
Contents

Preface to the Second Edition	ix
Preface to the First Edition	xi
Chapter I. Complex Numbers	1
§I.1. Definition of \mathbf{C}	2
§I.2. Field Axioms	2
§I.3. Embedding of \mathbf{R} in \mathbf{C} . The Imaginary Unit	3
§I.4. Geometric Representation	3
§I.5. Triangle Inequality	4
§I.6. Parallelogram Equality	5
§I.7. Plane Geometry via Complex Numbers	5
§I.8. \mathbf{C} as a Metric Space	6
§I.9. Polar Form	6
§I.10. De Moivre's Formula	7
§I.11. Roots	8
§I.12. Stereographic Projection	9
§I.13. Spherical Metric	10
§I.14. Extended Complex Plane	11
Chapter II. Complex Differentiation	13
§II.1. Definition of the Derivative	13
§II.2. Restatement in Terms of Linear Approximation	14
§II.3. Immediate Consequences	14

§II.4.	Polynomials and Rational Functions	15
§II.5.	Comparison Between Differentiability in the Real and Complex Senses	15
§II.6.	Cauchy-Riemann Equations	16
§II.7.	Sufficient Condition for Differentiability	17
§II.8.	Holomorphic Functions	17
§II.9.	Complex Partial Differential Operators	18
§II.10.	Picturing a Holomorphic Function	19
§II.11.	Curves in \mathbf{C}	20
§II.12.	Conformality	21
§II.13.	Conformal Implies Holomorphic	22
§II.14.	Harmonic Functions	23
§II.15.	Holomorphic Implies Harmonic	24
§II.16.	Harmonic Conjugates	24
Chapter III.	Linear-Fractional Transformations	27
§III.1.	Complex projective space	27
§III.2.	Linear-fractional transformations	28
§III.3.	Conformality	29
§III.4.	Fixed points	29
§III.5.	Three-fold transitivity	29
§III.6.	Factorization	30
§III.7.	Clircles	31
§III.8.	Preservation of clircles	31
§III.9.	Analyzing a linear-fractional transformation—an example	32
Chapter IV.	Elementary Functions	35
§IV.1.	Definition of e^z	35
§IV.2.	Law of Exponents	36
§IV.3.	e^z is holomorphic	36
§IV.4.	Periodicity	37
§IV.5.	e^z as a map	37
§IV.6.	Hyperbolic functions	38
§IV.7.	Zeros of $\cosh z$ and $\sinh z$.	38
§IV.8.	Trigonometric functions	39
§IV.9.	Logarithms	40
§IV.10.	Branches of $\arg z$ and $\log z$.	40

§IV.11.	$\log z$ as a holomorphic function	41
§IV.12.	Logarithms of holomorphic functions	42
§IV.13.	Roots	43
§IV.14.	Inverses of holomorphic functions	43
§IV.15.	Inverse trigonometric functions	44
§IV.16.	Powers	45
§IV.17.	Analytic continuation and Riemann surfaces	45
Chapter V.	Power Series	49
§V.1.	Infinite Series	49
§V.2.	Necessary Condition for Convergence	49
§V.3.	Geometric Series	50
§V.4.	Triangle Inequality for Series	50
§V.5.	Absolute Convergence	50
§V.6.	Sequences of Functions	51
§V.7.	Series of Functions	51
§V.8.	Power Series	52
§V.9.	Region of Convergence	53
§V.10.	Radius of Convergence	54
§V.11.	Limits Superior	54
§V.12.	Cauchy-Hadamard Theorem	55
§V.13.	Ratio Test	56
§V.14.	Examples	57
§V.15.	Differentiation of Power Series	58
§V.16.	Examples	60
§V.17.	Cauchy Product	61
§V.18.	Division of Power Series	63
Chapter VI.	Complex Integration	65
§VI.1.	Riemann Integral for Complex-Valued Functions	65
§VI.2.	Fundamental Theorem of Calculus	66
§VI.3.	Triangle Inequality for Integration	66
§VI.4.	Arc Length	67
§VI.5.	The Complex Integral	67
§VI.6.	Integral of a Derivative	68
§VI.7.	An Example	68

