcal{G}$ .

**Definition 5.3.**

If  $\mathcal{G}$  and  $\mathcal{H}$  are two groups of transformations such that  $\mathcal{H}$  is contained in  $\mathcal{G}$ , then  $\mathcal{H}$  is a **subgroup** of  $\mathcal{G}$ .

Thus  $\mathcal{E}_2$ , the group of isometries of the plane, is a subgroup of  $\mathcal{A}_2$ , the group of all transformations of the plane. Furthermore,  $\mathcal{E}_2$  itself has important subgroups. Here is a particularly important example.

**Theorem 5.4.**

*The collection  $\mathcal{T}_2$  of all translations of the plane is a subgroup of  $\mathcal{E}_2$ .*

*Proof.* Theorem 1.20 shows that  $\mathcal{T}_2$ , the collection of all translations of the plane, is closed under composition. We further claim that it is closed under inversion. For consider any translation  $\tau = \tau_{AB}$ . Then  $\tau$  is the product of two reflections,  $\tau = \sigma_{\ell_2}\sigma_{\ell_1}$ , so that the inverse of  $\tau$  is easily computed:

$$\begin{aligned}\tau^{-1} &= (\sigma_{\ell_2}\sigma_{\ell_1})^{-1} = \sigma_{\ell_1}^{-1}\sigma_{\ell_2}^{-1} \text{ by Proposition II.3.10,} \\ &= \sigma_{\ell_1}\sigma_{\ell_2} \text{ since reflections are their own inverses.}\end{aligned}$$

In particular, the inverse of a translation is a translation, giving  $\tau_{AB}^{-1} = \tau_{BA}$  for all points  $A$  and  $B$ . This proves  $\mathcal{T}_2$  is indeed a *subgroup* of  $\mathcal{E}_2$ .  $\square$

Further examples of groups of transformations will be developed in Exercise 5.1. However, we now show that the group of translations  $\mathcal{T}_2$  has an important property which is possessed by only certain groups: any two translations *commute* with each other, i.e., it does not matter in which order a product is formed. Having this property greatly simplifies the algebraic structure of a group, and hence it is worthy of a special name.

**Definition 5.5.**

A group  $\mathcal{G}$  is **abelian** (or **commutative**) if all its elements commute:

$$\phi\psi = \psi\phi \text{ for all } \phi, \psi \text{ in } \mathcal{G}.$$

**Proposition 5.6.**

*Any two translations of the plane commute, i.e.,*

$$\tau_2\tau_1 = \tau_1\tau_2 \text{ for all translations } \tau_1 \text{ and } \tau_2.$$

*Hence the group  $\mathcal{T}_2$  of translations of the plane is abelian.*

*Proof.* One approach to showing  $\tau_2\tau_1 = \tau_1\tau_2$  was outlined in Exercise 1.7: express  $\tau_1$  as a product of reflections in parallel lines, and then show that conjugation by any other translation  $\tau_2$  produces an equivalent product of reflections. Here is a second proof.

Consider any point  $A$  in the plane. We will show  $\tau_2\tau_1(A) = \tau_1\tau_2(A)$ . To do so, define the intermediate points  $B = \tau_1(A)$ ,  $C = \tau_2(A)$ , and  $D = \tau_2(B)$  as shown in Figure 5.7. We need only show  $D = \tau_1(C)$  to complete our proof.

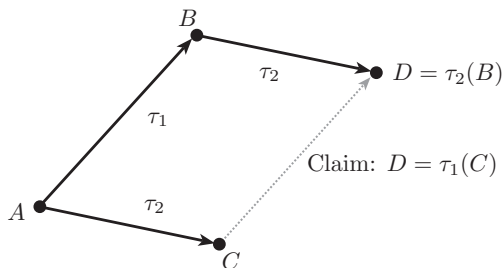


Figure 5.7. Need to show  $D = \tau_1(C)$ .

Since  $\tau_2(\overline{AB}) = \overline{CD}$ , then  $\overline{AB}$  and  $\overline{CD}$  have the same length and direction by Theorem II.2.11. Thus  $\tau_1 = \tau_{AB} = \tau_{CD}$  from Theorem II.1.13, proving  $\tau_1(C) = D$ , as desired.  $\square$

We end this section with an important property relating translations and rotations: two rotations with the same directed angle but about two distinct points differ by only a translation. This observation will prove exceedingly useful in the next two sections.

**Proposition 5.8.**

- (a) *The composition of a rotation  $\rho_{q,\theta}$  of angle  $\theta \neq 0 \pmod{2\pi}$  with a translation  $\tau$  is another rotation of angle  $\theta$ , i.e.,*

$$\rho_{q,\theta} \tau = \rho_{p,\theta} \quad \text{for some point } p.$$

- (b) *Conversely, given two rotations  $\rho_{q,\theta}$ ,  $\rho_{p,\theta}$  of any angle  $\theta$ , one is the composite of the other with a translation, i.e.,*

$$\rho_{q,\theta} \tau = \rho_{p,\theta} \quad \text{for a unique translation } \tau.$$

*Proof.* (a) (This result was previously given as Exercise 1.4.) We wish to prove  $\rho_{q,\theta} \tau = \rho_{p,\theta}$  for some point  $p$ . By Theorems 1.4 and 1.12 we know  $\rho_{q,\theta} = \sigma_{m_2} \sigma_{m_1}$  and  $\tau = \sigma_{\ell_2} \sigma_{\ell_1}$  where  $m_1$  and  $m_2$  intersect at  $q$  and make a directed angle of measure  $\theta/2$  from  $m_1$  to  $m_2$ , and  $\ell_1$  and  $\ell_2$  are parallel. Moreover, from Exercise 1.3 we can choose  $m_1$  and  $m_2$  so that  $m_1$  is parallel to  $\ell_1$  and  $\ell_2$ , as in the first frame of Figure 5.9. Since  $\theta \neq 0 \pmod{2\pi}$ , then  $m_1$  and  $m_2$  are not parallel; hence  $m_2$  intersects all lines parallel to  $m_1$ .

By Theorem 1.15 we can pairwise adjust  $\ell_1$  and  $\ell_2$  by translation to obtain new lines  $\ell'_1$  and  $\ell'_2$  where  $\ell'_2 = m_1$ . Thus, as illustrated in Figure 5.9,

$$\begin{aligned} \rho_{q,\theta} \tau &= \sigma_{m_2} \sigma_{m_1} \sigma_{\ell_2} \sigma_{\ell_1} \\ &= \sigma_{m_2} \sigma_{m_1} \sigma_{\ell'_2} \sigma_{\ell'_1} \\ &= \sigma_{m_2} \sigma_{\ell'_1} \quad \text{since } m_1 = \ell'_2. \end{aligned}$$

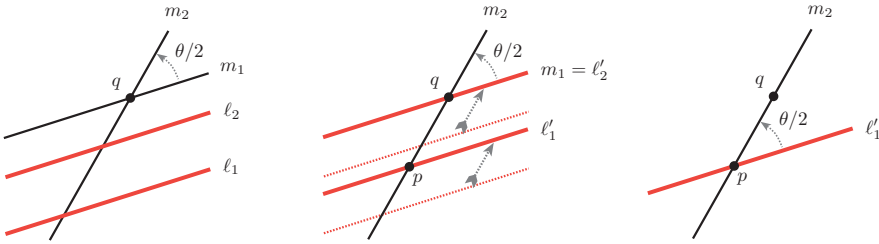


Figure 5.9.  $\rho_{q,\theta} \tau = \sigma_{m_2} \sigma_{m_1} \sigma_{l_2} \sigma_{l_1}$  reduces to  $\sigma_{m_2} \sigma_{l'_1} = \rho_{p,\theta}$ .

