CHAPTER 1

Basics on tensor norms

This first chapter presents the basics of tensor norms following the spirit of Grothendieck's presentation. To be sure, we've added examples and borrowed results (with proofs) from later chapters but otherwise we've followed the master's plan.

We start with a discussion of *reasonable crossnorms* paying particular attention to two examples: the injective and projective norms. This is followed with a discussion of some critical examples and Grothendieck's famous "computation" of the dual of the injective tensor product. A few illustrative examples follow, in particular, we compute the closed linear span of $(e_n \otimes e_n)_{n\geq 1}$ in $\ell^p \overset{\wedge}{\otimes} \ell^p$.

Our next section is devoted to *tensor norms* defined on the tensor product of finite dimensional Banach spaces. This is followed by a discussion of how to extend the definition of a tensor norm to the tensor product of infinite dimensional Banach spaces. This leads to the delicate issues of accessible spaces and tensor norms. Here we reproduce some of Grothendieck's *Memoir* to give these notions some grit.

In the fourth section, α-integral bilinear forms and α-integral linear operators make their appearance. It is through these classes of operators that Grothendieck's view of Banach space theory becomes clear. The fundamental facts about α-integral operators include their "ideal" properties as well as their finitary (or local) determination. This section ends with another visit to the delicate subject of accessibility and metric accessibility.

In the short fifth section α-nuclear forms and operators are introduced and their ideal structure noted.

We close this chapter with an exposition of Grothendieck's often overlooked paper on the celebrated Dvoretsky-Rogers theorem. The main result here is his generalization of their result with the conclusion that if $1 < p < \infty$ and X is an infinite dimensional Banach space, then

$$\ell^p \overset{\wedge}{\otimes} X \subsetneqq \ell^p_X \subsetneqq \ell^p \overset{\vee}{\otimes} X.$$

Crucial to the argumentation of this section is a discussion of Blaschke's selection principle, which we include as an appendix to the book.

The algebraic preliminaries

Let X, Y and Z be linear spaces (over the same scalar field $\mathbb{K}$, be it the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$). A function $\varphi : X \times Y \to Z$ is *bilinear* if $\varphi(x, \cdot) : Y \to Z$ is linear for each $x \in X$ and $\varphi(\cdot, y) : X \to Z$ is linear for each $y \in Y$. We will denote by $B(X, Y; Z)$ the linear space of all bilinear functions from $X \times Y$ to Z and by $B(X, Y)$ the space of all bilinear functions on $X \times Y$ into the scalar field.

We will denote by X' the algebraic dual of X, that is, the linear space of all linear functionals on X. $L(X;Z)$ will denote the linear space of all linear functions from X to Z.

The basic question answered by tensor product constructions is the following: Is there a linear space V such that $L(V;Z)$ coincides with (is isomorphic to, is *naturally* isomorphic to) $B(X,Y;Z)$? Rephrasing the question: Can we in some way linearize bilinear functions?

The answer is "yes" and the object we construct, the tensor product, $X \otimes Y$, of X and Y will do the job.

Since we are looking for a vector space V which, in particular, has a dual that is (isomorphic to) the space $B(X,Y)'$ of bilinear functionals, it is natural to look inside the dual $B(X,Y)'$ of $B(X,Y)$. Therein, we find a collection of functionals of the form $x \otimes y$, where $x \in X$ and $y \in Y$: $x \otimes y$ (called an elementary tensor) is the element of $B(X,Y)'$ whose value at $\varphi \in B(X,Y)$ is given by the evaluation

$$(x \otimes y)(\varphi) = \varphi(x,y).$$

The *tensor product* $X \otimes Y$ is the linear span of the collection of elementary tensors, $\{x \otimes y : x \in X, y \in Y\}$. So a typical $u \in X \otimes Y$ has the form

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i \tag{$*$}$$

where $\lambda_1, \ldots, \lambda_n$ are scalars, $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$ and $n \in \mathbb{N}$ is arbitrary.

The behavior of $X \otimes Y$ is worth emphasizing. $x \otimes y$ has itself a certain degree of bilinearity. Here's what's so:

(1) $(x_1 + x_2) \otimes y = (x_1 \otimes y) + (x_2 \otimes y)$,
(2) $x \otimes (y_1 + y_2) = (x \otimes y_1) + (x \otimes y_2)$,
(3) $\lambda(x \otimes y) = \lambda x \otimes y = x \otimes \lambda y$,
(4) $0 \otimes y = x \otimes 0 = 0_{X \otimes Y}$.

The reason why these relations are true is because of the way we are forced to determine when $u, v \in X \otimes Y$ are the same: $u = v$ precisely when $u(\varphi) = v(\varphi)$ for each $\varphi \in B(X,Y)$. It is also obvious from these relations that the representation $(*)$ of $u \in X \otimes Y$ is far from unique.

All we've said so far is easy enough to verify and soon leads to more insightful features of life inside $X \otimes Y$. Here's one: Let E be a linearly independent subset of X and let F be a linearly independent subset of Y, then the *set*

$$E \otimes F := \{e \otimes f : e \in E, f \in F\}$$

is linearly independent in $X \otimes Y$. Indeed, look at any finite linear combination of elements of $E \otimes F$. Without loss of generality, we may assume that this linear combination is of the form

$$\sum_{i \in I, j \in J} \lambda_{ij} e_i \otimes f_j,$$

where the sets I and J are finite subsets of E and F respectively. Now if

$$\sum_{i \in I, j \in J} \lambda_{ij} e_i \otimes f_j = 0$$

(in $X \otimes Y$), then, of course,

$$\Big(\sum_{i\in I, j\in J} \lambda_{ij} e_i \otimes f_j\Big)(\varphi) = 0$$

for each and every $\varphi \in B(X,Y)$. In particular, if $x' \in X'$ and $y' \in Y'$, then the bilinear functional $\varphi_{(x',y')}$ whose value at $(x,y) \in X \times Y$ is given by

$$\varphi_{(x',y')}(x,y) = x'(x)y'(y)$$

also satisfies

$$\Big(\sum_{i\in I, j\in J} \lambda_{ij} e_i \otimes f_j\Big)(\varphi_{(x',y')}) = 0.$$

That is,

$$\sum_{i\in I, j\in J} \lambda_{ij} x'(e_i) y'(f_j) = 0$$

for each $x' \in X'$ and $y' \in Y'$. Let's look more closely at what's going on here. For each $x' \in X'$ and $y' \in Y'$ we have

$$0 = \sum_{i\in I, j\in J} \lambda_{ij} x'(e_i) y'(f_j) = x'\Big(\sum_{i\in I, j\in J} \lambda_{ij} y'(f_j) e_i\Big).$$

Since this is so for each $x' \in X'$, it follows that $\sum_{i\in I, j\in J} \lambda_{ij} y'(f_j) e_i = 0$; hence

$$\sum_{i\in I} \Big(\sum_{j\in J} \lambda_{ij} y'(f_j)\Big) e_i = 0,$$

and by E's linear independence, we have

$$0 = \sum_{j\in J} \lambda_{ij} y'(f_j) = y'\Big(\sum_{j\in J} \lambda_{ij} f_j\Big), \text{ for each } i \in I.$$

Since this is so for each $y' \in Y'$, it follows that $\sum_{j\in J} \lambda_{ij} f_j = 0$ for each $i \in I$ and hence, thanks to F's linear independence, we have

$$\lambda_{ij} = 0 \text{ for each } j \in J \text{ and for each } i \in I.$$

By taking a close look at what we've done, one soon sees that the tensor product $X' \otimes Y'$ of the dual spaces X' and Y' can be identified with a subspace of $B(X,Y)$; just look at the linear extension of the mapping $x' \otimes y' \mapsto \varphi_{(x',y')}$ to all of $X' \otimes Y'$.

Now that we have a handle on $X \otimes Y$, however tenuous, it is time to establish that $X \otimes Y$ does the job it was created for: *linearizing bilinear functions*. To start, consider $\varphi \in B(X,Y)$ and define $U_\varphi : X \otimes Y \to \mathbb{K}$, first on the *elementary tensors* $x \otimes y$ by $U_\varphi(x \otimes y) = \varphi(x,y)$ and then extend it linearly to all of $X \otimes Y$. If $u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$, then $U_\varphi(u) = \sum_{i=1}^n \varphi(x_i, y_i) = \sum_{i=1}^n U_\varphi(x_i \otimes y_i)$. U_φ is easily seen to be linear and we have, in fact, factored φ as follows:

$$\begin{tikzcd} X \times Y \arrow[r, "\varphi"] \arrow[d, "\otimes"'] & \mathbb{K} \\ X \otimes Y \arrow[ur, "U_\varphi"'] & \end{tikzcd}$$

where the map from $X \times Y$ into $X \otimes Y$ is the bilinear function that takes the pair $(x,y) \in X \times Y$ to the elementary tensor $x \otimes y \in X \otimes Y$.

On the other hand, if we start with a linear functional U on $X \otimes Y$ and define φ_U on $X \times Y$ by the formula

$$\varphi_U(x, y) = U(x \otimes y),$$

then it is plain and easy to see that φ_U is bilinear, thanks in large part to the already noted bilinearity of $x \otimes y$.

If we step back and take stock of what's been done, we should realize that

- starting with the bilinear φ and passing to the linear U_φ, if we now look at φ_{U_φ}, we're back at φ;
- starting with the linear U and passing to the bilinear φ_U, if we next look at U_{φ_U}, we're back at U.

In other words, $B(X, Y)$ and $X \otimes Y$, are naturally isomorphic with the diagram

$$\begin{tikzcd}
X \times Y \arrow[dd, "\otimes"'] \arrow[dr, "\varphi"] & \\
& \mathbb{K} \\
X \otimes Y \arrow[ur, "U"'] &
\end{tikzcd}$$

telling the whole story. This story is often called the *universal mapping property* of tensor products.

It is noteworthy that the fact that the linear and/or bilinear functions took values in the scalar field $\mathbb{K}$ was unimportant to the argument. In fact, if Z is any linear space (over the same field as X and Y), then the Universal Mapping Property has a byproduct; the diagram

$$\begin{tikzcd}
X \times Y \arrow[dd, "\otimes"'] \arrow[dr, "\varphi"] & \\
& Z \\
X \otimes Y \arrow[ur, "U"'] &
\end{tikzcd}$$

establishes a natural isomorphism between $B(X, Y; Z)$ and $L(X \times Y; Z)$ with roots in the formula

$$U(x \otimes y) = \varphi(x, y).$$

Lest there be concern of precisely what we have constructed, be assured that the tensor product we've built is uniquely qualified to not only do the job we set for it (to linearize bilinear functionals), but more so, to linearize bilinear operators. Indeed we have the following:

Theorem. *Let X and Y be linear spaces and let W be a linear space and $\tau : X \times Y \to Z$ be a bilinear map with the property that for any linear space Z and any bilinear function $\varphi : X \times Y \to Z$, there is a unique linear function $L : W \to Z$ such that $\varphi = L \circ \tau$. Then there is a linear isomorphism $J : X \otimes Y \to W$ such that $J(x \otimes y) = \tau(x, y)$ for each $x \in X, y \in Y$.*

In other words, $X \otimes Y$ is unique to the extent of being able to linearize bilinear maps with minimal muss and fuss.

PROOF. First note that obviously $X \otimes Y$ satisfies the claims of the theorem. To establish the claims of the theorem for an arbitrary Z, we start by noting that the uniqueness of the linear map L for each φ ensures that $\tau(X \times Y)$ must span W. If we apply the defining property of W and τ to the bilinear function $X \times Y \to X \otimes Y : (x, y) \mapsto x \otimes y$, we find a linear function $L : W \to X \otimes Y$ for which $L(\tau(x, y)) = x \otimes y$ for all $x \in X, y \in Y$.

On the other hand, the bilinear function τ factors through $X \otimes Y$ to give a linear function $T : X \otimes Y \to W$ for which $T(x, y) = \tau(x, y)$ for all $x \in X, y \in Y$.

So we have for any $x \in X, y \in Y$ that

$$L \circ T(\tau(x, y)) = \tau(x, y)$$

and

$$L \circ T(x \otimes y) = x \otimes y.$$

Since $X \otimes Y$ is spanned by vectors of the form $x \otimes y$ and W is spanned by vectors of the form $\tau(x, y)$, where x roams freely through X and y wanders around in Y, it must be that $J = T$ is the required isomorphism. $\square$

One soon realizes that a tensor product of two linear spaces has different realizations. Depending on the specific component spaces, their tensor product sits naturally inside a number of well-known linear spaces. Here is a summary of the most important of these:

(1) $X' \otimes Y' \subseteq (X \otimes Y)'$; $\left(\sum_{i=1}^n x_i' \otimes y_i'\right)(x \otimes y) = \sum_{i=1}^n x_i'(x) y_i'(y)$,

(2) $X \otimes Y \subseteq B(X', Y')$; $\left(\sum_{i=1}^n x_i \otimes y_i\right)(x', y') = \sum_{i=1}^n x'(x_i) y'(y_i)$,

(3) $X' \otimes Y' \subseteq B(X, Y)$; $\left(\sum_{i=1}^n x_i' \otimes y_i'\right)(x, y) = \sum_{i=1}^n x_i'(x) y_i'(y)$,

(4) $X \otimes Y \subseteq L(X'; Y)$; $\left(\sum_{i=1}^n x_i \otimes y_i\right)(x') = \sum_{i=1}^n x'(x_i) y_i$,

(5) $X' \otimes Y' \subseteq L(X; Y')$; $\left(\sum_{i=1}^n x_i' \otimes y_i'\right)(x) = \sum_{i=1}^n x_i'(x) y_i'$,

(6) $X' \otimes Y \subseteq L(X; Y)$; $\left(\sum_{i=1}^n x_i' \otimes y_i\right)(x) = \sum_{i=1}^n x_i'(x) y_i$.

Remarks:

- If both X and Y are finite dimensional, then all of the mentioned inclusions are equalities. This can be easily verified by just checking the dimensions of the spaces involved.
- It is also easy to check that the operators in $L(X; Y)$ in (6) that arise from $X' \otimes Y$ are precisely the finite rank linear functions from X to Y.