§VI.8. Reparametrization	69
§VI.9. The Reverse of a Curve	70
§VI.10. Estimate of the Integral	71
§VI.11. Integral of a Limit	71
§VI.12. An Example	71
Chapter VII. Core Versions of Cauchy's Theorem, and Consequences	75
§VII.1. Cauchy's Theorem for a Triangle	75
§VII.2. Cauchy's Theorem for a Convex Region	78
§VII.3. Existence of a Primitive	78
§VII.4. More Definite Integrals	79
§VII.5. Cauchy's Formula for a Circle	79
§VII.6. Mean Value Property	81
§VII.7. Cauchy Integrals	82
§VII.8. Implications for Holomorphic Functions	83
§VII.9. Cauchy Product	84
§VII.10. Converse of Goursat's Lemma	85
§VII.11. Liouville's Theorem	86
§VII.12. Fundamental Theorem of Algebra	86
§VII.13. Zeros of Holomorphic Functions	87
§VII.14. The Identity Theorem	89
§VII.15. Weierstrass Convergence Theorem	89
§VII.16. Maximum Modulus Principle	90
§VII.17. Schwarz's Lemma	91
§VII.18. Existence of Harmonic Conjugates	93
§VII.19. Infinite Differentiability of Harmonic Functions	94
§VII.20. Mean Value Property for Harmonic Functions	94
§VII.21. Identity Theorem for Harmonic Functions	94
§VII.22. Maximum Principle for Harmonic Functions	95
§VII.23. Harmonic Functions in Higher Dimensions	95
Chapter VIII. Laurent Series and Isolated Singularities	97
§VIII.1. Simple Examples	97
§VIII.2. Laurent Series	98
§VIII.3. Cauchy Integral Near ∞	99
§VIII.4. Cauchy's Theorem for Two Concentric Circles	100

§VIII.5. Cauchy's Formula for an Annulus	101
§VIII.6. Existence of Laurent Series Representations	101
§VIII.7. Isolated Singularities	102
§VIII.8. Criterion for a Removable Singularity	105
§VIII.9. Criterion for a Pole	105
§VIII.10. Casorati-Weierstrass Theorem	106
§VIII.11. Picard's Theorem	106
§VIII.12. Residues	106
Chapter IX. Cauchy's Theorem	109
§IX.1. Continuous Logarithms	109
§IX.2. Piecewise C^1 Case	110
§IX.3. Increments in the Logarithm and Argument Along a Curve	110
§IX.4. Winding Number	111
§IX.5. Case of a Piecewise- C^1 Curve	111
§IX.6. Contours	113
§IX.7. Winding Numbers of Contours	114
§IX.8. Separation Lemma	115
§IX.9. Addendum to the Separation Lemma	117
§IX.10. Cauchy's Theorem	118
§IX.11. Homotopy	119
§IX.12. Continuous Logarithms—2-D Version	119
§IX.13. Homotopy and Winding Numbers	120
§IX.14. Homotopy Version of Cauchy's Theorem	121
§IX.15. Runge's Approximation Theorem	121
§IX.16. Second Proof of Cauchy's Theorem	122
§IX.17. Sharpened Form of Runge's Theorem	123
Chapter X. Further Development of Basic Complex Function	125
§X.1. Simply Connected Domains	125
§X.2. Winding Number Criterion	126
§X.3. Cauchy's Theorem for Simply Connected Domains	126
§X.4. Existence of Primitives	127
§X.5. Existence of Logarithms	127
§X.6. Existence of Harmonic Conjugates	128
§X.7. Simple Connectivity and Homotopy	128

§X.8. The Residue Theorem	129
§X.9. Cauchy's Formula	130
§X.10. More Definite Integrals	130
§X.11. The Argument Principle	137
§X.12. Rouché's Theorem	138
§X.13. The Local Mapping Theorem	140
§X.14. Consequences of the Local Mapping Theorem	140
§X.15. Inverses	141
§X.16. Conformal Equivalence	141
§X.17. The Riemann Mapping Theorem	142
§X.18. An Extremal Property of Riemann Maps	143
§X.19. Stieltjes-Osgood Theorem	144
§X.20. Proof of the Riemann Mapping Theorem	146
§X.21. Simple Connectivity Again	148
Appendix 1. Sufficient condition for differentiability	151
Appendix 2. Two instances of the chain rule	153
Appendix 3. Groups, and linear-fractional transformations	155
Appendix 4. Differentiation under the integral sign	157
References	159
Index	161