Let  $p$  be the intersection of  $m_2$  and  $l'_1$ . By Theorem 1.12 we know  $\sigma_{m_2} \sigma_{l'_1}$  is a rotation about  $p$ . Moreover, since  $l'_1$ ,  $l_1$ , and  $m_1$  are all parallel, the directed angle from  $l'_1$  to  $m_2$  will have the same directed angle measure  $\theta/2$  as the directed angle from  $m_1$  to  $m_2$  (a careful proof of this claim using the translational invariance of directed angle measure is given in Exercise 5.10). Thus  $\sigma_{m_2} \sigma_{l'_1} = \rho_{p,\theta}$ , as desired.

(b) We need to prove  $\rho_{q,-\theta} \rho_{p,\theta}$  is a translation. If  $p = q$ , this composite is merely the identity, so we can assume  $p \neq q$ . From Theorem 1.12 we know  $\rho_{q,-\theta} = \sigma_{m_2} \sigma_{m_1}$  and  $\rho_{p,\theta} = \sigma_{l_2} \sigma_{l_1}$  where  $m_1, m_2$  produce a directed angle at  $q$  of measure  $-\theta/2$  and  $l_1, l_2$  produce a directed angle at  $p$  of measure  $\theta/2$ . Moreover, from Exercise 1.3 we can choose  $m_1$  and  $l_2$  to contain both  $p$  and  $q$ ; hence  $m_1 = l_2$ . Denote this line as  $\ell$ , as shown in Figure 5.10. Therefore

$$\rho_{q,-\theta} \rho_{p,\theta} = \sigma_{m_2} \sigma_{m_1} \sigma_{l_2} \sigma_{l_1} = \sigma_{m_2} \sigma_{l_1}.$$

We claim  $m_2$  and  $l_1$  are parallel lines, so that  $\sigma_{m_2} \sigma_{l_1}$  is a translation from Proposition 1.2. This will finish the proof of Proposition 5.8.

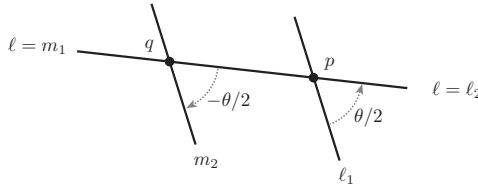


Figure 5.10. Proving  $\rho_{q,-\theta} \rho_{p,\theta}$  is a translation.

To prove  $m_2$  and  $l_1$  are parallel, choose  $A \in m_1 = \ell$  and  $B \in m_2$  such that  $m\angle ApB = -\theta/2$ , as shown in Figure 5.11.

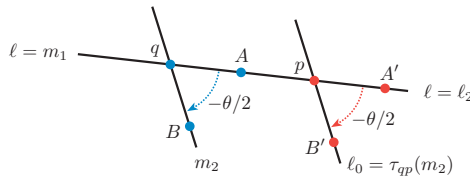


Figure 5.11. We prove  $m_2$  and  $l_1$  parallel by showing  $l_1 = l_0$ .

Since  $\ell = \overleftrightarrow{pq}$ , then  $\tau_{qp}$  fixes  $\ell$ , and the point  $A' = \tau_{qp}(A)$  is on  $\ell$ . Then define  $B' = \tau_{qp}(B)$ . The line  $\ell_0 = \overleftrightarrow{pB'}$  is the image under  $\tau_{qp}$  of  $m_2$  and thus  $\ell_0$  is parallel to  $m_2$  by Exercise II.2.8. Moreover,

$$m\angle A'pB' = m\angle ApB = -\frac{\theta}{2}.$$

But thus  $\sigma_{\ell_0}\sigma_{\ell} = \rho_{p,-\theta}$  by Proposition 1.8, giving

$$\sigma_{\ell}\sigma_{\ell_0} = \rho_{p,\theta} = \sigma_{\ell_2}\sigma_{\ell_1} = \sigma_{\ell_2}\sigma_{\ell}.$$

Thus  $\sigma_{\ell_0} = \sigma_{\ell_1}$ , proving  $\ell_1 = \ell_0$ , a line parallel to  $m_2$ , as desired.  $\square$

### Exercises III.5

#### Exercise 5.1.

Each set listed below contains *all* the transformations of the indicated type. Determine if each collection is a group (some you already know). If not, describe what goes wrong. Determine all the subgroup relationships which exist among the collections which you identify as groups.

- (a)  $\mathcal{A}_2$  – transformations.
- (b)  $\mathcal{E}_2$  – isometries.
- (c)  $\mathcal{E}_2^+$  – orientation-preserving isometries.
- (d)  $\mathcal{E}_2^-$  – orientation-reversing isometries.
- (e)  $\text{SO}_2(p)$  – rotations about a fixed point  $p$ .
- (f)  $\mathcal{R}_2$  – rotations (i.e., about any point).
- (g)  $\mathcal{T}_2$  – translations.
- (h)  $\text{O}_2(p)$  – isometries with fixed point  $p$ .
- (i)  $\mathcal{E}_2(\ell)$  – isometries with fixed line  $\ell$ .
- (j)  $\mathcal{E}_2(p, \ell)$  – isometries with fixed point  $p$  and fixed line  $\ell$ .
- (k)  $\mathcal{A}_2(p)$  – transformations with fixed point  $p$ .
- (l)  $\mathcal{A}_2^0$  – transformations with *at least one* fixed point.
- (m)  $\mathcal{A}_2(\ell)$  – transformations with fixed line  $\ell$ .
- (n)  $\mathcal{A}_2(p, \ell)$  – transformations with fixed point  $p$  and fixed line  $\ell$ .
- (o)  $\Sigma_2$  – reflections.
- (p)  $\mathcal{G}_2$  – glide reflections.
- (q)  $\mathcal{V}_2$  – point inversions.
- (r)  $\mathcal{T}_2 \cup \mathcal{V}_2$  – transformations and point inversions.

#### Exercise 5.2.

- (a) Prove the *intersection* of two subgroups of a group is a subgroup.
- (b)  $\mathcal{A}_2(\ell)$ , the set of all transformations of the plane with fixed line  $\ell$ , is a subgroup of  $\mathcal{A}_2$ . Based on this fact, show that  $\mathcal{E}_2(\ell)$ , the set of all *isometries* of the plane with fixed line  $\ell$ , is a subgroup of  $\mathcal{A}_2$ .
- (c) Is the *union* of two subgroups of a group necessarily a subgroup?