1.1. Reasonable crossnorms, including the norms $\wedge$ and $\vee$

1.1.1. Definitions. Let X and Y be Banach spaces (over the same scalar field). A norm α on $X \otimes Y$ will be called a *reasonable crossnorm* if α satisfies the following conditions:

(a) for $x \in X$ and $y \in Y$,

$$\alpha(x \otimes y) \leq \|x\| \|y\|,$$

(b) for $x^* \in X^*$ and $y^* \in Y^*$, $x^* \otimes y^* \in (X \otimes Y, \alpha)^*$, and

$$\|x^* \otimes y^*\|_{(X \otimes Y, \alpha)^*} \leq \|x^*\| \|y^*\|.$$

First, an elementary, but important, fact that follows from the definition:

PROPOSITION 1.1.1. *If α is a reasonable crossnorm on $X \otimes Y$, then α satisfies the following conditions:*

(a′) *for $x \in X$ and $y \in Y$,*

$$\alpha(x \otimes y) = \|x\|\|y\|,$$

(b′) *for $x^* \in X^*$ and $y^* \in Y^*$,*

$$\|x^* \otimes y^*\|_{(X\otimes Y,\alpha)^*} = \|x^*\|\|y^*\|.$$

PROOF. (a′) Let $x \in X$ and $y \in Y$ be given. Pick $x^* \in B_{X^*}$ and $y^* \in B_{Y^*}$ so that

$$x^*(x) = \|x\| \quad \text{and} \quad y^*(y) = \|y\|.$$

By (b), $x^* \otimes y^*$ has norm ≤ 1 as a member of $(X \otimes Y, \alpha)^*$ and so

$$\|x\|\|y\| = |x^*(x)||y^*(y)| = |(x^* \otimes y^*)(x \otimes y)| \leq \alpha(x \otimes y).$$

A quick look at (a) gives (a′).

(b′) Let $x^* \in X^*$ and $y^* \in Y^*$ be given. Choose $(x_n) \subseteq S_X$ and $(y_n) \subseteq S_Y$ so that

$$\|x^*\| = \lim_n |x^*(x_n)| \quad \text{and} \quad \|y^*\| = \lim_n |y^*(y_n)|.$$

By (a′), $\alpha(x_n \otimes y_n) = 1$ for all n; it follows from (b) that

$$|(x^* \otimes y^*)(x_n \otimes y_n)| \leq \|x^* \otimes y^*\|_{(X\otimes Y,\alpha)^*}$$

regardless of n. But now we see that

$$\begin{aligned} \|x^*\|\|y^*\| &= \lim_n |x^*(x_n)| \lim_n |y^*(y_n)| \\ &= \lim_n |(x^* \otimes y^*)(x_n \otimes y_n)| \\ &\leq \|x^* \otimes y^*\|_{(X\otimes Y,\alpha)^*}. \end{aligned}$$

This, in light of (b), gives (b′). □

Since each $u^* \in X^* \otimes Y^*$ is a linear combination of elementary tensors $x^* \otimes y^*$, (b) above implies that each $u^* \in X^* \otimes Y^*$ is a member of $(X \otimes Y, \alpha)^*$.

PROPOSITION 1.1.2. *$\|\cdot\|_{(X\otimes Y,\alpha)^*}$ restricted to $X^* \otimes Y^*$ is a reasonable crossnorm.*

PROOF. First, we note that condition (b) of the definition, as imposed to ensure that α is a reasonable cross-norm, is precisely what's needed to assure us that $\|.\|_{(X\otimes Y,\alpha)^*}$ satisfies (a). So all we need to do is establish (b) for $\|\cdot\|_{(X\otimes Y,\alpha)^*}$.

Let $x^{**} \in X^{**}$ and $y^{**} \in Y^{**}$ be given. By Goldstine's theorem we can find nets $(x_d)_{d\in D}$ and $(y_{d'})_{d'\in D'}$ in X and Y respectively, such that for $x^* \in X^*$ and $y^* \in Y^*$ we have

$$x^{**}(x^*) = \lim_d x^*(x_d) \quad \text{and} \quad y^{**}(y^*) = \lim_{d'} y^*(y_{d'}),$$

where for $d \in D$ and $d' \in D'$,

$$\|x_d\| \leq \|x^{**}\| \quad \text{and} \quad \|y_{d'}\| \leq \|y^{**}\|.$$

Take any $u^* \in X^* \otimes Y^*$, say $u^* = \sum_{i \le n} x_i^* \otimes y_i^*$. Then (b), for $\|.\|_{(X \otimes Y, \alpha)^*}$, follows from

$$\begin{aligned} |(x^{**} \otimes y^{**})(u^*)| &= |\sum_{i \le n} x^{**}(x_i^*) y^{**}(y_i^*)| \\ &= |\sum_{i \le n} \lim_d x_i^*(x_d) \lim_{d'} y_i^*(y_{d'})| \\ &= \lim_{d,d'} |\sum_{i \le n} x_i^*(x_d) y_i^*(y_{d'})| \\ &= \lim_{d,d'} |(x_d \otimes y_{d'})(u^*)| \\ &\le \overline{\lim}_{d,d'} \|x_d\| \|y_{d'}\| \|u^*\|_{(X \otimes Y, \alpha)^*} \\ &\le \|x^{**}\| \|y^{**}\| \|u^*\|_{(X \otimes Y, \alpha)^*}. \end{aligned}$$

□

If X and Y are Banach spaces and α is a reasonable crossnorm on $X \otimes Y$, then we will denote by $X \overset{\alpha}{\otimes} Y$ the completion of $X \otimes Y$ equipped with the norm α. $\mathcal{B}(X, Y; Z)$ will denote the (Banach) space of all bounded bilinear operators from $X \times Y$ into the Banach space Z. If $Z = \mathbb{K}$, then we'll use the notation $\mathcal{B}(X, Y)$. $\mathcal{L}(X; Y)$ is the space of all bounded linear operators from X to Y.

THEOREM 1.1.3 (Grothendieck (1953/1956a), Theorem 1, p. 8). *Let X and Y be Banach spaces.*

(1) *On $X \otimes Y$ there exists a least reasonable crossnorm $|\cdot|_\vee$ and a greatest reasonable crossnorm $|\cdot|_\wedge$.*
(2) *The norm $|\cdot|_\vee$ is the norm induced on $X \otimes Y$ by viewing $X \otimes Y$ as a subspace of $\mathcal{B}(X^*, Y^*)$, that is, for $u \in X \otimes Y$,*
$$|u|_\vee = \sup\{|u(x^*, y^*)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\}.$$
(3) *The norm $|\cdot|_\wedge$ is the norm induced on $X \otimes Y$ by duality with $\mathcal{B}(X, Y)$, that is, if $u \in X \otimes Y$, then*
$$|u|_\wedge = \sup\{|v(u)| : v \in \mathcal{B}(X, Y), \|v\| \le 1\}.$$
More to the point (and hinting at the use of the notation "∧"): If $u \in X \otimes Y$, then
$$|u|_\wedge = \inf \Big\{ \sum_{i \le n} \|x_i\| \|y_i\| \Big\},$$
where the infimum is to be taken over all (finite) representations of u of the form $u = \sum_{i \le n} x_i \otimes y_i$.

PROOF. First we show that the functional on $X \otimes Y$ given by

$$|u|_\vee = \sup\{|u(x^*, y^*)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\}$$

is a reasonable, indeed the least reasonable, crossnorm on $X \otimes Y$. It *is* easy to realize that $|\cdot|_\vee$ is a seminorm on $X \otimes Y$; of course, $|u|_\vee = 0$ implies that $x^* \otimes y^*(u) = 0$ for each $x^* \in X^*$ and $y^* \in Y^*$ and so $u = 0$ as a member of $X \otimes Y$; and so $|\cdot|_\vee$ is a norm. Let's look for $|\cdot|_\vee$'s reasonability.

Let $x \in X$ and $y \in Y$ be given. Then

$$\begin{aligned} |x \otimes y|_\vee &= \sup\{|(x \otimes y)(x^* \otimes y^*)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\} \\ &= \sup\{|x^*(x)| \cdot |y^*(y)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\} \\ &\le \|x\| \|y\|. \end{aligned}$$

$|\cdot|_\vee$ satisfies (a).

Let $x^* \in X^*$ and $y^* \in Y^*$ be given. Then for any $u \in X \otimes Y$ we have

$$\begin{aligned}|(x^* \otimes y^*)(u)| &= \|x^*\|\|y^*\|\Big|\Big(\frac{x^*}{\|x^*\|} \otimes \frac{y^*}{\|y^*\|}\Big)(u)\Big| \\ &= \|x^*\|\|y^*\|\Big|u\Big(\frac{x^*}{\|x^*\|}, \frac{y^*}{\|y^*\|}\Big)\Big| \\ &\leq \|x^*\|\|y^*\||u|_\vee,\end{aligned}$$

and so we see that $|\cdot|_\vee$ also satisfies (b). $|\cdot|_\vee$ is a reasonable crossnorm on $X \otimes Y$.

Let α be any reasonable crossnorm on $X \otimes Y$ and let $u \in X \otimes Y$. Then for any $x^* \in X^*$ and $y^* \in Y^*$ we have

$$\|x^* \otimes y^*\|_{(X \otimes Y, \alpha)^*} = \|x^*\|\|y^*\|;$$

it follows that

$$\begin{aligned}|u|_\vee &= \sup\{|u(x^*, y^*)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\} \\ &= \sup\{|(x^* \otimes y^*)(u)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\} \\ &\leq \alpha(u).\end{aligned}$$

$|\cdot|_\vee$ is the least of the reasonable crossnorms.

Next we will show that the functional $|\cdot|_\wedge$ given by

$$|u|_\wedge = \sup\{|v(u)| : v \in \mathcal{B}(X, Y), \|v\| \leq 1\}$$

is a reasonable crossnorm on $X \otimes Y$, the greatest of all reasonable crossnorms on $X \otimes Y$, in fact.

It is easy to see that $|\cdot|_\wedge$ is a seminorm on $X \otimes Y$. Since

$$\|x^* \otimes y^*\|_{\mathcal{B}(X,Y)} = \|x^*\|\|y^*\|$$

for any $x^* \in X^*$ and $y^* \in Y^*$, we see that for $u \in X \otimes Y$,

$$\begin{aligned}|u|_\vee &= \sup\{|(x^* \otimes y^*)(u)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\} \\ &\leq \sup\{|u(v)| : v \in \mathcal{B}(X, Y), \|v\| \leq 1\} \\ &= |u|_\wedge.\end{aligned}$$

It follows that $|\cdot|_\wedge$ is a norm on $X \otimes Y$. Moreover, if $x^* \in S_{X^*}$ and $y^* \in S_{Y^*}$, then for any $u \in X \otimes Y$,

$$|(x^* \otimes y^*)(u)| \leq |u|_\wedge;$$

from this it follows that for any $x^* \in X^*$ and $y^* \in Y^*$ we have

$$\|x^* \otimes y^*\|_{(X \otimes Y, |\cdot|_\wedge)^*} \leq \|x^*\|\|y^*\|;$$

(b) is satisfied by $|\cdot|_\wedge$. How about (a)? Well, take $x \in X$ and $y \in Y$ and note that

$$\begin{aligned}|x \otimes y|_\wedge &= \sup\{|(x \otimes y)(v)| : v \in \mathcal{B}(X, Y), \|v\| \leq 1\} \\ &= \sup\{|v(x, y)| : v \in \mathcal{B}(X, Y), \|v\| \leq 1\} \\ &\leq \sup\{\|v\|\|x\|\|y\| : v \in \mathcal{B}(X, Y), \|v\| \leq 1\} \\ &= \|x\|\|y\|.\end{aligned}$$

$|\cdot|_\wedge$ is a reasonable crossnorm on $X \otimes Y$.

Let α be any reasonable crossnorm on $X \otimes Y$ and let $u \in X \otimes Y$. Choose u^* from $(X \otimes Y, \alpha)^*$ so that

$$\|u^*\|_{(X \otimes Y, \alpha)^*} = 1 \quad \text{and} \quad u^*(u) = \alpha(u).$$

Define $v \in \mathcal{B}(X, Y)$ by $v(x, y) = u^*(x \otimes y)$.

Compute $\|v\|_{\mathcal{B}(X,Y)}$:

$$\begin{aligned}\|v\| &= \sup\{|v(x, y)| : x \in B_X, y \in B_Y\}\\ &= \sup\{|u^*(x \otimes y)| : x \in B_X, y \in B_Y\}\\ &\leq \|u^*\| = 1.\end{aligned}$$

Therefore,

$$\alpha(u) = |u^*(u)| = |v(u)| \leq |u|_\wedge.$$

$|\cdot|_\wedge$ is the greatest of all reasonable crossnorms on $X \otimes Y$.

Finally, we come to the alternative description of $|\cdot|_\wedge$ given in (3). Temporarily, let $\alpha(u)$ be the quantity mentioned in (3), that is, for $u \in X \otimes Y$ define

$$\alpha(u) = \inf\Big\{\sum_{i \leq n} \|x_i\|\|y_i\|\Big\},$$

where the infimum is taken over all (finite) representations of u in the form

$$u = \sum_{i \leq n} x_i \otimes y_i.$$

It is easy to guess that α is a seminorm on $X \otimes Y$ and even easier to see that

$$\alpha(x \otimes y) \leq \|x\|\|y\|.$$

If $x^* \in X^*$ and $y^* \in Y^*$ and $u \in X \otimes Y$, say $u = \sum_{i \leq n} x_i \otimes y_i$, then

$$\begin{aligned}|x^* \otimes y^*(u)| &= \Big|\sum_{i \leq n} x^*(x_i) y^*(y_i)\Big|\\ &\leq \sum_{i \leq n} |x^*(x_i)| \cdot |y^*(y_i)|\\ &\leq \sum_{i \leq n} \|x^*\|\|x_i\|\|y^*\|\|y_i\|\\ &= \|x^*\|\|y^*\| \sum_{i \leq n} \|x_i\|\|y_i\|;\end{aligned}$$

it follows that

$$|(x^* \otimes y^*)(u)| \leq \|x^*\|\|y^*\|\alpha(u),$$

and so α satisfies (b) in the definition of reasonable crossnorms, since plainly $u = 0$ whenever $\alpha(u) = 0$, α is a reasonable crossnorm on $X \otimes Y$ and so, like all such, must satisfy $\alpha(u) \leq |u|_\wedge$ for all $u \in X \otimes Y$. On the other hand, if $u = \sum_{i \leq n} x_i \otimes y_i$, then

$$|u|_\wedge = \Big|\sum_{i \leq n} x_i \otimes y_i\Big|_\wedge \leq \sum_{i \leq n} |x_i \otimes y_i|_\wedge = \sum_{i \leq n} \|x_i\|\|y_i\|,$$

which, thanks to the arbitrary representation of u in the form $u = \sum_{i \leq n} x_i \otimes y_i$, gives us

$$|u|_\wedge \leq \alpha(u).$$

$\alpha = |\cdot|_\wedge$ and the proof of our theorem is, at last, finished. □

The norm $|\cdot|_\vee$ or $\vee$ is called the *injective norm* and $|\cdot|_\wedge$ or $\wedge$ is usually called the *projective norm*. If X and Y are Banach spaces, then the space $X \overset{\vee}{\otimes} Y$, the completion of $X \otimes Y$ with respect to $|\cdot|_\vee$, is usually called the *injective tensor product* of X and Y. Similarly, the space $X \overset{\wedge}{\otimes} Y$, the completion of $X \otimes Y$ with respect to $|\cdot|_\wedge$, is usually called the *projective tensor product* of X and Y. The reason for this terminology will become clear soon.