**Exercise 5.3. Normal Subgroups.**

- (a) A subgroup  $\mathcal{H}$  of a group  $\mathcal{G}$  is said to be a **normal subgroup** if  $\phi\psi\phi^{-1}$  is in  $\mathcal{H}$  for all  $\psi \in \mathcal{H}$  and  $\phi \in \mathcal{G}$ . Is  $\mathcal{T}_2$ , the group of translations of the plane, a normal subgroup of  $\mathcal{E}_2$ , the group of isometries of the plane? *Hint:* Exercise 1.8.
- (b) Some of the following subsets of  $\mathcal{E}_2$  are subgroups (Exercise 5.1):  
 $\mathcal{E}_2^+$ ,  $\mathcal{E}_2^-$ ,  $\text{SO}_2(p)$ ,  $\mathcal{R}_2$ ,  $\mathcal{T}_2$ ,  $\text{O}_2(p)$ ,  $\mathcal{E}_2(\ell)$ ,  $\mathcal{E}_2(p, \ell)$ ,  $\Sigma_2$ ,  $\mathcal{G}_2$ .  
 Determine which of these subsets are *normal* subgroups of  $\mathcal{E}_2$ .

**Exercise 5.4. Centralizers.**

Given a group of transformations  $\mathcal{G}$ , the **centralizer** of  $\phi \in \mathcal{G}$  is the collection of transformations  $\psi$  in  $\mathcal{G}$  which commute with  $\phi$ , i.e.,

$$\mathcal{Z}_{\mathcal{G}}(\phi) = \{ \psi \in \mathcal{G} \mid \phi\psi = \psi\phi \}.$$

- (a) Prove that  $\mathcal{Z}_{\mathcal{G}}(\phi)$  is a subgroup of  $\mathcal{G}$ .
- (b) Describe the centralizers for the following isometries in  $\mathcal{G} = \mathcal{E}_2$ :  
 $\iota$ ,  $\sigma_{\ell}$ ,  $\tau_{AB}$ ,  $\nu_p$ ,  $\rho_{p,\theta}$  ( $\theta \neq 0 \pmod{\pi}$ ),  $\gamma_{\ell,s}$ .  
 Justify your answers. *Hint:* Exercises 1.7 and 1.8.

**Exercise 5.5.**

For each collection in Exercise 5.1 that is a group, determine if it is abelian. Prove your claims.

**Exercise 5.6. Symmetry Groups.**

Given a subset  $F$  (a figure or a *configuration*, a union of figures) in the plane, an isometry  $\phi$  is a **symmetry** of  $F$  if  $\phi$  maps  $F$  onto  $F$ , i.e.,  $\phi(F) = F$ . The collection of all symmetries of  $F$  is denoted by  $\text{Sym}(F)$ .

- (a) Determine the symmetries of a circle  $\mathcal{C}$  with center  $p$  and radius  $r$ .
- (b) Determine all the symmetries of a line  $\ell$ .
- (c) For any figure  $F$  prove that  $\text{Sym}(F)$  is a group. For this reason  $\text{Sym}(F)$  is called the **symmetry group of  $F$** . This exceedingly important result will play a central role in Chapters VII and VIII.
- (d) Let  $F_1$  be the union of two distinct lines  $\ell_1$  and  $\ell_2$  that intersect at a point  $p$ . Let  $F_2$  be the set of four points obtained by intersecting  $F_1$  with a circle whose center is  $p$ . *Without* explicitly determining the symmetry groups of  $F_1$  and  $F_2$ , prove that  $\text{Sym}(F_1) = \text{Sym}(F_2)$ .  
*Hint:* First show that any isometry in  $\text{Sym}(F_1)$  or  $\text{Sym}(F_2)$  fixes  $p$ .
- (e) Determine  $\text{Sym}(F_2)$ , thereby also obtaining  $\text{Sym}(F_1)$ . ( $F_2$  is a *minimal configuration* for  $F_1$  in the sense that it is the simplest subset of  $F_1$  that still retains all the symmetry information of  $F_2$ . Finding a *minimal configuration* for a given figure is often the best way

to determine a desired symmetry group.) *Hint:* There is a special case, i.e., a configuration of two intersecting lines (or four points) that has more symmetry than the generic case.

- (f) Suppose  $F_3$  is the union of two distinct parallel lines  $\ell_1$  and  $\ell_2$ . Determine  $\text{Sym}(F_3)$ .

**Exercise 5.7. Dilatations.**

A transformation  $\phi$  of the plane is called a **dilatation** if the image  $\phi(\ell)$  of any line  $\ell$  is a line parallel to  $\ell$  or equal to  $\ell$ . In Exercise II.2.8 you showed that all translations are dilatations.

- (a) Show that the collection  $\mathcal{D}_0$  of *all* dilatations is a subgroup of  $\mathcal{A}_2$ .  
 (b) An *isometric dilatation* is an isometry which is a dilatation. Prove the collection  $\mathcal{D}_1$  of all isometric dilatations is a subgroup of  $\mathcal{E}_2$ .  
 (c) By using our classification of all possible isometries of the plane, explicitly determine all *isometric dilatations*.

More facts about dilatations will be established in Exercise IV.5.10.

**Exercise 5.8. Isometries of the Line.**

Consider isometries of the line and their orthogonal extensions to isometries of the plane as analyzed in Exercises 3.3 and 3.4.

- (a) Show that  $\mathcal{E}_1$ , the collection of all isometries of the line, is a group.  
 (b) Let  $\mathcal{E}_2^*(\ell_0)$  denote the collection of all isometries that fix *all* lines parallel to  $\ell_0$ . Show  $\mathcal{E}_2^*(\ell_0)$  consists precisely of those isometries of the plane that are orthogonal extensions of isometries of the line  $\ell_0$ .  
 (c) Define the mapping  $\Phi$  from  $\mathcal{E}_1$  to  $\mathcal{E}_2^*(\ell_0)$  by  $\Phi(\phi) = \phi^e$  where  $\phi^e$  is the orthogonal extension of the line isometry  $\phi$ . Show  $\Phi$  is one-to-one and onto. Also show  $\Phi(\phi\psi) = \Phi(\phi)\Phi(\psi)$  and  $\Phi(\phi^{-1}) = \Phi(\phi)^{-1}$  for all  $\phi$  and  $\psi$  in  $\mathcal{E}_1$ . Explain why the existence of such a map  $\Phi$  shows that  $\mathcal{E}_1$  and  $\mathcal{E}_2^*(\ell_0)$  are essentially “the same” group!

**Exercise 5.9.**

Fix a reflection  $\sigma$  in the plane and define the group  $\mathbb{Z}_2 = \{\iota, \sigma\}$ . Further define a function  $\mathcal{F} : \mathcal{E}_2 \rightarrow \mathbb{Z}_2$  by

$$\mathcal{F}(\phi) = \begin{cases} \iota & \text{if } \phi \text{ is orientation-preserving,} \\ \sigma & \text{if } \phi \text{ is orientation-reversing.} \end{cases}$$

- (a) Prove  $\mathcal{F}$  is a *group homomorphism* from  $\mathcal{E}_2$  onto  $\mathbb{Z}_2$ , i.e.,  
 $\mathcal{F}(\phi\psi) = \mathcal{F}(\phi)\mathcal{F}(\psi)$  for all  $\phi, \psi \in \mathcal{E}_2$ , and  
 $\mathcal{F}(\phi^{-1}) = (\mathcal{F}(\phi))^{-1}$  for all  $\phi \in \mathcal{E}_2$ .  
 (b) Prove  $\mathcal{F}$  is the *unique* group homomorphism of  $\mathcal{E}_2$  onto  $\mathbb{Z}_2$ .  
*Hint:* Suppose  $\mathcal{T} : \mathcal{E}_2 \rightarrow \mathbb{Z}_2$  is any group homomorphism of  $\mathcal{E}_2$  onto

$\mathbb{Z}_2$ . Using Exercise 1.7, prove  $\mathcal{T}$  will map *all* reflections onto  $\iota$  or *all* reflections onto  $\sigma$ . Use this to show  $\mathcal{T} = \mathcal{F}$ .