Critical to the understanding of the Résumé is the realization that a reasonable crossnorm is defined for one pair of Banach spaces at a time. While it's true that some reasonable crossnorms are defined in such a way that any pair of Banach spaces fits cleanly into their definition, that is because of the special character of those norms.

Though hardly to be counted as one of Grothendieck's deepest results, the following is one of his better-known (a curious happenstance to be sure):

Proposition 1.1.4 (Grothendieck (1953/1956a), Theorem 1, p. 55). *Let X and Y be any Banach spaces and $u \in X \overset{\wedge}{\otimes} Y$. Then u is representable in the form $u = \sum_{n=1}^\infty x_n \otimes y_n$ and*

$$|u|_\wedge = \inf\left\{\sum_{n=1}^\infty \|x_n\|\|y_n\| : u = \sum_{n=1}^\infty x_n \otimes y_n\right\}$$

where any finite or infinite representation of u is allowable.

Proof. Once u finds itself in $X \overset{\wedge}{\otimes} Y$ it is because there are u_n's in $X \otimes Y$ such that

$$|u - u_n|_\wedge < \epsilon/2^{n+2} \text{ for each } n$$

where $\epsilon > 0$ is the usual preordained obstacle. We write

$$u_1 = \sum_{i \le i(1)} x_i \otimes y_i$$

with the representation chosen so that

$$\sum_{i \le i(1)} \|x_i\|\|y_i\| \le |u_1|_\wedge + \epsilon/2^4 \le |u|_\wedge + \epsilon/2^3.$$

For $u_{n+1} - u_n \in X \otimes Y$, we have

$$|u_{n+1} - u_n|_\wedge \le |u - u_{n+1}|_\wedge + |u - u_n|_\wedge < \epsilon/2^{n+4} + \epsilon/2^{n+3} < \epsilon/2^{n+2},$$

so we can write $u_{n+1} - u_n$ in the form

$$u_{n+1} - u_n = \sum_{i(n) < i \le i(n+1)} x_i \otimes y_i$$

where the x_i's and y_i's are chosen to satisfy

$$\sum_{i(n) < i \le i(n+1)} \|x_i\|\|y_i\| < \epsilon/2^{n+2}.$$

$\sum_{n=1}^\infty x_n \otimes y_n = u_1 + \sum_{n=1}^\infty (u_{n+1} - u_n)$ converges absolutely to u and

$$|u|_\wedge \le \sum_{n=1}^\infty \|x_n\|\|y_n\| \le |u|_\wedge + \epsilon. \qquad \square$$

Concerning $\vee$ we take note of the frequently employed interpretation of members of the injective tensor product as operators. To be specific:

PROPOSITION 1.1.5. *If X and Y are Banach spaces, then $X^* \overset{\vee}{\otimes} Y$ is a closed linear subspace of the space $\mathcal{L}(X;Y)$ of all bounded linear operators from X to Y.*

PROOF. If $u \in X^* \otimes Y$, then u suggests the operator $\tilde{u} : X \to Y$ is given by

$$\tilde{u}(x) = \sum_{i \le n} x_i^*(x) y_i,$$

where $u = \sum_{i \le n} x_i^* \otimes y_i$. It's easy to see that$\|\tilde{u}\| = |u|_\vee$:

$$\begin{aligned} |u|_\vee &= \sup\Big\{\Big|\sum_{i\le n} x^{**}(x_i^*) y^*(y_i)\Big| : x^{**} \in B_{X^{**}}, y^* \in B_{Y^*}\Big\} \\ &= \sup\Big\{\Big|x^{**}\Big(\sum_{i\le n} y^*(y_i) x_i^*\Big)\Big| : x^{**} \in B_{X^{**}}, y^* \in B_{Y^*}\Big\} \end{aligned}$$

which, thanks to Goldstine's theorem,

$$\begin{aligned} &= \sup\Big\{\Big|\Big(\sum_{i\le n} y^*(y_i) x_i^*\Big)(x)\Big| : y^* \in B_{Y^*}, x \in B_X\Big\} \\ &= \sup\Big\{\Big|\sum_{i\le n} y^*(y_i) x_i^*(x)\Big| : y^* \in B_{Y^*}, x \in B_X\Big\} \\ &= \sup\Big\{\Big|y^*\Big(\sum_{i\le n} x_i^*(x) y_i\Big)\Big| : y^* \in B_{Y^*}, x \in B_X\Big\} \\ &= \sup\{|y^*\Big(\tilde{u}(x)\Big)| : y^* \in B_{Y^*}, x \in B_X\} \\ &= \sup\{\|\tilde{u}(x)\| : x \in B_X\} \\ &= \|\tilde{u}\|. \end{aligned}$$

The map $u \to \tilde{u}$ now extends to an isometric isomorphism of $X^* \overset{\vee}{\otimes} Y$ into $\mathcal{L}(X;Y)$. □

Also noteworthy is the fact that $X^* \otimes Y$ corresponds to the linear space $\mathcal{F}(X;Y)$ of all finite rank bounded linear operators from X to Y and so $X^* \overset{\vee}{\otimes} Y$ corresponds to the closure of $\mathcal{F}(X;Y)$ in $\mathcal{L}(X;Y)$; it follows that *the operators $\tilde{u} : X \to Y$ that correspond to the member u of $X^* \overset{\vee}{\otimes} Y$ are all limits in $\mathcal{L}(X;Y)$ of finite rank bounded linear operators and, as such, are compact.* The intriguing possibility that $X^* \overset{\vee}{\otimes} Y = \mathcal{K}(X;Y)$, the space of compact linear operators from X to Y, will be pursued later.

1.1.2. Injectivity of $\vee$ and projectivity of $\wedge$.

PROPOSITION 1.1.6 (The injectivity of $\vee$). *Let X and Y be Banach spaces. If Z is a closed linear subspace of X, then $Z \overset{\vee}{\otimes} Y$ is a closed linear subspace of $X \overset{\vee}{\otimes} Y$.*

PROOF. Indeed, take a vector $\sum_{i\le n} z_i \otimes y_i$ in $Z \overset{\vee}{\otimes} Y$ and compute the norm

$$\begin{aligned}
|\sum_{i\le n} z_i \otimes y_i|_{Z\overset{\vee}{\otimes}Y} &= \sup\Big\{\Big|\sum_{i\le n} z^*(z_i)y^*(y_i)\Big| : z^* \in B_{Z^*}, y^* \in B_{Y^*}\Big\} \\
&= \sup\Big\{\Big|z^*\Big(\sum_{i\le n} y^*(y_i)z_i\Big)\Big| : z^* \in B_{Z^*}, y^* \in B_{Y^*}\Big\} \\
&= \sup\Big\{\Big\|\sum_{i\le n} y^*(y_i)z_i\Big\|_Z : y^* \in B_{Y^*}\Big\} \\
&= \sup\Big\{\Big\|\sum_{i\le n} y^*(y_i)z_i\Big\|_X : y^* \in B_{Y^*}\Big\} \\
&= \sup\Big\{\Big|x^*\Big(\sum_{i\le n} y^*(y_i)z_i\Big)\Big| : x^* \in B_{X^*}, y^* \in B_{Y^*}\Big\} \\
&= \sup\Big\{\Big|\sum_{i\le n} x^*(z_i)y^*(y_i)\Big| : x^* \in B_{X^*}, y^* \in B_{Y^*}\Big\} \\
&= |\sum_{i\le n} z_i \otimes y_i|_{X\overset{\vee}{\otimes}Y}.
\end{aligned}$$

It is plain, and easy to see that $Y \overset{\vee}{\otimes} Z$ is also a subspace of $Y \overset{\vee}{\otimes} X$. □

It is worth noting that this injective behavior of $\vee$ is not shared by $\wedge$ and, in fact, the generation of norms on tensor products that enjoy injectivity, be it on the left or right, will be a major theme later on.

PROPOSITION 1.1.7 (The projectivity of $\wedge$). *If Y is a closed linear subspace of the Banach space X, then for any Banach space $Z, Z \overset{\wedge}{\otimes} (X/Y)$ is a quotient of $Z \overset{\wedge}{\otimes} X$.*

PROOF. Let $q : X \to X/Y$ be the canonical quotient map, $\epsilon > 0$ and $\hat{u} \in Z \overset{\wedge}{\otimes} (X/Y)$. There are sequences (z_n) and (q_n) in Z and X/Y, respectively, such that $\|z_n\| \le 1$, $\hat{u} = \sum_n z_n \otimes q_n$ and

$$\|\hat{u}\|_{Z\overset{\wedge}{\otimes}(X/Y)} \le \sum_n \|z_n\|\|q_n\| \le \|\hat{u}\|_{Z\overset{\wedge}{\otimes}(X/Y)} + \varepsilon/2.$$

For each n there is an $x_n \in X$ so that $q(x_n) = q_n$ and

$$\|q_n\|_{X/Y} \le \|x_n\| \le \|q_n\|_{X/Y} + (\varepsilon/2^{n+1}).$$

Look at $\sum_n z_n \otimes x_n = u \in Z \overset{\wedge}{\otimes} X$. It is plain and easy to see that $(id_Z \otimes q)(u) = \hat{u}$ and

$$\begin{aligned}
\|\hat{u}\|_{Z\overset{\wedge}{\otimes}(X/Y)} &\le \|u\|_{Z\overset{\wedge}{\otimes}X} \\
&\le \sum_n \|z_n\|\|x_n\| \\
&\le \sum_n \|z_n\|\Big(\|q_n\|_{X/Y} + \varepsilon/2^{n+1}\Big) \\
&\le \sum_n \|z_n\|\|q_n\|_{X/Y} + \sum_n \varepsilon/2^{n+1} \\
&\le \|\hat{u}\|_{Z\overset{\wedge}{\otimes}(X/Y)} + \varepsilon/2 + \varepsilon/2.
\end{aligned}$$

□

1.1.3. The universal mapping property of $\overset{\wedge}{\otimes}$ and the dual of $X \overset{\wedge}{\otimes} Y$. A point of fact must be raised and treated: The space $(X \overset{\wedge}{\otimes} Y)^*$ is not only naturally isometric to a subspace of $\mathcal{B}(X,Y)$, these spaces are isometrically isomorphic. Indeed, more can be said, namely we have the following *universal mapping principle*.

THEOREM 1.1.8 (Grothendieck (1953/1956a), Theorem 2, p. 8′). *For any Banach spaces X, Y and Z, the space $\mathcal{L}(X \overset{\wedge}{\otimes} Y; Z)$ of all bounded linear operators from $X \overset{\wedge}{\otimes} Y$ to Z is isometrically isomorphic to the space $\mathcal{B}(X,Y;Z)$ of all bounded bilinear transformations taking $X \times Y$ to Z. The natural correspondence establishing this isometric isomorphism is given by*

$$v \in \mathcal{L}(X \overset{\wedge}{\otimes} Y; Z) \Leftrightarrow \varphi \in \mathcal{B}(X,Y;Z)$$

via

$$v(x \otimes y) = \varphi(x,y).$$

PROOF. From the (algebraic) definition of tensor products of vector spaces it follows that there is a natural correspondence between $L(X \otimes Y; Z)$, the space of linear maps from $X \otimes Y$ to Z, and $B(X,Y;Z)$, the space of bilinear maps from $X \times Y$ to Z. This correspondence is exactly that mentioned above. We only need to show that v is bounded as a linear operator if and only if φ is bounded as a bilinear transformation, with equality of norms.

Take $u \in X \otimes Y$, say $u = \sum_{i \le n} x_i \otimes y_i$. Then

$$\begin{aligned}\|v(u)\| &= \Big\| v\Big(\sum_{i\le n} x_i \otimes y_i\Big)\Big\| = \Big\| \sum_{i \le n} v(x_i \otimes y_i)\Big\| = \Big\|\sum_{i\le n} \varphi(x_i, y_i)\Big\| \\ &\le \sum_{i \le n} \|\varphi(x_i,y_i)\| \le \sum_{i\le n} \|\varphi\|\|x_i\|\|y_i\| = \|\varphi\| \sum_{i \le n} \|x_i\|\|y_i\|;\end{aligned}$$

from this it follows that $\|v(u)\| \le \|\varphi\| \cdot |u|_\wedge$ for any $u \in X \otimes Y$. It is a short step from this to the conclusion that $\|v(u)\| \le \|\varphi\| \cdot |u|_\wedge$ for any $u \in X \overset{\wedge}{\otimes} Y$. So $\|v\|_{\mathcal{L}} \le \|\varphi\|_{\mathcal{B}}$.

The reverse inequality is even easier. For $x \in X$ and $y \in Y$,

$$\|\varphi(x,y)\| = \|v(x \otimes y)\| \le \|v\|_{\mathcal{L}} |x \otimes y|_\wedge = \|v\|_{\mathcal{L}} \|x\|\|y\|,$$

and so $\|\varphi\|_{\mathcal{B}} \le \|v\|_{\mathcal{L}}$. □

COROLLARY 1.1.9 (Grothendieck (1953/1956a), p. 8′). *$\mathcal{B}(X,Y)$ is isometrically isomorphic to $(X \overset{\wedge}{\otimes} Y)^*$. The correspondence between $\varphi \in \mathcal{B}(X,Y)$ and $u^* \in (X \overset{\wedge}{\otimes} Y)^*$ is given by*

$$\varphi \leftrightarrow u^*$$

via

$$\varphi(x,y) = u^*(x \otimes y).$$

1.1.4. Examples: $C(K)\overset{\vee}{\otimes}X$ and $L^1(\mu)\overset{\wedge}{\otimes}X$. The following examples help illustrate the nature of the tensor norms $\wedge$ and $\vee$ as well as indicate, in part, why they fit so easily within the landscape of Banach space theory.

THEOREM 1.1.10 (Grothendieck (1953/1956a), Theorem 3, p. 21). *Let X be a Banach space.*

If K is a compact Hausdorff space, then $C(K)\overset{\vee}{\otimes}X$ is isometrically isomorphic to the space $C_X(K)$ of X-valued continuous functions on K equipped with the supremum norm.

If μ is a measure on some measurable space (Ω,Σ), then $L^1(\mu)\overset{\wedge}{\otimes}X$ is isometrically isomorphic to the space $L^1_X(\mu)$ (of equivalence classes) of X-valued Bochner μ-integrable functions defined on Ω equipped with the norm $\|f\|=\int\|f(\omega)\|d\mu(\omega)$.