- (c) The *kernel* of a homomorphism  $\mathcal{T} : \mathcal{E}_2 \rightarrow \mathbb{Z}_2$  is the collection

$$\ker \mathcal{T} = \{\phi \mid \mathcal{T}(\phi) = \iota\}.$$

What is the kernel for the homomorphism  $\mathcal{T}$  of (b)? Use this to give another definition for the orientation of an isometry  $\phi$ .

### Exercise 5.10.

Suppose  $m_2$  is a transversal of the two parallel lines  $m_1$  and  $\ell'_1$ , where  $m_1$  intersects  $m_2$  at  $q$  and  $\ell'_1$  intersects  $m_2$  at  $p$ . Choose  $A \in m_1$  and  $B \in m_2$  such that  $m\angle AqB = \theta/2$  as shown in Figure 5.12. Thus  $\sigma_{m_2}\sigma_{m_1} = \rho_{q,\theta}$ . We wish to prove  $\sigma_{m_2}\sigma_{\ell'_1} = \rho_{p,\theta}$ .

- (a) Prove  $\tau_{qp}$  fixes the line  $m_2$  and maps  $m_1$  to  $\ell'_1$ . *Hint*: Exercise II.2.8.  
 (b) If  $A' = \tau_{qp}(A)$ ,  $B' = \tau_{qp}(B)$ ,  $\theta'/2 = m\angle A'pB'$ , prove  $\sigma_{m_2}\sigma_{\ell'_1} = \rho_{p,\theta'}$ .  
 (c) Use the translational invariance of directed angle measure (Definition I.11.14) to prove  $\theta' = \theta$ , and hence  $\sigma_{m_2}\sigma_{\ell'_1} = \rho_{p,\theta}$ , as desired.

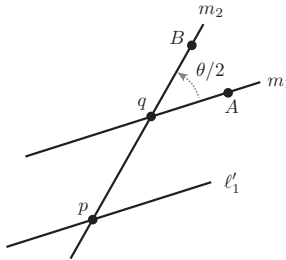


Figure 5.12.  $\rho_{q,\theta}\tau = \sigma_{m_2}\sigma_{m_1}\sigma_{\ell'_1}\sigma_{\ell'_1}$  reduces to  $\sigma_{m_2}\sigma_{\ell'_1} = \rho_{p,\theta}$ .

## §III.6 The Second Structure Theorem

The *First Structure Theorem* (Theorem II.4.5) tells a lot about isometries: it shows they are built in simple ways out of reflections, it leads to a qualitative classification of isometries, and it allows us to compute the product of two isometries. However, the *First Structure Theorem* has the unavoidable drawback that the factorization of an isometry is redundant — somewhat so for rotations and translations (as seen in Theorem 1.15) and highly so for glide reflections (a consequence of Theorem 2.10).

We used this redundancy to find an efficient factorization of a glide reflection into the product of reflections in two parallel lines and reflection in a third line orthogonal to the first two. Even this is not quite unique, however, as the two parallel lines can be (simultaneously) translated in the direction of the third line without changing the product. This redundancy prevents us from getting a clear picture of the whole group of plane isometries.

Therefore we present a *Second Structure Theorem* which gives a *unique* factorization of a general isometry into simpler pieces. This give us an economical, global view of what the set of all isometries looks like.

**Theorem 6.1.**    *The Second Structure Theorem.*

*Fix line  $\ell$  and point  $p$ . Then every isometry  $\phi$  has a unique factorization*

$$\phi = \sigma_\ell^\epsilon \rho_{p,\theta} \tau$$

*where  $\rho_{p,\theta}$  is a rotation about  $p$  and  $\tau$  is a translation.*

- *The exponent  $\epsilon$  is the parity of  $\phi$  and determines if  $\sigma_\ell$  is included in the factorization. It is independent of the choices of  $\ell$  and  $p$ .*
- *If  $\phi$  is orientation-preserving, then the directed angle measure  $\theta$  is independent of the choices of  $\ell$  and  $p$ .*

*Proof.* We begin with three given objects: a line  $\ell$ , a point  $p$ , and an isometry  $\phi$ . Also determined at the start is the parity  $\epsilon$  for  $\phi$  since no isometry can be both orientation-preserving and orientation-reversing (recall that the parity of  $\phi$  is 0 if  $\phi$  is orientation-preserving and 1 if  $\phi$  is orientation-reversing). We must find the angle measure  $\theta$  and the translation  $\tau_{AB}$ .

*Existence of Decomposition.* The angle  $\theta$  and translation  $\tau$  must satisfy  $\phi = \sigma_\ell^\epsilon \rho_{p,\theta} \tau$ . It is thus convenient to define the isometry  $\phi_0 = \sigma_\ell^\epsilon \phi$ , i.e.,

$$\phi_0 = \begin{cases} \phi & \text{if } \phi \text{ is orientation-preserving,} \\ \sigma_\ell \phi & \text{if } \phi \text{ is orientation-reversing.} \end{cases}$$

Notice that, no matter what the parity of  $\phi$ , the isometry  $\phi_0$  is orientation-preserving and hence is either a rotation or a translation.

Since we need an angle  $\theta$  and translation  $\tau$  such that  $\phi_0 = \rho_{p,\theta} \tau$ , then  $\tau$  must be such that  $\phi_0 \tau^{-1}$  fixes  $p$ . But  $\phi_0 \tau^{-1}(p) = p$  is equivalent to

$$\tau^{-1}(p) = \phi_0^{-1}(p).$$

Defining  $q = \phi_0^{-1}(p)$ , it follows that the desired  $\tau$  has to be  $\tau = \tau_{qp}$ .

Thus  $\phi_0 \tau^{-1}$  is an orientation-preserving isometry fixing  $p$ ; by Theorem 3.2 (*Fixed Points and Fixed Lines*) it must therefore be a rotation about  $p$ , i.e.,

$$\phi_0 \tau^{-1} = \rho_{p,\theta}$$

for some angle  $\theta$ . But this gives  $\phi = \sigma_\ell^\epsilon \rho_{p,\theta} \tau$ , the desired factorization.

*Uniqueness of the Decomposition.* Given an isometry  $\phi$ , we must show that for each choice of a point  $p$  and a line  $\ell$  there is only one rotation  $\rho_{p,\theta}$  and one translation  $\tau$  such that  $\phi = \sigma_\ell^\epsilon \rho_{p,\theta} \tau$ . Thus suppose

$$\phi = \sigma_\ell^\epsilon \rho_{p,\theta_1} \tau_1 = \sigma_\ell^\epsilon \rho_{p,\theta_2} \tau_2.$$

This is equivalent to  $\rho_{p,\theta_1}\tau_1 = \rho_{p,\theta_2}\tau_2$ , which can be reorganized as

$$\tau_2\tau_1^{-1} = \rho_{p,\theta_2}^{-1}\rho_{p,\theta_1} = \rho_{p,\theta_1-\theta_2}.$$

However, from Theorem 5.4 we know  $\tau_2\tau_1^{-1}$  is another translation. Thus we have a translation equaling a rotation, which can happen only if both equal the identity. Hence

$$\rho_{p,\theta_1-\theta_2} = \iota = \tau_2\tau_1^{-1},$$

proving  $\rho_{p,\theta_1} = \rho_{p,\theta_2}$  and  $\tau_1 = \tau_2$ , the desired uniqueness.