PROOF. First, we establish that $C(K)\overset{\vee}{\otimes}X$ is isometrically isomorphic to $C_X(K)$. To this end we define $J:C(K)\otimes X\to C_X(K)$ by $(Ju)(k)=\sum_{i\le n}f_i(k)x_i$ for $u=\sum_{i\le n}f_i\otimes x_i\in C(K)\otimes X$. The function Ju is independent of the particular representation chosen for u. In fact, if $\sum_{j\le m}g_j\otimes x'_j$ is another representation of u, then for any $x^*\in X^*$ we have for any $k\in K$ that

$$\sum_{i\le n}f_i(k)x^*(x_i)=(\delta_k\otimes x^*)(u)=\sum_{j\le m}g_j(k)x^*(x'_j),$$

so that for any $x^*\in X^*$ we have for all $k\in K$ that

$$x^*\Big(\sum_{i\le n}f_i(k)x_i\Big)=x^*\Big(\sum_{j\le m}g_j(k)x'_j\Big);$$

therefore, $\sum_{i\le n}f_i(k)x_i$ must agree with $\sum_{j\le m}g_j(k)x'_j$ at each and every k in K. J is an isometry. Indeed, if $u=\sum_{i\le n}f_i\otimes x_i$ is a member of $C(K)\otimes X$, then

$$\begin{aligned}
\Big\|Ju\Big\|_\infty &= \sup_{k\in K}\Big\|\sum_{i\le n}f_i(k)x_i\Big\| \\
&= \sup_{k\in K,\ x^*\in B_{X^*}}\Big|x^*\Big(\sum_{i\le n}f_i(k)x_i\Big)\Big| \\
&= \sup_{k\in K,\ x^*\in B_{X^*}}\Big|\sum_{i\le n}f_i(k)x^*(x_i)\Big| \\
&= \sup_{x^*\in B_{X^*}}\Big\|\sum_{i\le n}x^*(x_i)f_i\Big\|_\infty \\
&= \sup_{x^*\in B_{X^*},\ \nu\in B_{C(K)^*}}\Big|\nu\Big(\sum_{i\le n}x^*(x_i)f_i\Big)\Big| \\
&= \sup_{x^*\in B_{X^*},\ \nu\in B_{C(K)^*}}\Big|\sum_{i\le n}x^*(x_i)\nu(f_i)\Big| \\
&= \sup_{x^*\in B_{X^*},\ \nu\in B_{C(K)^*}}\Big|(\nu\otimes x^*)\Big(\sum_{i\le n}f_i\otimes x_i\Big)\Big| \\
&= \Big|\sum_{i\le n}f_i\otimes x_i\Big|_\vee .
\end{aligned}$$

J's range is dense, too. In fact, let $f : K \to X$ be continuous and let $\varepsilon > 0$ be given. $f(K)$ is compact so there are points $k_1, \ldots, k_n \in K$ such that for any $k \in K$ there's a $j : 1 \leq j \leq n$ for which $\|f(k) - f(k_j)\| \leq \varepsilon/2$, say. Let $U_j = \{k : \|f(k) - f(k_j)\| < \varepsilon\}$. Then $\{U_1, \ldots, U_n\}$ is a finite open cover of K and, therefore, there is a continuous partition of unity $\{f_1, \ldots, f_n\}$ subordinate to $\{U_1, \ldots, U_n\}$, that is, there are continuous real-valued functions $f_1, \ldots, f_n$ on K each having values in $[0,1]$ with $\sum_{i \leq n} f_i(k) \equiv 1$ and $f_i(k) = 0$ when k is outside U_i. Define $g : K \to X$ by $g(k) = \sum_{j \leq n} f_j(k) f(k_j)$. Plainly $g = J(\sum_{j \leq n} f_j \otimes f(k_j))$ and if $k \in K$, then

$$\begin{aligned}
\Big\|g(k) - f(k)\Big\| &= \Big\|\sum_{j \leq n} f_j(k) f(k_j) - f(k)\Big\| \\
&= \Big\|\sum_{j \leq n} f_j(k)[f(k_j) - f(k)]\Big\| \\
&= \Big\|\sum_{j : k \in U_j} f_j(k)[f(k_j) - f(k)]\Big\| \\
&< \varepsilon;
\end{aligned}$$

it follows that $\|g - f\|_\infty \leq \varepsilon$ and with this the density of J's range is plain.

We turn now to the claim about $L^1(\mu) \overset{\wedge}{\otimes} X$. Look at the bilinear operator

$$J : L^1(\mu) \times X \to L^1_X(\mu)$$

defined by $J(f, x) = f(\cdot)x$; plainly, $\|J\| \leq 1$. Thanks to the Universal Mapping Property of the projective norm, J induces a bounded linear operator, still denoted by J, from $L^1(\mu) \overset{\wedge}{\otimes} X$ to $L^1_X(\mu)$, with $\|J\| \leq 1$. J is, in fact, an isometry. Of course, we've seen that $\|Ju\|_{L^1_X(\mu)} \leq \|u\|_{L^1(\mu) \overset{\wedge}{\otimes} X}$ holds for all $u \in L^1(\mu) \overset{\wedge}{\otimes} X$.

To establish J's claim to isometry it will suffice to show that J is an isometry on a dense subset of $L^1(\mu) \overset{\wedge}{\otimes} X$ and that J's range is (at least) dense. Which subset of $L^1(\mu) \overset{\wedge}{\otimes} X$ will do the trick? $L^1(\mu) \otimes X$ comes to mind, but it is too big; in fact, the easiest subset to work with is the set of vectors of the form $\sum_{i \leq n} s_i \otimes x_i$, where s_i is a simple function. An easy $\frac{\varepsilon}{n} + \ldots (n - \text{terms}) \ldots + \frac{\varepsilon}{n} = \varepsilon$ argument will quickly convince anyone that this subset is indeed dense in $L^1(\mu) \otimes X$ and what's even lovelier is that the values J attains on this subset are also dense, this time in $L^1_X(\mu)$, consisting as they do of all the Σ-simple functions from Ω to X. So let's take a vector of the form $\sum_{i \leq n} s_i \otimes x_i$ in $L^1(\mu) \otimes X$ and note that it can be represented in the form $\sum_{j \leq m} \chi_{A_j} \otimes x'_j$ where $A_1, \ldots, A_m$ are pairwise disjoint members of Σ; this is the form we want because

$$\big\|J\big(\sum_{j \leq m} \chi_{A_j} \otimes x'_j\big)\big\|_{L^1_X(\mu)} = \int \big\|\sum_{j \leq m} \chi_{A_j}(\omega) x'_j\big\| d\mu(\omega)$$

which, thanks to the disjointness of the A_j's, is

$$\begin{aligned}
&= \sum_{j \leq m} \mu(A_j) \|x'_j\| \\
&\geq \Big|\sum_{j \leq m} \chi_{A_j} \otimes x'_j\Big|_\wedge
\end{aligned}$$

thanks again to our alternative description of $|\cdot|_\wedge$. It follows for vectors of the form $v = \sum_{i\le n} s_i \otimes x_i$, that

$$\|Jv\|_{L^1_X(\mu)} \ge |v|_\wedge,$$

and so ends the proof. $\square$

It is noteworthy that if K_1, K_2 are compact Hausdorff spaces, then $C(K_1) \overset{\vee}{\otimes} C(K_2)$ can be identified with $C_{C(K_2)}(K_1)$ by Theorem 1.1.10; but $C_{C(K_2)}(K_1)$ is naturally identifiable with $C(K_1 \times K_2)$.

Before we continue we have to introduce some notation.

A sequence $(x_i)_i$ of members of a Banach space X is called *absolutely p-summable* if $\sum_i \|x_i\|^p < \infty$. We denote the space of all absolutely p-summable sequences in X by $\ell^p(X)$ or ℓ^p_X and define the $\ell^p(X)$-norm of $(x_i)_i$ by

$$\|(x_i)_i\|_{\ell^p(X)} = (\sum_i \|x_i\|^p)^{\frac{1}{p}}.$$

$\ell^p(X)$ is a Banach space.

The space of norm null sequences of X will be denoted by $c_0(X)$ and the space of bounded sequences will be denoted by $\ell^\infty(X)$. Note that both of these spaces are Banach spaces with norm defined by $\|(x_i)\| = \sup_i \|x_i\|$.

It is straightforward to show that if $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\ell^p(X)^* = \ell^{p'}(X^*) \quad \text{(isometrically)}$$

where the evaluation of a member of $(x_i^*)_i$ of $\ell^{p'}(X^*)$ at a member $(x_i)_i$ of $\ell^p(X)$ is given by

$$\sum_i x_i^*(x_i),$$

a series that is easily seen to be absolutely convergent.

We also consider the space $\ell^p_{\text{weak}}(X)$ of *weakly p-summable sequences* of vectors in X; here a sequence (x_n) of members of X is "weakly p-summable" if $(x^*(x_n))_n \in \ell^p$ for each $x^* \in X^*$ and the norm of $(x_n)_n$ is given by

$$\|(x_n)_n\|_{\ell^p_{\text{weak}}(X)} = \sup_{x^* \in B_{X^*}} \Big(\sum_n |x^*(x_n)|^p\Big)^{\frac{1}{p}}.$$

We'll denote by $\check{\ell}^p_{\text{weak}}(X)$ the subspace of $\ell^p_{\text{weak}}(X)$, consisting of all sequences (x_n) of vectors in X, such that $\lim_{n\to\infty} \|(0,\ldots,0,x_n,x_{n+1},\ldots)\|_{\ell^p_{\text{weak}}} = 0$

We will frequently need to work with *finite sums* like $(\sum_{i\le n} |x^*(x_i)|^p)^{\frac{1}{p}}$ and $(\sum_{i\le n} \|(x_n)\|^p)^{\frac{1}{p}}$. This in mind, we use the same notation as before:

$$\|(x_1,\ldots,x_n)\|_{\ell^p(X)} = (\sum_{i\le n} \|x_i\|^p)^{\frac{1}{p}}$$

and

$$\|(x_1,\ldots,x_n)\|_{\ell^p_{\text{weak}}(X)} = \sup_{x^* \in B_{X^*}} \sum_{i\le n} |x^*(x_n)|.$$

If we write $\|(x_n)\|_{\ell^p(X)}$ or $\|(x_n)\|_{\ell^p_{\text{weak}}(X)}$, it will be clear from the context, whether (x_n) is a finite or an infinite sequence of vectors in a Banach space X.

Note that the space c_0 is isomorphic to a $C(K)$-space but *not* isometrically isomorphic to a $C(K)$. It is with sheer delight that we can still say the following:

THEOREM 1.1.11 (Grothendieck (1953/1956b), p. 88). *For any Banach space X we have*

$$c_0 \overset{\vee}{\otimes} X = c_0(X).$$

PROOF. A quick look will convince you that the natural inclusion of $(c_0 \otimes X, |\cdot|_\vee)$ into $\ell^\infty(X)$ finds itself inside $c_0(X)$ with norms preserved:

$$\begin{aligned} |\sum_{j\le m}(\lambda_i^j)_i \otimes x_j|_\vee &= \sup_{\gamma\in B_{\ell^1},\ x^*\in B_{X^*}} |\sum_{j\le n}\gamma(\lambda^j)x^*(x_j)| \\ &= \sup_{\gamma\in B_{\ell^1},\ x^*\in B_{X^*}} |\big(\gamma(\sum_{j\le n}x^*(x_j)\lambda_i^j)\big)_i| \\ &= \sup_{x^*\in B_{X^*}} \|\big(\sum_{j\le n}x^*(x_j)\lambda_i^j\big)_i\|_{c_0} \\ &= \|(\sum_{j\le n}\lambda_i^j x_i)_i\|_{c_0(X)}. \end{aligned}$$

Since this inclusion has dense range, the completion of $(c_0 \otimes X, |\cdot|_\vee)$ is $c_0(X)$. □

It is now easy to see that if one of the coordinates is ℓ^p, then the injective tensor product behaves well, too.

COROLLARY 1.1.12. *Let X be any Banach space. Then $\ell^p \overset{\vee}{\otimes} X$ can be identified with the space $\check{\ell}^p_{\text{weak}}(X)$.*

PROOF. As in the proof of Theorem 1.1.11 it is easy to see that the natural inclusion of $(\ell^p \otimes X, |\cdot|_\vee)$ into $\ell^p_{\text{weak}}(X)$ preserve norms:

$$\begin{aligned} |\sum_{j\le m}(\lambda_i^j)_i \otimes x_j|_\vee &= \sup_{\gamma\in B_{\ell^q},\ x^*\in B_{X^*}} |\sum_{j\le n}\gamma(\lambda^j)x^*(x_j)| \ (\frac{1}{p}+\frac{1}{q}=1) \\ &= \sup_{\gamma\in B_{\ell^q},\ x^*\in B_{X^*}} |\big(\gamma(\sum_{j\le n}x^*(x_j)\lambda_i^j)\big)_i| \\ &= \sup_{x^*\in B_{X^*}} \|\big(\sum_{j\le n}x^*(x_j)\lambda_i^j\big)_i\|_{\ell^p} \\ &= \|(\sum_{j\le n}\lambda_i^j x_i)_i\|_{\ell^p_{\text{weak}}(X)}. \end{aligned}$$

It is easy to see that $\check{\ell}^p_{\text{weak}}(X)$ is contained in the range of the natural inclusion of $\ell^p \overset{\vee}{\otimes} X$ into $\ell^p_{\text{weak}}(X)$ and to check that under this natural inclusion the sequences in $\check{\ell}^p_{\text{weak}}(X)$ can be approximated by members of $\ell^p \otimes X$. Hence $\check{\ell}^p_{\text{weak}}(X) \subseteq \ell^p \overset{\vee}{\otimes} X$. To see that $\check{\ell}^p_{\text{weak}}(X) = \ell^p \overset{\vee}{\otimes} X$ one only needs to verify that the elementary tensors $\lambda \otimes x$ in $\ell^p \otimes X$ are indeed contained in $\check{\ell}^p_{\text{weak}}(X)$. □

There is another interesting interpretation of $\ell^p \overset{\vee}{\otimes} X$. First take note of the following very classical result: If $1 < p < \infty$ and if $\frac{1}{p} + \frac{1}{p'} = 1$, the space $\mathcal{L}(\ell^{p'}; X)$ of bounded linear operators from ℓ^p to X is isometrically isomorphic to the space $\ell^p_{\text{weak}}(X)$. The proof is straightforward once it is realized that an operator $u \in \mathcal{L}(\ell^{p'}; X)$ corresponds to the sequence $(x_i)_i$ that is weakly p-summable via

$$u(e_i) = x_i.$$

It is plain that under this isometry, the compact operators (which can be approximated by operators of finite rank in this case) correspond to sequences in $\check{\ell}^p{}_{\mathrm{weak}}(X)$. So $\ell^p \overset{\vee}{\otimes} X$ is isometrically isomorphic to the space $\mathcal{K}(\ell^{p'}, X)$ of compact operators from $\ell^{p'}$ to X.

Also note that for $p = 1$ we have that the space $\mathcal{L}(c_0; X)$ of bounded linear operators from c_0 to X is isometrically isomorphic to the space $\ell^1_{\mathrm{weak}}(X)$, of weakly summable sequences of vectors in X and that $\ell^1 \overset{\vee}{\otimes} X$ is isometrically isomorphic to the space $\mathcal{K}(c_0, X)$ of compact operators from c_0 to X.

So for $p = 1$ we have yet another interpretation of the injective tensor product. The well-known Orlicz-Pettis theorem says that in Banach spaces the classes of weakly subseries convergent and norm unconditionally convergent series coincide. So we can rephrase the previous corollary:

Corollary 1.1.13 (Grothendieck (1953/1956b), Proposition 3, p. 88). *Let X be any Banach space. Then $\ell^1 \overset{\vee}{\otimes} X$ can be identified with the space* $\mathrm{uc}(X)$ *of unconditionally summable sequences in X, where the norm of an $(x_n)_{n\le 1} \in \mathrm{uc}(X)$ is given by*

$$\|(x_n)\| = \sup_{x^* \in B_{X^*}} \sum_n |x^* x_n|.$$

It should be noted that this corollary actually says that $\mathrm{uc}(X) = \check{\ell}^1_{\mathrm{weak}}(X)$.

Note: There are many misconceptions about the structure of tensor products. With the Universal Mapping Property and other basics about the projective tensor product, we're able to issue fair warnings about the linear topological structure of the projective product at least, as well as, illustrate how to compute some norms. We start with the following simply established, yet instructive, fact.

Proposition 1.1.14. *The closed linear span of $(e_n \otimes e_n)_{n\ge 1}$ in $\ell^p \overset{\wedge}{\otimes} \ell^p$ is isometric to ℓ^1, whenever $1 \le p \le 2$.*

Proof. Suppose $u = \sum_{i\le n} a_i e_i \otimes e_i \in \ell^p \otimes \ell^p$; of course, $\|u\|_\wedge \le \sum_{i\le n} |a_i|$. On the other hand, if we define $\varphi_u \in B(\ell^p, \ell^p)$ by $\varphi_u(x, y) = \sum_{i\le n} \mathrm{sign}(a_i) x_i y_i$, then $\|\varphi_u\| \le 1$, and since φ_u achieves the value 1 on $B_{\ell^p} \times B_{\ell^p}$, $\|\varphi_u\| = 1$. It follows that

$$\sum_{i\le n} |a_i| = |\sum_{i\le n} a_i \varphi_u(e_i, e_i)| = |\varphi_u(u)| \le \|u\|_\wedge,$$

and that's all that's needed. $\square$

A quick corollary follows:

Corollary 1.1.15. *If $1 < p \le 2$, then $\ell^p \overset{\wedge}{\otimes} \ell^p$ is not reflexive.*

Proposition 1.1.16. *The span of $(e_n \otimes e_n)_{n\ge 1}$ in $\ell^p \overset{\wedge}{\otimes} \ell^p$ is $\ell^{\frac{p}{2}}$, if $2 < p < \infty$.*

Proof. We use the Rademacher functions and their orthonormal character: consider any $u = \sum_{i\le n} a_i e_i \otimes e_i \in \ell^p \otimes \ell^p$; rewrite u:

$$u = \int_0^1 (\sum_{i\le n} \mathrm{sign}(a_i)|a_i|^{\frac{1}{2}} r_i(t) e_i) \otimes (\sum_{i\le n} |a_i|^{\frac{1}{2}} r_i(t) e_i) dt.$$

Then

$$\begin{aligned}
\|u\|_\wedge &= \| \int_0^1 (\sum_{i\le n} \operatorname{sign}(a_i)|a_i|^{\frac{1}{2}} r_i(t)e_i) \otimes (\sum_{i\le n} |a_i|^{\frac{1}{2}} r_i(t)e_i) dt \|_\wedge \\
&\le \int_0^1 \| \sum_{i\le n} \operatorname{sign}(a_i)|a_i|^{\frac{1}{2}} r_i(t)e_i\|_{\ell^p} \| \sum_{i\le n} |a_i|^{\frac{1}{2}} r_i(t)e_i\|_{\ell^p} dt \\
&\le \sup_{0\le t\le 1} \| \sum_{i\le n} \operatorname{sign}(a_i)|a_i|^{\frac{1}{2}} r_i(t)e_i\|_{\ell^p} \| \sum_{i\le n} |a_i|^{\frac{1}{2}} r_i(t)e_i\|_{\ell^p} \\
&\le (\sum_{i\le n} |a_i|^{\frac{p}{2}})^{\frac{1}{p}} (\sum_{i\le n} |a_i|^{\frac{p}{2}})^{\frac{1}{p}} \\
&= \|(a_i)_{i\le n}\|_{\ell^{\frac{p}{2}}}.
\end{aligned}$$

To see the reverse, we call on the Universal Mapping Property and argue by duality: look at the bilinear functional φ_u on ℓ^∞ given by

$$\varphi_u(x,y) = \sum_{i\le n} \operatorname{sign}(a_i)|a_i|^{\frac{p}{2}-1} x_i y_i.$$

Notice that $\varphi_u(u) = \sum_{i\le n} |a_i|^{\frac{p}{2}}$. Further, if $r,s,t \ge 1$ and $\frac{1}{r}+\frac{1}{s}+\frac{1}{t}=1$, then

$$|\varphi_u(x,y)| \le \|(\operatorname{sign}(a_i)|a_i|^{\frac{p}{2}-1})\|_r \|x\|_s \|y\|_t.$$

Letting r,s,t take on special values will finish the task at hand. Which values? Well, let s and t be p, respectively, and let r be what is left over, namely, $r = \frac{p}{p-2}$. The result is that

$$\sum_{i\le n} |a_i|^{\frac{p}{2}} = |\varphi_u(u)| \le \|u\|_\wedge \|\varphi_u\|.$$

But

$$\begin{aligned}
\|\varphi_u\| &\le \|(\operatorname{sign}(a_i)|a_i|^{\frac{p}{2}-1})\|_{\frac{p}{p-2}} \\
&= (\sum_{i\le n} |a_i|^{\frac{p}{2}})^{\frac{p-2}{p}}.
\end{aligned}$$

Putting this all on one line shows

$$\sum_{i\le n} |a_i|^{\frac{p}{2}} \le \|u\|_\wedge (\sum_{i\le n} |a_i|^{\frac{p}{2}})^{1-\frac{2}{p}}.$$

A bit of long division tells the tale:

$$\|(a_i)\|_{p/2} \le \|u\|_\wedge.$$

□

Finally,

Proposition 1.1.17. *The closed linear span of* $(e_n \otimes e_n)$ *in* $\ell^\infty \overset{\wedge}{\otimes} \ell^\infty$ *is* c_0.

Proof. Let $(a_i) \in c_0$. Then, looking at

$$\begin{aligned}
u = \sum_{i\le m} a_i(e_i \otimes e_i) &= \sum_{i\le n} (a_i e_i) \otimes e_i \\
&= \int_0^1 (\sum_{i\le n} a_i r_i(t) e_i) \otimes (\sum_{i\le n} r_i(t) e_i) dt,
\end{aligned}$$

we quickly see that, as above,

$$\begin{aligned}\|u\|_\wedge &\le \sup_{0\le t\le 1}(\|\sum_{i\le n} a_i r_i(t) e_i\|_\infty \cdot \|\sum_{i\le n} r_i(t)e_i\|_\infty)\\ &= \|(a_i)\|_\infty.\end{aligned}$$

If we choose an index i_0 so that $\|(a_i)\|_\infty = |a_{i_0}|$ and look at the bilinear functional φ_u on $\ell^\infty \times \ell^\infty$ given by

$$\varphi_u(x,y) = \operatorname{sign}(a_{i_o}) x_{i_0} y_{i_0},$$

then we can quickly verify that $\|\varphi_u\| = 1$ and that

$$\varphi_u(u) = |a_{i_0}| = \|(a_i)\|_\infty.$$

By duality,

$$\|u\|_\wedge \ge \|(a_i)\|_\infty.$$

Case closed. □

To summarize, *the closed linear span D_p of the sequence $(e_n \otimes e_n)_{n\ge 1}$ in $\ell^p \overset{\wedge}{\otimes} \ell^p$ is isometrically isomorphic to*

$$\begin{cases} \ell^1, \text{ for } 1 \le p \le 2,\\ \ell^{\frac{p}{2}}, \text{ for } 2 \le p < \infty,\\ c_0, \text{ for } p = \infty.\end{cases}$$

Next, we give an application of the second part of Theorem 1.1.10; we'll compute the closed linear space of the sequences $(r_n \otimes r_n)_{n\ge 1}$ in $L^1(0,1) \overset{\wedge}{\otimes} L^p(0,1), 1 \le p \le \infty$.

First:

PROPOSITION 1.1.18. *If $1 \le p < \infty$, then the closed linear span of $(r_n \otimes r_n)_{n\ge 1}$ in $L^1(0,1) \overset{\wedge}{\otimes} L^p(0,1)$ is isomorphic to ℓ^2.*

PROOF. We start with the identification of $L^1(0,1) \overset{\wedge}{\otimes} L^p(0,1)$ with $L^1_{L^p(0,1)}(0,1)$ and look at $\sum_{i\le n} a_i r_i \otimes r_i$, therein; using Khinchin's inequalities, we see that

$$\begin{aligned}\|\sum_{i\le n} a_i r_i \otimes r_i\|_{L^1(0,1)\overset{\wedge}{\otimes} L^p(0,1)} &= \int_0^1 \|\sum_{i\le n} a_i r_i(t) r_i\|_{L^p(0,1)} dt\\ &= \int_0^1 \Big(\int_0^1 |\sum_{i\le n} a_i r_i(t) r_i(s)|^p ds\Big)^{\frac{1}{p}} dt\\ &\asymp \int_0^1\int_0^1 |\sum_{i\le n} a_i r_i(t) r_i(s)| ds dt\\ &= \int_0^1\int_0^1 |\sum_{i\le n} a_i r_i(s) r_i(t)| dt ds\\ &= \int_0^1 |\sum_{i\le n} a_i r_i(t)| dt\\ &\asymp \|(a_i)\|_{\ell^2},\end{aligned}$$

where "$\asymp$" indicates equivalence up to universal constants. □

For $p = \infty$ the computation is even easier:

PROPOSITION 1.1.19. *The closed linear span of* $(r_n \otimes r_n)_{n\geq 1}$ *in* $L^1(0,1) \overset{\wedge}{\otimes} L^\infty(0,1)$ *is isomorphic (isometric in the real case) to* ℓ^1.

PROOF. Again we use of the identification of $L^1(0,1) \overset{\wedge}{\otimes} L^\infty(0,1)$ with $L^1_{L^\infty(0,1)}(0,1)$ to see that

$$\begin{aligned} \|\sum_{i\leq n} a_i r_i \otimes r_i\|_{L^1(0,1)\overset{\wedge}{\otimes} L^\infty(0,1)} &= \int_0^1 \|\sum_{i\leq n} a_i r_i(s) r_i\|_{L^\infty(0,1)} ds \\ &\asymp \int_0^1 \|(a_i r_i(s))_{i\leq n}\|_{\ell^1} ds \\ &= \int_0^1 \sum_{i\leq n} |a_i r_i(s)| ds = \sum_{i\leq n} |a_i|. \end{aligned}$$

□

We've seen that $L^1(\mu) \overset{\wedge}{\otimes} X$ can be identified with $L^1_X(\mu)$ (Theorem 1.1.10); it follows that $\ell^1 \overset{\wedge}{\otimes} X$ is nothing else than ℓ^1_X. Moreover, $\ell^1 \overset{\vee}{\otimes} X$ can be identified with the space $uc(X)$ of unconditionally summable sequences in X. Further, the structure of $\ell^p \overset{\wedge}{\otimes} \ell^p$ is possessed of subtleties galore. How do $\ell^p \overset{\wedge}{\otimes} X$ and $\ell^p \overset{\vee}{\otimes} X$ compare if $1 < p$? Here's a first response.

THEOREM 1.1.20 (Grothendieck (1953/1956b), Proposition 4, p. 89). *Let* X *be any Banach space. If* $1 < p < \infty$*, then*

$$\ell^p \overset{\wedge}{\otimes} X \subseteq \ell^p_X \subseteq \ell^p \overset{\vee}{\otimes} X,$$

with all inclusions having norm $= 1$.

Comment: After the results of Section 1.4, we'll see that each inclusion is injective as well, thanks to ℓ^p's accessibility.

PROOF. For $\lambda = (\lambda_i)_i \in \ell^p$ and $x \in X$ denote by $\lambda \cdot x$ the sequence $(\lambda_i x)_i$ of vectors in X; plainly, $\|\lambda \cdot x\|_p$ is $\leq \|\lambda\|_p \|x\|$. Hence the natural map $(\lambda, x) \to \lambda \cdot x$ is a bilinear operator from $\ell^p \times X$ into ℓ^p_X of norm ≤ 1. The Universal Mapping Property assures us of the existence of a (natural) linear operator from $\ell^p \overset{\wedge}{\otimes} X$ into ℓ^p_X having norm ≤ 1.

It is plain that if $(x_n)_n \in \ell^p_X$, then $\sum_i |x^* x_i|^p < \infty$ for each $x^* \in X^*$ and that

$$\sup_{x^*\in B^*} (\sum_i |x^*(x_i)|^p)^{\frac{1}{p}} < \infty,$$

too; in fact, for any $x^* \in X^*$ we have

$$(\sum_i |x^*(x_i)|^p)^{\frac{1}{p}} \leq (\sum_i \|x_i\|^p)^{\frac{1}{p}} \|x^*\|.$$

To pin things down, notice that this inequality holds on $\ell^p \otimes X$, a dense linear subspace of ℓ^p_X. So for $(x_i) \in \ell^p \otimes X$, we have

$$\sup_{x^*\in B_{X^*}} (\sum_i |x^* x_i|^p)^{\frac{1}{p}} \leq (\sum_i \|x_i\|^p)^{\frac{1}{p}}.$$

However, the quantity on the left is easily seen to be the $\ell^p \overset{\vee}{\otimes} X$-norm of any $(x_i)_i \in \ell^p \otimes X$ and so the inclusion $\ell^p_X \hookrightarrow \ell^p \overset{\vee}{\otimes} X$ is of norm ≤ 1. □

1.1.5. Integral bilinear forms and the dual of $X \overset{\vee}{\otimes} Y$. Our identification of $(X \overset{\wedge}{\otimes} Y)^*$ with the space $\mathcal{B}(X, Y)$ of all bounded bilinear functionals on $X \times Y$ begs the question: What is the dual of $X \overset{\vee}{\otimes} Y$? Naturally, we look among the bounded bilinear functionals on $X \times Y$ to sort out the dual of $X \overset{\vee}{\otimes} Y$; if we follow Grothendieck's excellent directions ([Grothendieck (1953/1956a), pp. 19 and 20]), here's where we are led.