*Rotation Angle.* When  $\phi$  is orientation-preserving, we claim the rotation angle  $\theta$  in the factorization is independent of the initial choices of  $\ell$  and  $p$ . Independence of  $\ell$  is clear since  $\ell$  does not appear in the factorization if  $\phi$  is orientation-preserving. Independence of  $p$  is more substantial.

Suppose we have two factorizations for  $\phi$  about two points,  $p$  and  $q$ :

$$\rho_{p,\theta_1}\tau_1 = \rho_{q,\theta_2}\tau_2.$$

We need to prove  $\theta_1 = \theta_2 \pmod{2\pi}$ . We can rewrite our equality as

$$\rho_{p,\theta_1} = \rho_{q,\theta_2}\tau_2\tau_1^{-1}.$$

Since a product of translations is a translation by Theorem 5.4, the isometry  $\tau_2\tau_1^{-1}$  is a translation  $\tau_3$ , and we thus have shown

$$\rho_{p,\theta_1} = \rho_{q,\theta_2}\tau_3.$$

But by Proposition 5.8 we know there exists a translation  $\tau_4$  such that

$$\rho_{p,\theta_1} = \rho_{q,\theta_1}\tau_4.$$

Hence both  $\rho_{p,\theta_1} = \rho_{q,\theta_2}\tau_3$  and  $\rho_{p,\theta_1} = \rho_{q,\theta_1}\tau_4$  are factorizations of  $\rho_{p,\theta_1}$  around the point  $q$ , and we have already established the uniqueness of such a factorization. Thus

$$\rho_{q,\theta_1} = \rho_{q,\theta_2},$$

proving  $\theta_1 = \theta_2 \pmod{2\pi}$ , as desired.  $\square$

### Exercises III.6

#### Exercise 6.1.

Fix a vertical line  $\ell$ , a point  $q$  on  $\ell$ , and a point  $p$  to the left of  $\ell$  such that  $\overline{pq}$  is perpendicular to  $\ell$ . Let  $m$  be another line through  $q$ , passing below the point  $p$  and making an angle of  $45^\circ$  with the line  $\ell$  — see Figure 6.2.

- (a) Given this configuration, take the reflection  $\phi = \sigma_m$  and determine its *Second Structure Theorem* decomposition with respect to  $p$  and  $\ell$ , i.e., determine the angle  $\theta$  and the translation  $\tau$  such that

$$\phi = \sigma_m = \sigma_\ell \rho_{p,\theta} \tau,$$

where  $\epsilon$  is the parity of  $\phi$ . Justify your answers. In particular, express  $\tau$  in the form  $\tau_{qq_0}$  and draw the line segment  $\overline{qq_0}$  in Figure 6.2. *Hints:*  $\sigma_\ell^\epsilon \sigma_m = \rho_{p,\theta} \tau$ , and  $\tau = \tau_{qq_0}$  if  $q_0 = \tau(q)$ .

- (b) For each of the points  $q, p, a, b$  shown in Figure 6.2, draw in the movements as obtained from the decomposition of  $\sigma_m$ . (For  $q$  this means drawing the points  $q_1 = \tau(q)$ ,  $q_2 = \rho_{p,\theta}(q_1)$ , and  $q_3 = \sigma_\ell^\epsilon(q_2)$ , and similarly for the other three points.)

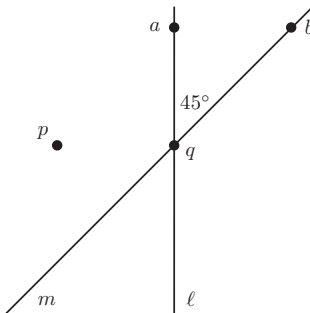


Figure 6.2. Illustration for Exercise 6.1.

**Exercise 6.2.**

If  $\phi$  is an *orientation-reversing* isometry, prove that the angle  $\theta$  in the *Second Structure Theorem* factorization for  $\phi$  is *not* independent of the choices of  $\ell$  and  $p$ . This means showing that varying the choices of  $\ell$  and  $p$  will produce different angles  $\theta$  in the factorization  $\phi = \sigma_\ell \rho_{p,\theta} \tau$ .

*Hint:* You need only vary  $\ell$ . In particular, consider two distinct lines  $\ell$  and  $m$  which intersect at  $p$  and compare the factorizations of  $\phi$  with respect to  $p, \ell$  and  $p, m$ .

**Exercise 6.3.**

Show that the factorization for  $\phi$  obtained in the *Second Structure Theorem* could also have been done in the reverse order, i.e., given  $\ell, p$ , and  $\phi$ , there exist an angle  $\theta'$  and a translation  $\tau'$  such that

$$\phi = \tau' \rho_{p,\theta'} \sigma_\ell^\epsilon.$$

How do  $\theta'$  and  $\tau'$  relate to the  $\theta$  and  $\tau$  of the original factorization?

**Exercise 6.4.**

Fix a line  $\ell$  and a point  $p$  in the plane. In terms of  $\ell$  and  $p$ , determine the *Second Structure Theorem* factorization for the following isometries:

- (a)  $\tau_{py} \rho_{p,\theta}$  where  $y$  is a point in the plane,
- (b)  $\tau_{py} \sigma_\ell$  where  $y$  is a point in the plane,
- (c)  $\rho_{p,\theta} \sigma_\ell$ .

*Hint:* Use Exercises 1.7 and 1.8.

**Exercise 6.5.**

Fix a line  $\ell$  and a point  $p \in \ell$ . Given two isometries  $\phi_1$  and  $\phi_2$ , each has a *Second Structure Theorem* factorization in terms of  $\ell$  and  $p$ ,

$$\phi_1 = \sigma_\ell^{\epsilon_1} \rho_{p,\theta_1} \tau_1 \quad \text{and} \quad \phi_2 = \sigma_\ell^{\epsilon_2} \rho_{p,\theta_2} \tau_2.$$

The composite isometry  $\phi_1\phi_2 = \phi_3$  also has its own factorization,

$$\phi_1\phi_2 = \phi_3 = \sigma_\ell^{\epsilon_3} \rho_{p,\theta_3} \tau_3.$$

- Suppose  $\phi_1$  and  $\phi_2$  are both orientation-preserving. Express each of the factorization components of the composite  $\phi_1\phi_2$  (i.e.,  $\sigma_\ell^{\epsilon_3}$ ,  $\rho_{p,\theta_3}$ , and  $\tau_3$ ) in terms of the factorization components of the separate isometries  $\phi_1$  and  $\phi_2$ . *Hint:* Exercises 1.7 and 1.8.
- Repeat (a) for the case where  $\phi_1$  is orientation-reversing.
- Repeat (a) for the case where  $\phi_1$  is an arbitrary isometry but  $\phi_2$  is orientation-reversing. *Hint:* It helps a lot that  $p$  is on  $\ell$ .

**Exercise 6.6.**

Isometries of the line were studied in Exercises 3.3 and 4.8. State and prove a version of the *Second Structure Theorem* for such mappings.

**Exercise 6.7.**

Let  $p$  be a point in the plane and define  $O_2(p)$  to be the collection of all isometries that have  $p$  as a fixed point.