THEOREM 1.1.21. *A bounded bilinear functional φ on $X \times Y$ defines a member of $(X \overset{\vee}{\otimes} Y)^*$ if and only if φ is of integral type, that is, there is a regular Borel measure μ on the compact space $(B_{X^*}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*)$ such that for $x \in X$ and $y \in Y$ we have*

$$\varphi(x, y) = \int_{B_{X^*} \times B_{Y^*}} x^*(x) y^*(y) d\mu(x^*, y^*).$$

PROOF. By the very definition of $|u|_\vee$,

$$|u|_\vee = \sup\{|u(x^*, y^*)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\};$$

so it is plain that $(X \otimes Y, |\cdot|_\vee)$ is isometrically isomorphic to a linear subspace of the Banach space $C\Big((B_{X^*}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*)\Big)$ of continuous scalar-valued functions defined on the compact space $(B_{X^*}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*)$. Therefore, $X \overset{\vee}{\otimes} Y$, the completion of $(X \otimes Y, |\cdot|_\vee)$, is also isometrically isomorphic to a (closed) subspace of the space $C\Big((B_{X^*}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*)\Big)$. If φ is a member of $(X \overset{\vee}{\otimes} Y)^*$, then the Hahn-Banach theorem allows us to extend φ to a member $\tilde{\varphi}$ of $C\Big((B_{X^*}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*)\Big)^*$ without changing norms. Of course, the Riesz theorem now comes into play; the result is a regular Borel measure μ on the space $(B_{X^*}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*)$ such that

$$\tilde{\varphi}(f) = \int_{B_{X^*} \times B_{Y^*}} f(x^*, y^*) d\mu(x^*, y^*)$$

for all $f \in C\Big((B_{X^*}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*)\Big)$. Restricting our attention to members $x \otimes y$ of $X \otimes Y$ and taking into account $\tilde{\varphi}$'s action on $x \otimes y$ we see that

$$\varphi(x, y) = \int_{B_{X^*} \times B_{Y^*}} x^*(x) y^*(y) d\mu(x^*, y^*)$$

holds for any $x \in X$ and $y \in Y$. □

It is worth noting that in the above argument, $\|\varphi\|$ and $\|\mu\|$ coincide.

Naturally, the bilinear functionals found in $(X \overset{\vee}{\otimes} Y)^*$ are called *integral*. The space of all such bilinear forms is denoted by $\mathcal{B}^\wedge(X, Y)$ and the norm on this space is called the *integral norm* and will be denoted by $\|.\|_\wedge$.

A slightly different view of Grothendieck's description of $(X \overset{\vee}{\otimes} Y)^*$ will find use later.

THEOREM 1.1.22 (Grothendieck (1953/1956a), Theorem 1, p. 20). *Let φ be a bilinear functional on $X \times Y$. For φ to be integral with $\|\varphi\|_\wedge \leq 1$ it is necessary and sufficient that there is a compact Hausdorff space K, a regular Borel probability measure μ on K, bounded linear operators $a : X \to L^\infty(\mu), b : Y \to L^\infty(\mu)$ of norm ≤ 1 such that for $x \in X$ and $y \in Y$,*

$$\varphi(x, y) = \int_K ax(k)by(k)d\mu(k).$$

PROOF. We start by embedding $X \overset{\vee}{\otimes} Y$ into $C(K)$, where K is the compact Hausdorff space $(B_{X^*}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*)$; after all, the injective norm was defined for just such a role:

$$\begin{aligned} |u|_\vee &= \sup\Big\{|(x^* \otimes y^*)(u)| : x^* \in B_{X^*}, y^* \in B_{Y^*}\Big\} \\ &= \|u(\cdot)\|_{C\big((B_{X^*}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*)\big)}. \end{aligned}$$

If $\varphi \in (X \overset{\vee}{\otimes} Y)^*$, then φ extends to a $\nu \in C(K)^*$ where $\|\nu\| = \|\varphi\|_\wedge$. Of course, ν is a regular Borel measure on K; let $\mu \in C(K)^*$ be given by

$$\mu(\cdot) = \frac{|\nu|(\cdot)}{\|\nu\|}$$

and let $f \in L^\infty(\mu)$ denote $\frac{d\nu}{d\mu}$. Notice that $\|f\|_{L^\infty(\mu)} = \|\nu\| = \|\varphi\|_\wedge$. If we assume $\|\varphi\|_\wedge \leq 1$ and define $a : X \to L^\infty(\mu)$ by $ax(x^*, y^*) = x^*(x)$ and $b : Y \to L^\infty(\mu)$ by $by(x^*, y^*) = f(x^*, y^*)y^*(y)$, then plainly $\|a\|, \|b\| \leq 1$ and

$$\begin{aligned} \int_K ax(k)\text{by}(k)d\mu(k) &= \int_K x^*(x)f(x^*, y^*)y^*(y)d\mu(x^*, y^*) \\ &= \int_K x^*(x)y^*(y)f(x^*, y^*)d\mu(x^*, y^*) \\ &= \int_K x^*(x)y^*(y)d\nu(x^*, y^*) \\ &= \varphi(x, y). \end{aligned}$$

Conversely, suppose φ is represented in the form described in the theorem. Let ψ be the bilinear continuous form on $L^\infty(\mu) \times L^\infty(\mu)$ given by $\psi(f, g) = \int fgd\mu$. Since $\varphi = \psi \circ (a \otimes b)$, to show $\|\varphi\|_\wedge \leq 1$ it will be enough to show that $\|\psi\|_\wedge \leq 1$. Let S be the Stone space of the Boolean algebra of equivalence classes of μ-measurable sets mod μ-null sets. $C(S)$ is isometrically isomorphic to $L^\infty(\mu)$, thanks to the Stone-Weierstrass theorem; indeed, if given a μ-measurable set E we denote by $\tilde{E}$ the clopen subset of the totally disconnected compact Hausdorff space S corresponding to E's equivalence class under the Stone isomorphism, then the map $\chi_E \to \chi_{\tilde{E}}$ is quickly seen to extend to a linear isometry of $L^\infty(\mu)$ with a dense subalgebra (and hence all) of $C(S)$. Along with this isometry, which we'll call σ, comes an induced measure $\tilde{\mu}$ whose defining property is $\tilde{\mu}(\tilde{E}) = \mu(E)$. If we consider $\rho \in \mathcal{B}\Big(C(S), C(S)\Big)$ given by $\rho(f, g) = \int fg \ d\tilde{\mu}$, then it is easily seen and plain that $\psi = \rho \circ (\sigma \otimes \sigma)$ and so, to show that $\|\psi\|_\wedge \leq 1$, it is enough to see our way to $\|\rho\|_\wedge \leq 1$. Now, let $s \in S$ and denote by δ_s the point mass at $s : \delta_s(f) = f(s)$ for $f \in C(S)$. Consider the function $G : S \to \mathcal{B}^\wedge\Big(C(S), C(S)\Big)$ given by $G(s) = \delta_s \otimes \delta_s$.

Notice that $\|\delta_s \otimes \delta_s\|_\wedge = 1$ and that G is continuous from S into $\Big(C(S) \overset{\vee}{\otimes} C(S)\Big)^*$ where the latter is equipped with the weak* topology. Therefore, G is Gelfand integrable! Moreover, G's Gelfand integral is nothing else than ρ! What is more,

$$\|\rho\|_\wedge = \Big\|\text{Gelfand} \int G(s) d\tilde{\mu}(s)\Big\|_\wedge \le \|\tilde{\mu}\| = \|\mu\| \le 1.$$

What more can we say? □

In our proof we made use of a powerful abstract tool for averaging: the *Gelfand integral*. Let (Ω, Σ, μ) be a finite measure space and X a Banach space. A function $f : \Omega \to X^*$ is *Gelfand integrable* if $f(\cdot)(x) \in L^1(\mu)$ for each $x \in X$; in this case, for each $E \in \Sigma$ there is $x^*_E \in X^*$ such that for any $x \in X$,

$$x^*_E(x) = \int_E f(\omega)(x) \mathrm{d}\mu(\omega).$$

x^*_E is called the *Gelfand integral of f over E*. If Ω is a compact Haussdorff space and Σ is the Borel σ-field, then any $f : \Omega \to X^*$ that is weak*-continuous is Gelfand integrable with respect to any $\mu \in C(\Omega)^*$.

Note: The above representation of the dual of $X \overset{\vee}{\otimes} Y$ pays handsome dividends that we'll appreciate in later chapters. Already we can understand weak convergence for *sequences* in $X \overset{\vee}{\otimes} Y$; in this we follow Dan Lewis's lead.

THEOREM 1.1.23 (Lewis (1973)). *Let (u_n) be a sequence in $X \overset{\vee}{\otimes} Y$. Then (u_n) is weakly Cauchy if and only if for each $x^* \in X^*$ and each $y^* \in Y^*$ the sequence $(x^* \otimes y^*)(u_n)_n$ is convergent. (u_n) is weakly convergent to $u_0 \in X \overset{\vee}{\otimes} Y$ precisely when given $x^* \in X^*$ and $y^* \in Y^*$, then $(x^* \otimes y^*)(u_0) = \lim_n (x^* \otimes y^*)(u_n)$.*

PROOF. We'll show that (u_n) is weakly null if and only if for each $x^* \in X^*$ and each $y^* \in Y^*$, we have $\lim_n (x^* \otimes y^*)(u_n) = 0$. On the one hand, we see that $x^* \otimes y^*$ is a member of $(X \overset{\vee}{\otimes} Y)^*$ and so whenever (u_n) is weakly null, $\lim_n (x^* \otimes y^*)(u_n) = 0$; on the other hand, it is precisely the point of this theorem to show that one can test the weak nullity of (u_n) simply by checking the u_n's against φ's from $(X \overset{\vee}{\otimes} Y)^*$ of this very elementary and simple form. So let's get on with the meatier aspect of this theorem of Dan Lewis.

Suppose $\lim_n (x^* \otimes y^*)(u_n) = 0$ for each $x^* \in X^*$ and each $y^* \in Y^*$. Let $\varphi \in (X \overset{\vee}{\otimes} Y)^*$. Then φ is of integral type so there is a regular Borel measure μ on the space $((B_{X^*}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*))$ such that for $u \in X \overset{\vee}{\otimes} Y$ we have

$$\varphi(u) = \int_{B_{X^*} \times B_{Y^*}} (x^* \otimes y^*)(u) d\mu(x^*, y^*).$$

Viewing the u_n's as members of $C\Big((B_{X^*}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*)\Big)$, they are uniformly bounded — this follows from the Banach-Steinhaus theorem. The u_n's also converge pointwise to zero on $B_{X^*} \times B_{Y^*}$ — this is our hypothesis. Therefore, by Lebesgue's bounded convergence theorem we have

$$\begin{aligned} \lim_n \varphi(u_n) &= \lim_n \int_{B_{X^*} \times B_{Y^*}} u_n(x^*, y^*) d\mu(x^*, y^*) \\ &= 0. \end{aligned}$$

(u_n) is weakly null in $X \overset{\vee}{\otimes} Y$.

The first assertion of the theorem now follows from what we've proved if we reflect on a very old pearl of wisdom muttered by Pełczynski to the effect that the sequence (z_n) in a Banach space Z is weakly Cauchy if and only if given any increasing sequences (k_n) and (j_n) of positive integers, the sequence $(z_{k_n} - z_{j_n})$ is weakly null. The second assertion is easy; after all, a sequence (z_n) converges weakly to z_0 if and only if $(z_n - z_0)$ is weakly null. $\square$

1.2. Definition of ⊗-norms

We will denote by $\mathcal{F}$ the class of all finite dimensional Banach spaces over the scalar field $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$. If X is a Banach space, then $\mathcal{F}(X)$ will denote the set of all finite dimensional subspaces of X. Here we identify isometrically isomorphic spaces.

A *tensor norm* ($\otimes$-norm, for short) α is a method of ascribing to any pair (E, F) of objects from $\mathcal{F}$ a reasonable crossnorm α for $E \otimes F$ in such a way that should $E, F, G, H \in \mathcal{F}$ and $u : E \to F$ and $v : G \to H$ be (bounded) linear operators, then $u \otimes v : E \otimes G \to F \otimes H$ has bound

$$\|u \otimes v\|_{\mathcal{L}(E \overset{\alpha}{\otimes} G; F \overset{\alpha}{\otimes} H)} \leq \|u\|\|v\|.$$

Here we hasten to point out that when $E, F \in \mathcal{F}$ and α is a $\otimes$-norm, then $(E \otimes F, \alpha)$ is a finite dimensional normed linear space and so is complete already; consequently, $E \overset{\alpha}{\otimes} F = (E \otimes F, \alpha)$.

Our old standbys $\vee$ and $\wedge$ provide us with important examples of $\otimes$-norms.

Proposition 1.2.1. *$\vee$ and $\wedge$ are $\otimes$-norms.*

Proof. We only need to check for the *uniform crossnorm* property.

Let $u : E \to F$ and $v : G \to H$ be linear operators where $E, F, G, H \in \mathcal{F}$. Let $\sum_{i \leq n} e_i \otimes g_i$ be a typical member of $E \otimes G$ and compute:

$$\begin{aligned}
&|(u \otimes v)(\sum_{i \leq n} e_i \otimes g_i)|_{F \overset{\vee}{\otimes} H} \\
&\qquad = \sup_{f^* \in B_{F^*},\ h^* \in B_{H^*}} \{|(f^* \otimes h^*)((u \otimes v)(\sum_{i \leq n} e_i \otimes g_i))|\} \\
&\qquad = \sup_{f^* \in B_{F^*},\ h^* \in B_{H^*}} \{|(f^* \otimes h^*)(\sum_{i \leq n} u(e_i) \otimes v(g_i))|\} \\
&\qquad = \sup_{f^* \in B_{F^*},\ h^* \in B_{H^*}} \{|\sum_{i \leq n} f^*(u(e_i))h^*(v(g_i))|\} \\
&\qquad = \sup_{f^* \in B_{F^*},\ h^* \in B_{H^*}} \{|\sum_{i \leq n} (u^* f^*)(e_i)(v^* h^*)(g_i)|\}
\end{aligned}$$

which, with the usual apologies offered in case either u or v is $= 0$, is

$$\begin{aligned}
&= \sup_{f^* \in B_{F^*},\ h^* \in B_{H^*}} \{\|u^*\|\|v^*\| |\sum_{i \leq n} (\frac{u^*}{\|u^*\|} f^*)(e_i)(\frac{v^*}{\|v^*\|} h^*)(g_i)|\} \\
&= \|u^*\|\|v^*\| \sup_{f^* \in B_{F^*},\ h^* \in B_{H^*}} \{|\sum_{i \leq n} (\frac{u^* f^*}{\|u^*\|})(e_i)(\frac{v^* h^*}{\|v^*\|})(g_i)|\}
\end{aligned}$$

$$\leq \|u\|\|v\| \sup_{e^* \in B_{E^*},\ g^* \in B_{G^*}} \{|\sum_{i \leq n} e^*(e_i) g^*(g_i)|\}$$
$$= \|u\|\|v\| |\sum_{i \leq n} e_i \otimes g_i|_\vee.$$

Okay?