- Prove that every isometry  $\phi$  can be *uniquely* factored as  $\phi = \phi_p \tau$  where  $\phi_p$  is in  $O_2(p)$  and  $\tau$  is a translation. The isometry  $\phi_p$  is called the ***p*-component** of  $\phi$ .
- If  $\phi_p$  is the *p*-component of  $\phi$  and  $\psi_p$  is the *p*-component of  $\psi$ , what is the *p*-component of  $\phi\psi$ ? What is the *p*-component of  $\phi^{-1}$ ?  
*Hint:* Factor  $\phi$  and  $\psi$  into  $\phi = \phi_p\tau_1$  and  $\psi = \psi_p\tau_2$ . Then analyze  $\phi\psi$  by use of Exercise 1.7b.
- Completely describe the isometries in  $O_2(p)$ .

**Exercise 6.8.**

Fix a line  $\ell$  and a point  $p \in \ell$ . Let  $\mathcal{E}_2(\ell)$  be the group of isometries that leave  $\ell$  invariant, and let  $\mathcal{E}_2(p, \ell)$  be the group of isometries that leave both  $p$  and  $\ell$  invariant.

- Interpret Theorem 6.1, the *Second Structure Theorem*, for  $\phi \in \mathcal{E}_2(\ell)$ . What can you say about the translational factor? What about the factor  $\phi_p$  that fixes  $p$ , i.e., the *p*-component of Exercise 6.7?
- Let  $m_p$  be the line perpendicular to  $\ell$  at  $p$ . Prove

$$\mathcal{E}_2(p, \ell) = \{\iota, \sigma_\ell, \sigma_{m_p}, \nu_p\}.$$

- (c) Prove every  $\phi \in \mathcal{E}_2(\ell)$  can be factored as  $\phi = \sigma_\ell^\epsilon \sigma_{m_p}^\eta \tau$ , where  $\tau$  is a translation parallel to  $\ell$  and  $\epsilon$  and  $\eta$  are each equal to 0 or 1.
- (d) The reflection  $\sigma_\ell$  acts as the identity on  $\ell$ : it fixes every point of  $\ell$ , and it interchanges the two half planes bounded by  $\ell$ . Show  $\sigma_\ell$  commutes with every element of  $\mathcal{E}_2(\ell)$ .
- (e) Conversely, if  $\sigma_\ell$  commutes with an isometry  $\phi$ , prove  $\phi$  belongs to  $\mathcal{E}_2(\ell)$ , i.e.,  $\phi$  preserves  $\ell$ .
- (f) Let  $\mathcal{E}_2^\epsilon(\ell)$  denote the collection of isometries in  $\mathcal{E}_2(\ell)$  that preserve each of the half planes bounded by  $\ell$ . Prove every  $\phi \in \mathcal{E}_2(\ell)$  can be factored in the form  $\phi = \sigma_\ell^\epsilon \psi$  where  $\psi \in \mathcal{E}_2^\epsilon(\ell)$  and  $\epsilon$  equals 0 or 1.
- (g) Compare the factorizations of  $\phi \in \mathcal{E}_2(\ell)$  given in (a), (c), and (f).
- (h) *Isometries of a line*  $\ell$  were defined in Exercise 3.3. Let  $\mathcal{E}_1(\ell)$  denote the collection of all isometries of  $\ell$ ; this is easily seen to form a group. Show that the mapping  $\phi \rightarrow \phi|_\ell$ , which takes an isometry  $\phi$  in  $\mathcal{E}_2(\ell)$  and restricts its domain to  $\ell$ , defines a one-to-one and onto mapping from  $\mathcal{E}_2^\epsilon(\ell)$  to  $\mathcal{E}_1(\ell)$ . *Hint*: Consider Exercise 3.4.

### §III.7 Rotation Angles

Given any fixed choice of a line  $\ell$  and a point  $p$ , every isometry  $\phi$  of the plane has a unique *Second Structure Theorem* factorization of the form

$$\phi = \sigma_\ell^\epsilon \rho_{p,\theta} \tau,$$

where  $\epsilon$  is the parity of  $\phi$ . When  $\phi$  is orientation-reversing, the angle  $\theta$  in this factorization varies with the choices of  $\ell$  and  $p$ , i.e.,  $\theta = \theta(\ell, p)$ , as seen in Exercise 6.2. However, as stated in the *Second Structure Theorem*, there is no such dependence when  $\phi$  is orientation-preserving. This allows us to define the *rotation angle* for  $\phi$  to be the angle obtained in the *Second Structure Theorem* factorization for  $\phi$  for *any* choice of  $\ell$  and  $p$ .

Physically, anyone who “observes” an orientation-preserving isometry will sense that “part” of the isometry is a rotation. Moreover, the angle of this rotation will be given the same measured value by *all* observers. (As will be seen in Exercise 7.3, there is **no** reasonable definition for a rotation angle of an *orientation-reversing* isometry.)

#### Definition 7.1.

If  $\phi$  is an orientation-preserving isometry of the plane, then the **rotation angle** for  $\phi$ , denoted by  $\angle\phi$ , is defined as that directed angle measure  $\theta$  which appears in any factorization of the form  $\phi = \rho_{p,\theta} \tau$ .

Since we know all the orientation-preserving isometries of the plane — rotations and translations — the following is a natural result.

**Proposition 7.2.**

Suppose  $\phi$  is an orientation-preserving isometry of the plane.

- (a)  $\phi$  is a translation  $\tau$  if and only if  $\angle\phi = 0 \pmod{2\pi}$ .  
 (b)  $\phi$  is a non-trivial rotation  $\rho_{p,\theta}$  if and only if  $\angle\phi = \theta \not\equiv 0 \pmod{2\pi}$ .

*Proof.* (a) If  $\angle\phi = 0 \pmod{2\pi}$ , then  $\phi = \rho_{q,0} \tau = \tau$ , a translation as desired. Conversely, if  $\phi = \tau$ , then for any choice of  $p$ , the *Second Structure Theorem* factorization is  $\phi = \tau = \rho_{p,0} \tau$ . This is true because this factorization is of the appropriate form for the point  $p$  and, given a choice of  $p$ , the *Second Structure Theorem* factorization for  $\phi$  is unique. Hence, since  $\phi = \rho_{p,0} \tau$  is of the appropriate form, it must be the *Second Structure Theorem* factorization! Hence the rotation angle must be  $\angle\phi = 0 \pmod{2\pi}$ , as desired.

(b) If  $\angle\phi = \theta \pmod{2\pi}$ , then  $\phi = \rho_{q,\theta} \tau$  for some translation  $\tau$ . But we know by Proposition 4.1 that this product of four reflections reduces to a product of two reflections, i.e., a rotation or a translation. Since the rotation angle is non-trivial, we must get a rotation, as claimed.

For the converse, if  $\phi = \rho_{p,\theta}$ , then the *Second Structure Theorem* factorization for  $\phi$  about the point  $p$  is merely  $\phi = \rho_{p,\theta}$ . Hence  $\angle\phi$  obviously equals  $\theta \pmod{2\pi}$ , as claimed. (We have used our freedom to choose the center of rotation  $p$  as the point in the *Second Structure Theorem* factorization.)  $\square$

It may seem strange to define the rotation angle in so “indirect” a fashion — using the *Second Structure Theorem* — when we only have rotations and translations, and they can easily be assigned a rotation angle as described in Proposition 7.2. However, our method can be more easily generalized to other dimensions, and it also makes short work of the following basic result: *the rotation angle is additive with respect to composition:*

**Theorem 7.3.**

Suppose  $\phi$  and  $\psi$  are orientation-preserving isometries. Then the rotation angle of  $\psi\phi$  is the sum of the rotation angles of  $\psi$  and  $\phi$ , i.e.,

$$\angle\psi\phi = \angle\psi + \angle\phi \pmod{2\pi}.$$

*Remark:* This result is closely related to Exercise 6.7b.

*Proof.* Fix  $p$  and factor the isometries  $\phi$  and  $\psi$  according to the *Second Structure Theorem*:  $\phi = \rho_{p,\theta_1} \tau_1$  and  $\psi = \rho_{p,\theta_2} \tau_2$ , where  $\angle\phi = \theta_1 \pmod{2\pi}$  and  $\angle\psi = \theta_2 \pmod{2\pi}$ . Hence

$$\begin{aligned} \psi\phi &= \rho_{p,\theta_2} \tau_2 \rho_{p,\theta_1} \tau_1 \\ &= \rho_{p,\theta_2} (\rho_{p,\theta_1} \rho_{p,\theta_1}^{-1}) \tau_2 \rho_{p,\theta_1} \tau_1 \\ &= \rho_{p,\theta_2} \rho_{p,\theta_1} (\rho_{p,\theta_1}^{-1} \tau_2 \rho_{p,\theta_1}) \tau_1 \\ &= \rho_{p,\theta_1+\theta_2} (\rho_{p,\theta_1}^{-1} \tau_2 \rho_{p,\theta_1}) \tau_1 \end{aligned}$$

where we have used the equality  $\rho_{p,\theta_2} \rho_{p,\theta_1} = \rho_{p,\theta_1+\theta_2}$ , as developed in Example II.3.2b. Moreover, we claim that the isometry in parentheses,  $\rho_{p,\theta_1}^{-1} \tau_2 \rho_{p,\theta_1}$ , is merely another translation. (This is a consequence of Exercise 1.7b, but for the sake of completeness we will give the full argument.) First note that this isometry is orientation-preserving since it is the product of only orientation-preserving isometries. Hence it is a rotation or a translation. However, it cannot have a fixed point unless it is trivial. For suppose it *did* have a fixed point  $q$ . Then

$$\rho_{p,\theta_1}^{-1} \tau_2 \rho_{p,\theta_1}(q) = q,$$

which would imply

$$\tau_2(\rho_{p,\theta_1}(q)) = \rho_{p,\theta_1}(q).$$

This would make  $\rho_{p,\theta_1}(q)$  a fixed point of the translation  $\tau_2$ , which is impossible unless  $\tau_2$  equals  $\iota$ , in which case  $\rho_{p,\theta_1}^{-1} \tau_2 \rho_{p,\theta_1}$  also equals  $\iota$ . Hence  $\rho_{p,\theta_1}^{-1} \tau_2 \rho_{p,\theta_1}$  is the identity transformation or an orientation-preserving isometry with no fixed points. In either case it is a translation  $\tau_3$ , and we obtain

$$\psi\phi = \rho_{p,\theta_1+\theta_2} \tau_3 \tau_1.$$

Since the product of translations is a translation, this equality shows that the rotation angle of  $\psi\phi$  is indeed the sum  $\theta_1 + \theta_2 \pmod{2\pi}$ , as desired.  $\square$

#### Corollary 7.4.

*Suppose  $\phi_1, \dots, \phi_n$  are orientation-preserving isometries whose rotation angles sum to  $0 \pmod{2\pi}$ , i.e.,*

$$\angle\phi_1 + \dots + \angle\phi_n = 0 \pmod{2\pi}.$$

*Then the product  $\phi_n \dots \phi_1$  is a translation.*

*Proof.* Applying Theorem 7.3 recursively  $n - 1$  times to the products

$$\phi_2\phi_1, \quad \phi_3(\phi_2\phi_1), \quad \phi_4(\phi_3\phi_2\phi_1), \dots, \phi_n(\phi_{n-1} \dots \phi_1)$$

shows that the rotation angle of the product  $\phi_n \dots \phi_1$  is the sum of the rotation angles of the individual isometries. Hence this angle is  $0 \pmod{2\pi}$ . But by Proposition 7.2a, a rotation angle of zero means that the isometry must be a translation (which, in the trivial case, might be the identity).  $\square$

#### Corollary 7.5.

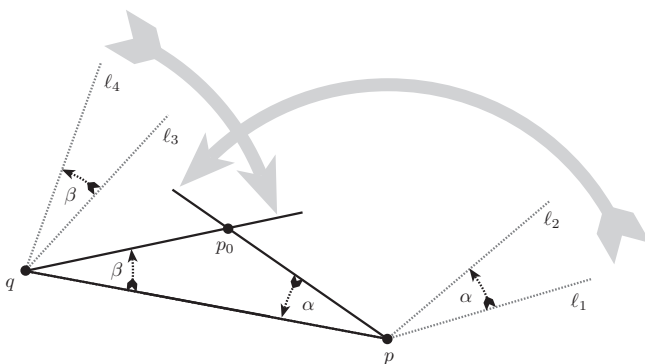
*Suppose  $\phi = \rho_{p,\theta_1}$  and  $\psi = \rho_{q,\theta_2}$  are two rotations in the plane.*

(a) *If  $\theta_1 + \theta_2 = 0 \pmod{2\pi}$ , then  $\psi\phi$  is a translation.*

(b) *If  $\theta_1 + \theta_2 \neq 0 \pmod{2\pi}$ , then  $\psi\phi$  is a rotation of angle  $\theta = \theta_1 + \theta_2$ .*

*Proof.* This follows by combining Proposition 7.2 and Theorem 7.3.  $\square$

The conclusions of this proposition are what you should have drawn in Exercise 4.2. It may give you a better intuitive understanding of the result to recall the explicit geometric argument outlined in Exercise 4.2. The proof consisted of applying Proposition 1.8 (rotations as products of reflections) to both  $\phi = \rho_{p,\theta_1}$  and  $\psi = \rho_{q,\theta}$ , so that  $\phi = \sigma_{\ell_2}\sigma_{\ell_1}$  and  $\psi = \sigma_{\ell_4}\sigma_{\ell_3}$ , where  $\ell_1$  and  $\ell_2$  intersect at  $p$  with angle  $\alpha = (1/2)\theta_1$  and  $\ell_3$  and  $\ell_4$  intersect at  $q$  with angle  $\beta = (1/2)\theta_2$  — see Figure 4.13, as reproduced here. Using Theorem 1.15b, rotate  $\ell_1, \ell_2$  about  $p$  until  $\ell_2$  passes through  $q$  and rotate  $\ell_3, \ell_4$  about  $q$  until  $\ell_3$  passes through  $p$  without changing the product  $\psi\phi$ .



Copy of Figure 4.13. Altering the reflections for a product of rotations.

The reflections in  $\ell_2$  and  $\ell_3$  thus eliminate each other, leaving  $\psi\phi$  as the product of two reflections in (the rotated images of) lines  $\ell_1$  and  $\ell_4$ .