The computation for $\wedge$ is even quicker. If $E, F, G, H \in \mathcal{F}$ and $u : E \to F$ and $v : G \to H$ are the objects in question, then for a typical $\sum_{i \leq n} e_i \otimes g_i \in E \otimes G$ we have

$$\begin{aligned} |(u \otimes v)(\sum_{i \leq n} e_i \otimes g_i)|_\wedge &= |\sum_{i \leq n} (ue_i) \otimes (vg_i)|_\wedge \\ &\leq \sum_{i \leq n} |u(e_i) \otimes v(g_i)|_\wedge \\ &= \sum_{i \leq n} \|u(e_i)\|\|v(g_i)\| \\ &\leq \|u\|\|v\| \sum_{i \leq n} \|e_i\|\|g_i\|. \end{aligned}$$

It follows that

$$|(u \otimes v)(\sum_{i \leq n} e_i \otimes g_i)|_\wedge \leq \|u\|\|v\| |\sum_{i \leq n} e_i \otimes g_i|_\wedge,$$

as wanted. □

1.2.1. Fundamental operations on $\otimes$-norms. If G and H are vector spaces, then the map $t : G \otimes H \to H \otimes G$ is the isomorphism generated by $t(g \otimes h) = h \otimes g$. For $u \in G \otimes H$ we denote by ${}^t u$ the image $t(u)$ in $H \otimes G$.

Suppose α is a $\otimes$-norm. We define the ${}^t\alpha$ of α as follows: If $E, F \in \mathcal{F}$, then for $u \in E \otimes F$,

$${}^t\alpha(u) = \alpha({}^t u).$$

PROPOSITION 1.2.2. *If α is a $\otimes$-norm, then so, too, is ${}^t\alpha$.*

PROOF. It is plain that ${}^t\alpha$ is a norm on $E \otimes F$ whenever $E, F \in \mathcal{F}$. Moreover, if $e \in E$ and $f \in F$, then

$${}^t\alpha(e \otimes f) = \alpha(f \otimes e) = \|f\|\|e\|.$$

Further, if $e^* \in E^*$ and $f^* \in F^*$, then

$$\begin{aligned} \|e^* \otimes f^*\|_{(E \otimes F, {}^t\alpha)^*} &= \sup_{{}^t\alpha(u) \leq 1} |(e^* \otimes f^*)(u)| \\ &= \sup_{\alpha(v) \leq 1,\ v \in F \otimes E} |(e^* \otimes f^*)({}^t v)| \\ &= \sup_{\alpha(v) \leq 1,\ v \in F \otimes E} |(f^* \otimes e^*)(v)| \\ &= \|f^* \otimes e^*\|_{(F \otimes E, \alpha)^*} \\ &= \|f^*\|\|e^*\|, \end{aligned}$$

so ${}^t\alpha$ is a reasonable crossnorm on any $E \otimes F$ with $E, F \in \mathcal{F}$.

Let $E, F, G, H \in \mathcal{F}$ and consider linear operators $u : E \to F, v : G \to H$. Picture as such:

$$\begin{tikzcd}
(E \otimes G, {}^t\alpha) \arrow[r, "u\otimes v"] \arrow[d, leftrightarrow, "t"'] & (F \otimes H, {}^t\alpha) \arrow[d, leftrightarrow, "t"] \\
(G \otimes E, \alpha) \arrow[r, "v\otimes u"'] & (H \otimes F, \alpha)
\end{tikzcd}$$

t, regardless of which side you put him on, is destined to be an isometry in this scene. It follows that

$$\begin{aligned}
\|u \otimes v\|_{\mathcal{L}(E \overset{t_\alpha}{\otimes} G; F \overset{t_\alpha}{\otimes} H)} &= \|v \otimes u\|_{\mathcal{L}(G \overset{\alpha}{\otimes} E; H \overset{\alpha}{\otimes} F)} \\
&\le \|v\|\|u\|.
\end{aligned}$$

${}^t\alpha$ is a $\otimes$-norm. $\square$

Notice that *if α is a $\otimes$-norm, then* ${}^t({}^t\alpha) = \alpha$, since for any $E, F \in \mathcal{F}$ and any $u \in E \otimes F$ we have ${}^t({}^t\alpha)(u) = {}^t\alpha({}^tu) = \alpha({}^{tt}u) = \alpha(u)$.

Plainly, ${}^t\wedge = \wedge$ and ${}^t\vee = \vee$.

Now let α be a $\otimes$-norm and let $E, F \in \mathcal{F}$. Of course, $E \otimes F$ is algebraically identical to $(E^* \overset{\alpha}{\otimes} F^*)^*$! This is due to the finite dimensionality of all spaces involved. Take a $u \in E \otimes F$ and define the dual norm α^* by $\alpha^*(u) = \|u\|_{(E^* \overset{\alpha}{\otimes} F^*)^*}$. It should be noted that for $E, F \in \mathcal{F}$, the equality $E \overset{\alpha^*}{\otimes} F = (E^* \overset{\alpha}{\otimes} F^*)^*$ holds isometrically by definition.

PROPOSITION 1.2.3. *If α is a $\otimes$-norm, then so, too, is α^*.*

PROOF. We saw in Proposition 1.1.2 that α^* acts reasonably well. Let's check the all-important uniform crossnorm condition. Let $E, F, G, H \in \mathcal{F}$ and suppose $u : E \to F$ and $v : G \to H$ are linear operators. In diagrammatic fashion:

$$\begin{tikzcd}
E \overset{\alpha^*}{\otimes} G \arrow[r, "u\otimes v"] \arrow[d, leftrightarrow] & F \overset{\alpha^*}{\otimes} H \\
(E^* \overset{\alpha}{\otimes} G^*)^* \arrow[r, "(u^*\otimes v^*)^*"'] & (F^* \overset{\alpha}{\otimes} H^*)^* \arrow[u]
\end{tikzcd}$$

where the arrows $\updownarrow$ on the left and right denote the defining isometries that gave us our very definition of α^*. Of course, the isometric behavior of these vertical arrows allows us to conclude that

$$\begin{aligned}
\|u \otimes v\|_{\mathcal{L}(E \overset{\alpha^*}{\otimes} G; F \overset{\alpha^*}{\otimes} H)} &= \|(u^* \otimes v^*)^*\|_{\mathcal{L}((E^* \overset{\alpha}{\otimes} G^*)^*; (F^* \overset{\alpha}{\otimes} H^*)^*)} \\
&= \|u^* \otimes v^*\|_{\mathcal{L}(F^* \overset{\alpha}{\otimes} H^*; E^* \overset{\alpha}{\otimes} G^*)} \\
&\le \|u^*\|\|v^*\| = \|u\|\|v\|.
\end{aligned}$$

$\square$

A few easily established facts about α^*:

PROPOSITION 1.2.4. *If α is a $\otimes$-norm, then $(\alpha^*)^* = \alpha$.*

Proof. Indeed, if $E, F \in \mathcal{F}$, then

$$E \overset{(\alpha^*)^*}{\otimes} F = (E^* \overset{\alpha^*}{\otimes} F^*)^* = (E^{**} \overset{\alpha}{\otimes} F^{**})^{**} = E \overset{\alpha}{\otimes} F,$$

thanks to the omnipresence of finite dimensional Banach spaces. $\square$

Proposition 1.2.5. *If α is a $\otimes$-norm, then $({}^t\alpha)^* = {}^t(\alpha^*)$.*

Proof. In fact, if $E, F \in \mathcal{F}$, then

$$E \overset{{}^t(\alpha^*)}{\otimes} F = F \overset{\alpha^*}{\otimes} E = (F^* \overset{\alpha}{\otimes} E^*)^* = (E^* \overset{{}^t\alpha}{\otimes} F^*)^* = E \overset{({}^t\alpha)^*}{\otimes} F.$$

$\square$

Finally, adding one new fact and summarizing already-gained information we have:

Theorem 1.2.6. *$\vee$ is the least and $\wedge$ the greatest of the $\otimes$-norms. $\vee$ and $\wedge$ are symmetric (equal to their own transpose) and each is the other's dual: $\wedge^* = \vee$ and $\vee^* = \wedge$.*

Proof. Only the claim on dual norms need be verified and it is an easy consequence of the following natural isometric inclusions for any Banach spaces X, Y:

$$X \overset{\vee}{\otimes} Y \hookrightarrow \mathcal{B}(X^*, Y^*) = (X^* \overset{\wedge}{\otimes} Y^*)^*;$$

recall these as a part of the lesson of Section 1.1. In case E and F are members of $\mathcal{F}$, all in sight are finite dimensional and so are equal if you look at $E \overset{\vee}{\otimes} F$, $\mathcal{B}(E^*, F^*)$, $(E^* \overset{\wedge}{\otimes} F^*)^*$. It follows that $\vee = \wedge^*$, and from this we see that $\vee^* = \wedge^{**} = \wedge$. $\square$

Last, but not least, of the fundamental operations on $\otimes$-norms to be considered at this juncture is the *contragradient*:

Proposition 1.2.7. *If α is a tensor norm, then so, too, is the contragradient $\overset{\vee}{\alpha}$ of α given by $\overset{\vee}{\alpha} = {}^t(\alpha^*) = ({}^t\alpha)^*$.*

1.2.2. Order relations among $\otimes$-norms. Let α and β be $\otimes$-norms. We say $\alpha \leq \beta$ if for any $E, F \in \mathcal{F}$ and any $u \in E \otimes F$ we have $\alpha(u) \leq \beta(u)$.

Obviously, we have:

Proposition 1.2.8. *If α and β are $\otimes$-norms, then $\alpha \leq \beta$ holds if and only if ${}^t\alpha \leq {}^t\beta$.*

Not quite so obvious but easily established is:

Proposition 1.2.9. *If α and β are $\otimes$-norms, then $\alpha \leq \beta$ holds if and only if $\beta^* \leq \alpha^*$.*

Proof. Suppose $\alpha \leq \beta$ and $E, F \in \mathcal{F}$. Take $u \in E \otimes F$ and compute:

$$\begin{aligned} \beta^*(u) &= \|u\|_{(E^* \overset{\beta}{\otimes} F^*)^*} \\ &= \sup\{|u(v)| : v \in E^* \otimes F^*, \beta(v) \leq 1\} \\ &\leq \sup\{|u(v)| : v \in E^* \otimes F^*, \alpha(v) \leq 1\} \\ &= \|u\|_{(E^* \overset{\alpha}{\otimes} F^*)^*} \\ &= \alpha^*(u). \end{aligned}$$

$\square$

Suppose $\{\alpha_i : i \in I\}$ is a family of $\otimes$-norms. For $E, F \in \mathcal{F}$ and $u \in E \otimes F$ consider

$$(\vee\alpha_i)(u) \equiv \sup_{i\in I} \alpha_i(u) \leq |u|_\wedge.$$

It is plain that $\vee\alpha_i$ is a norm on $E\otimes F$ and naturally satisfies $(\vee\alpha_i)(e\otimes f) = \|e\|\|f\|$ whenever $e \in E$ and $f \in F$. If $i \in I$ and $e^* \in E^*$ and $f^* \in F^*$, then

$$|e^* \otimes f^*(u)| \leq \|e^* \otimes f^*\|_{(E\overset{\alpha_i}{\otimes}F)^*}\alpha_i(u) \leq \|e^*\|\|f^*\|(\vee\alpha_i)(u),$$

for any $u \in E \otimes F$; it follows that

$$\|e^* \otimes f^*\|_{(E\otimes F, \vee\alpha_i)^*} \leq \|e^*\|\|f^*\|$$

and $\vee\alpha_i$ is a reasonable crossnorm on $E\otimes F$. $\vee\alpha_i$ is now a candidate for $\otimes$-normhood; after all, $E, F \in \mathcal{F}$ were arbitrary and we formed $\vee\alpha_i$ to be a reasonable crossnorm on $E\otimes F$. All that's left is to show that $\vee\alpha_i$ inherits the uniform crossnorm property from its generating family $\{\alpha_i : i \in I\}$. This, too, is easy: Let $E, F, G, H \in \mathcal{F}$ and $u : E \to F, v : G \to H$ be linear operators; then, if $w \in E \otimes G$, we have

$$\begin{aligned}(\vee\alpha_i)(u \otimes v)(w) &= \sup_{i\in I} \alpha_i(u \otimes v)(w)\\ &\leq \sup_{i\in I} \|u\|\|v\|\alpha_i(w)\\ &= \|u\|\|v\| \sup_{i\in I} \alpha_i(w)\\ &= \|u\|(\vee\alpha_i)(w).\end{aligned}$$

Since $\alpha \leq \beta$ holds for the tensor norms α, β precisely when $\beta^* \leq \alpha^*$ and since $\alpha^{**} = \alpha$ for any tensor norm α, we've really just shown that

PROPOSITION 1.2.10. *The tensor norms constitute a complete lattice.*

1.3. Extension of $\otimes$-norms to spaces of infinite dimensions

In this section we will show how tensor norms generate reasonable uniform crossnorms on tensor products of arbitrary Banach spaces. We will further investigate the effects of the fundamental operations on $\otimes$-norms and in so doing we will come naturally to the question of accessibility.