If (the rotated images of)  $\ell_1$  and  $\ell_4$  intersect at a point  $p_0$  as in Figure 4.13 (which occurs when  $\alpha + \beta$  is not an integral multiple of  $\pi$ ), then the counterclockwise angle at  $p_0$  from the rotated  $\ell_1$  to the rotated  $\ell_4$  is seen to be  $\alpha + \beta = 1/2(\theta_1 + \theta_2)$ . Hence  $\psi\phi$  equals  $\rho_{p_0,\theta_1+\theta_2}$ , the desired result in (a).

If  $\alpha + \beta$  is an integral multiple of  $\pi$ , then (the rotated images of)  $\ell_1$  and  $\ell_4$  are parallel or equal and  $\psi\phi$  will be a translation, the desired result in (b).

### Exercises III.7

#### Exercise 7.1.

Show that the product of an even number of point inversions is a translation while the product of an odd number is another point inversion.

#### Exercise 7.2.

Suppose  $\phi$  is an orientation-preserving isometry and  $\ell$  is any *directed* line, i.e., pick a point  $p$  on  $\ell$  and specify one of the two half lines of  $\ell$  starting at  $p$  as the *positive* half line  $\ell^+(p)$ . Then  $\phi(\ell)$  is a line by Proposition II.2.9; it inherits a direction from  $\ell$  via  $\phi$  by defining a positive half line for  $\phi(\ell)$  to be  $\phi(\ell^+(p))$ . Show the following:

- (a) if  $\angle\phi \neq 0 \pmod{\pi}$ , then  $\ell$  and  $\phi(\ell)$  intersect and  $\angle\phi$  is the counterclockwise angle formed by rotating from the positive portion of  $\ell$  to the positive portion of  $\phi(\ell)$ ,
- (b) if  $\angle\phi = 0 \pmod{\pi}$ , then  $\ell$  and  $\phi(\ell)$  are parallel or identical. Moreover
- if  $\angle\phi = 0 \pmod{2\pi}$ , then  $\ell$  and  $\phi(\ell)$  have the same direction  
(thus  $\tau_{AB} = \tau_{\phi(A)\phi(B)}$ ),
  - if  $\angle\phi = \pi \pmod{2\pi}$ , then  $\ell$  and  $\phi(\ell)$  have opposite directions  
(thus  $\tau_{AB} = \tau_{\phi(B)\phi(A)}$ ).

This yields a physical characterization for the rotation angle  $\angle\phi$  for an orientation-preserving isometry  $\phi$ : it is that counterclockwise angle made by going from the positive portion of any directed line  $\ell$  to the positive portion of its image  $\phi(\ell)$ .

### Exercise 7.3.

Suppose  $\phi$  is an orientation-reversing isometry of the plane.

- (a) If  $\theta \neq 0 \pmod{\pi}$ , show there is a directed line  $\ell$  such that  $\ell$  and  $\phi(\ell)$  intersect with  $\theta$  as the counterclockwise angle formed by rotating from a positive half line of  $\ell$  to a positive half line of  $\phi(\ell)$ .
- (b) Show there exist directed lines  $\ell$  such that  $\ell$  and  $\phi(\ell)$  are parallel and of similar direction.
- (c) Further show there exist directed lines  $\ell$  such that  $\ell$  and  $\phi(\ell)$  are parallel and of opposite direction.
- (d) Is it possible to formulate a reasonable definition for the rotation angle of an orientation-reversing isometry? If so, state and justify your definition. If not, state why such a definition is impossible.

### Exercise 7.4.

Suppose  $\triangle v_1 v_2 v_3$  is a triangle with vertices  $v_1, v_2, v_3$  when listed in clockwise order.

- (a) If the interior angles at  $v_1, v_2, v_3$  have measures  $\theta_1, \theta_2, \theta_3$ , prove

$$\rho_{v_3, 2\theta_3} \rho_{v_2, 2\theta_2} \rho_{v_1, 2\theta_1} = \iota.$$

*Hint:* Apply Proposition 1.8 to each rotation.

- (b) Conversely, if  $\theta_1, \theta_2, \theta_3$  are positive numbers whose sum is  $\pi$  and

$$\rho_{v_3, 2\theta_3} \rho_{v_2, 2\theta_2} \rho_{v_1, 2\theta_1} = \iota,$$

prove that the corresponding interior angles at  $v_1, v_2, v_3$  have measures  $\theta_1, \theta_2, \theta_3$ . *Hint:* Compare  $\triangle = \triangle v_1 v_2 v_3$  with the triangle  $\triangle' = \triangle v_1 v_2 v'_3$  where  $v'_3$  is chosen on the side of  $\overline{v_1 v_2}$  that puts  $v_1, v_2, v'_3$  into clockwise order and makes the interior angles of  $\triangle'$  at  $v_1, v_2, v_3$  have measures  $\theta_1, \theta_2, \theta_3$ . Prove  $v_3$  must equal  $v'_3$ .

- (c) Prove that  $\triangle v_1 v_2 v_3$  is equilateral if and only if

$$\rho_{v_3, 120^\circ} \rho_{v_2, 120^\circ} \rho_{v_1, 120^\circ} = \iota.$$

- (d) If the interior angles at  $v_1$ ,  $v_2$ ,  $v_3$  have measures  $\theta_1$ ,  $\theta_2$ ,  $\theta_3$ , then completely describe the isometry

$$\phi = \rho_{v_1, 2\theta_1} \rho_{v_2, 2\theta_2} \rho_{v_3, 2\theta_3}.$$

*Hint:* Exercises 2.7 and 2.9.

**Exercise 7.5. Napoleon's Theorem.**

Consider any triangle with vertices  $v_1$ ,  $v_2$ , and  $v_3$ . On each side of the triangle erect an equilateral triangle as shown in Figure 7.6. Let  $c_1$ ,  $c_2$ , and  $c_3$  be the centers of these equilateral triangles. Prove  $\triangle c_1 c_2 c_3$  is equilateral. (There is a legend attributing this theorem to Napoleon Bonaparte. Nice story. Probably false.) *Hint:* Exercise 7.4c. In particular, apply  $\rho_{c_3, 120^\circ} \rho_{c_2, 120^\circ} \rho_{c_1, 120^\circ}$  to  $v_2$ . Napoleon's Theorem is formally stated as Theorem VI.6.17. Also see Exercise VI.6.2.

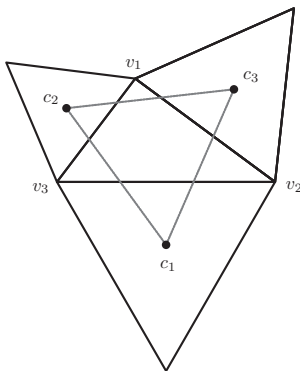


Figure 7.6. According to Napoleon's Theorem,  $\triangle c_1 c_2 c_3$  is equilateral.

**Exercise 7.6.**

Let  $\mathcal{E}_{\theta_0}^+$  denote the collection of all orientation-preserving isometries of the plane whose rotation angles  $\theta$  can be expressed as integral multiples of  $\theta_0$  (including the value  $\theta = 0$ ). Is  $\mathcal{E}_{\theta_0}^+$  a group? If so, is it a *normal* subgroup (as defined in Exercise 5.3) of  $\mathcal{E}_2^+$ , the collection of all orientation-preserving isometries of the plane? Is it a *normal* subgroup of  $\mathcal{E}_2$ , the collection of all isometries of the plane?