Let α be a $\otimes$-norm and let X and Y be arbitrary Banach spaces. It is plain that whenever $E \in \mathcal{F}(X)$ and $F \in \mathcal{F}(Y)$, then $E \otimes F$ is a member of $\mathcal{F}(X \otimes Y)$; furthermore, it is true that

$$X \otimes Y = \bigcup_{E\in\mathcal{F}(X)\ F\in\mathcal{F}(Y)} E \otimes F.$$

Suppose we direct pairs (E, F) from $\mathcal{F}(X) \times \mathcal{F}(Y)$ by $(E, F) \leq (\tilde{E}, \tilde{F})$ if $E \subseteq \tilde{E}$ and $F \subseteq \tilde{F}$. Should $(\tilde{E}, \tilde{F})$ be any pair following (E, F) in our direction and if we denote by $i_{(E,\tilde{E})}$ the natural inclusion $E \hookrightarrow \tilde{E}$ of E into $\tilde{E}$, then whenever $u \in X \otimes Y$ finds itself in $E \otimes F$ we would have

$$\begin{aligned}|u|_{\tilde{E}\overset{\alpha}{\otimes}\tilde{F}} &= |(i_{(E,\tilde{E})} \otimes i_{(F,\tilde{F})})(u)|_{\tilde{E}\overset{\alpha}{\otimes}\tilde{F}}\\ &\leq |u|_{E\overset{\alpha}{\otimes}F},\end{aligned}$$

where $|u|_{E\overset{\alpha}{\otimes}F}$ denotes the α-norm of u in $(E \overset{\alpha}{\otimes} F, \alpha)$. Consequently, the net $(|u|_{E\overset{\alpha}{\otimes}F})$ is a non-increasing monotone net and so for $u \in X \otimes Y$ we can define $\alpha(u)$ unambiguously by

(1) $$\alpha(u) = \inf\{|u|_{E\overset{\alpha}{\otimes}F} : E \in \mathcal{F}(X), F \in \mathcal{F}(Y), u \in E \otimes F\}$$

which agrees with

$$\lim_{(E,F)\in\mathcal{F}(X)\times\mathcal{F}(Y),\ u\in E\otimes F} |u|_{E\overset{\alpha}{\otimes}F}.$$

A small but important observation is the fact that should X and Y be themselves finite dimensional, then α as defined above is just what it ought to be: α.

By $\vee$'s injectivity it is easy to see that the above definition (1) is also consistent in case $\alpha = \vee$.

Before we show that definition (1) is also consistent in case $\alpha = \wedge$, we first prove the following:

PROPOSITION 1.3.1. *Regardless of the Banach spaces X and Y, the functional α so-determined as in* (1) *above is a reasonable crossnorm on $X \otimes Y$.*

PROOF. It is plain that $\alpha(\lambda u) = |\lambda|\alpha(u)$ for any scalar λ and any $u \in X \otimes Y$. Suppose $u, v \in X \otimes Y$. Then we can find $E \in \mathcal{F}(X)$ and $F \in \mathcal{F}(Y)$ such that $u, v \in E \otimes F$. Once this is done, whenever $(\tilde{E}, \tilde{F}) \in \mathcal{F}(X) \times \mathcal{F}(Y)$ follows (E, F) in the direction given $\mathcal{F}(X) \times \mathcal{F}(Y)$, we have $u, v \in \tilde{E} \otimes \tilde{F}$ and

$$|u+v|_{\tilde{E}\overset{\alpha}{\otimes}\tilde{F}} \leq |u|_{\tilde{E}\overset{\alpha}{\otimes}\tilde{F}} + |v|_{\tilde{E}\overset{\alpha}{\otimes}\tilde{F}}.$$

From this we see that $\alpha(u+v) \leq \alpha(u) + \alpha(v)$, hence α is a seminorm on $X \otimes Y$. If $u \in X \otimes Y$, then there are $E \in \mathcal{F}(X)$ and $F \in \mathcal{F}(Y)$ so that $u \in E \otimes F$; of course, $|u|_\vee \leq |u|_{E\overset{\alpha}{\otimes}F}$ holds for any such $(E, F) \in \mathcal{F}(X) \times \mathcal{F}(Y)$, where, as we've already noted, $|u|_\vee$ is quite well defined and unambiguous. It follows that $|u|_\vee \leq \alpha(u)$ and α is, in fact, a norm on $X \otimes Y$. Free-of-charge we also obtain that if $x^* \in X^*$ and $y^* \in Y^*$, then for any $u \in X \otimes Y$,

$$|(x^* \otimes y^*)(u)| \leq \|x^*\|\|y^*\||u|_\vee \leq \|x^*\|\|y^*\|\alpha(u).$$

α satisfies condition (b) required for its reasonability. α also satisfies (a) since given $x \in X$ and $y \in Y$ we obviously have $|x \otimes y|_{E\overset{\alpha}{\otimes}F} = \|x\|\|y\|$ once $E \in \mathcal{F}(X), F \in \mathcal{F}(Y)$ and $x \in E, y \in F$; this triviality also passes its blessings to α to give $\alpha(x \otimes y) \leq \|x\|\|y\|$, regardless of $x \in X$ and $y \in Y$. □

To see that definition (1) above is also unambiguous in case of $\wedge$, we temporarily let $|\cdot|'_\wedge$ denote the functional on $X \otimes Y$ defined via the procedure of this section; $|\cdot|_\wedge$ still denotes the reasonable crossnorm $\wedge$ defined as in section 1.1. Of course, if $u \in X \otimes Y$, then $|u|'_\wedge \leq |u|_\wedge$; after all, $|\cdot|_\wedge$ *is the greatest* of the reasonable crossnorms on $X \otimes Y$ and $|\cdot|'_\wedge$ is just one in the crowd of reasonable crossnorms. On the other hand, if $u \in X \otimes Y$ and $E \in \mathcal{F}(X), F \in \mathcal{F}(Y)$ are such that $u \in E \otimes F$, then supposing $u = \sum_{i\leq n} e_i \otimes f_i$, $e_i \in E$, $f_i \in F$, we have $|u|_\wedge \leq \sum_{i\leq n} \|e_i\|\|f_i\|$. But now using the fact that the representation $u = \sum_{i\leq n} e_i \otimes f_i$ is arbitrary among those representations of u as a member of $E \otimes F$ we see that $|u|_\wedge \leq |u|_{E\overset{\wedge}{\otimes}F}$. Of course, once this is seen to be true, we know that $|u|_\wedge \leq |u|'_\wedge$ is not too far behind and so $|\cdot|_\wedge = |\cdot|'_\wedge$ as anticipated.

Yet another property inherited by the reasonable crossnorms generated by a $\otimes$-norm α is the following uniformity:

PROPOSITION 1.3.2. *If X_1, X_2, Y_1 and Y_2 are Banach spaces, $u_1 : X_1 \to Y_1$ and $u_2 : X_2 \to Y_2$ are bounded linear operators, then $u_1 \otimes u_2$ defines a bounded linear operator from $X_1 \overset{\alpha}{\otimes} X_2$ to $Y_1 \overset{\alpha}{\otimes} Y_2$ with norm $\leq \|u_1\|\|u_2\|$.*

PROOF. Key to establishing this uniformity of α is estimating $\alpha((u_1 \otimes u_2)(w))$ for a typical $w \in X_1 \otimes X_2$. To this end, suppose the old faithful $\epsilon > 0$ has been given to us and we choose $E \in \mathcal{F}(X_1)$ and $F \in \mathcal{F}(X_2)$ such that $w \in E \otimes F$ and

$$\Big|w|_{E \overset{\alpha}{\otimes} F} \leq (1 + \epsilon)\alpha(w).$$

Then

$$\begin{aligned}
\alpha((u_1 \otimes u_2)(w)) &\leq |(u_1 \otimes u_2)(w)|_{u_1(E) \overset{\alpha}{\otimes} u_2(F)} \\
&\leq \|u_1\|\|u_2\| \cdot |w|_{E \overset{\alpha}{\otimes} F} \\
&\leq \|u_1\|\|u_2\|(1 + \epsilon)\alpha(w).
\end{aligned}$$

It follows that $u_1 \otimes u_2$ is a bounded linear operator from $(X_1 \otimes X_2, \alpha)$ to $(Y_1 \otimes Y_2, \alpha)$ with norm $\leq \|u_1\|u_2\|$; $u_1 \otimes u_2$ has but one bounded linear extension to an operator that takes $X_1 \overset{\alpha}{\otimes} X_2$ to $Y_1 \overset{\alpha}{\otimes} Y_2$, an extension of the same norm as $u_1 \otimes u_2$. $\square$

It is plain that ${}^t\alpha$ extends to $X \otimes Y$ in the most straightforward way: ${}^t\alpha(u) = \alpha({}^tu)$, for $u \in X \otimes Y$. One needs only to look at what happens in those $E \otimes F$'s where $E \in \mathcal{F}(X), F \in \mathcal{F}(Y)$ and $u \in E \otimes F$ to see this as a triviality.

Of more concern is how (or whether) to use duality in our extension procedure. The issue is a deep one. Recall that if $E, F \in \mathcal{F}$, then

$$E \overset{\alpha}{\otimes} F = E \overset{(\alpha^*)^*}{\otimes} F \hookrightarrow (E^* \overset{\alpha^*}{\otimes} F^*)^*$$

with all three spaces being, in fact, equal and isometric, thanks to the finite and equal dimensionality of everything in sight. This says that $E \overset{\alpha}{\otimes} F$ and $E^* \overset{\alpha^*}{\otimes} F^*$ are in perfect duality.

In infinite dimensions this duality is not so secure; indeed, some very subtle issues are involved. Suppose X and Y are infinite dimensional Banach spaces and let α be any $\otimes$-norm. As a subspace of $(X \otimes Y, \alpha)^*$, the space $X^* \otimes Y^*$ inherits the norm $\|\cdot\|_{(X\otimes Y,\alpha)^*}$; in section 1.1 this norm was already used in the definition of a reasonable crossnorm. However, the $\otimes$-norm α^* is also well defined on $X^* \otimes Y^*$. So for $u \in X^* \otimes Y^*$, both $\|u\|_{(X\otimes Y,\alpha)^*}$ and $\alpha^*(u)$ make perfect sense and may even coincide; but maybe not! *In the case of our two old friends $\wedge$ and $\vee$, it will become clear that we always have*

$$\|u\|_{(X\otimes Y,\wedge)^*} = |u|_{\wedge^*} = |u|_{\vee};$$

but there are very delicate situations *in which $\|\cdot\|_{(X\otimes Y,\vee)^*}$ and $|u|_{\wedge}$ do not coincide. Be forewarned that the equality $\|u\|_{(X\otimes Y,\vee)^*} = |u|_{\wedge}$ is intimately related to questions of accessibility (also known as approximability) and as such goes right to the heart and soul of structural aspects of Banach space theory.*

The best that can generally be said is this:

PROPOSITION 1.3.3. *If $u \in X \otimes Y$, then u acts in a natural continuous linear manner on $(X^* \otimes Y^*, \alpha^*)$ with $\|u\|_{(X^*\otimes Y^*,\alpha^*)^*} \leq \alpha(u)$.*

(Of course, this inequality, unsatisfactory though it may be to some, extends to any member u of $X \overset{\alpha}{\otimes} Y$ viewed as a member of $(X^* \overset{\alpha^*}{\otimes} Y^*)^*$.)

PROOF. Take a $u^* \in X^* \otimes Y^*$ for which $\alpha^*(u^*) < 1$. Remember how α^* is defined: from α^*'s values on $E \otimes F$'s where $E \in \mathcal{F}(X^*)$ and $F \in \mathcal{F}(Y^*)$. So choose an $E \in \mathcal{F}(X^*)$ and an $F \in \mathcal{F}(Y^*)$ such that $u^* \in E \otimes F$ and $|u^*|_{E \overset{\alpha^*}{\otimes} F} < 1$, too. Let's estimate $|u(u^*)|$; after all, we want to find out how big a functional u is on $(X^* \otimes Y^*, \alpha^*)$, don't we?

$$|u(u^*)| = |(u|_{E\otimes F})(u^*)| \leq \|(u\,|_{E\otimes F})\|_{(E\otimes F, \alpha^*)^*}.$$

But $E \in \mathcal{F}(X^*)$, and $F \in \mathcal{F}(Y^*)$, so if we let

$$E_\perp = \{x \in X : e(x) = 0, \quad \text{for all} \quad e \in E\}$$

and

$$F_\perp = \{y \in Y : f(y) = 0, \quad \text{for all} \quad y \in F\},$$

then $E^* = X/E_\perp$ and $F^* = Y/F_\perp$. If we let

$$q_{E_\perp} : X \twoheadrightarrow X/E_\perp = E^* \quad \text{and} \quad q_{F_\perp} : Y \twoheadrightarrow Y/F_\perp = F^*$$

be the canonical quotient maps, then $\|q_{E_\perp}\|, \|q_{F_\perp}\| \leq 1$ and, because the adjoint of $q_{E_\perp}$ is the natural inclusion of E into X^* and the adjoint of $q_{F_\perp}$ is the natural inclusion of F into Y^*,

$$u\,|_{E\otimes F} = (q_{E_\perp} \otimes q_{F_\perp})(u).$$

So

$$\begin{aligned} |u(u^*)| &\leq \|u\,|_{E\otimes F}\,\|_{(E\otimes F,\alpha^*)^*} \\ &= \|(q_{E_\perp} \otimes q_{F_\perp})(u)\|_{(E\otimes F,\alpha^*)^*} \\ &= \|(q_{E_\perp} \otimes q_{F_\perp})(u)\|_{(E \overset{\alpha^*}{\otimes} F)^*} \\ &= |(q_{E_\perp} \otimes q_{F_\perp})(u)|_{E^* \overset{\alpha}{\otimes} F^*} \\ &\leq \|q_{E_\perp}\|\|q_{F_\perp}\|\|u\|_{(X\otimes Y,\alpha)} \\ &\leq \alpha(u), \end{aligned}$$

thanks to the uniformity enjoyed by the extension of $\otimes$-norms.

The claim is established. $\square$

1.3.1. Metric accessibility and accessibility. We now introduce the notion of a bilinear form having type α, a notion which will be studied in detail later. If α is a $\otimes$-norm and X and Y are Banach spaces, then there is a canonical inclusion (not necessarily an injection) of $X \overset{\wedge}{\otimes} Y$ into $X \overset{\alpha}{\otimes} Y$ of norm at most one; after all, $\alpha \leq |\cdot|_\wedge$. The adjoint of the inclusion is a mapping taking $(X \overset{\alpha}{\otimes} Y)^*$ into $(X \overset{\wedge}{\otimes} Y)^* = \mathcal{B}(X, Y)$, a linear operator of norm at most one. It follows that members $\varphi \in (X \overset{\alpha}{\otimes} Y)^*$ may be viewed as continuous bilinear forms on $X \times Y$ via the formula $\varphi(x, y) =$ evaluation of φ, as a member of $(X \overset{\alpha}{\otimes} Y)^*$, at $x \otimes y$. Preliminaries aside, we say that $\varphi \in \mathcal{B}(X, Y)$ is of *type* α if φ is in the image of $(X \overset{\alpha^*}{\otimes} Y)^*$ under the canonical inclusion

$$(X \overset{\alpha^*}{\otimes} Y)^* \hookrightarrow \mathcal{B}(X, Y)$$

discussed above (albeit in the guise of α). The space of all (bounded) bilinear forms on $X \times Y$ of type α will be denoted by $\mathcal{B}^\alpha(X, Y)$; the α-norm of a $\varphi \in \mathcal{B}^\alpha(X, Y)$, written $\|\varphi\|_\alpha$, is the norm of φ as a member of $(X \overset{\alpha^*}{\otimes} Y)^*$.

Of course, $X \overset{\alpha}{\otimes} Y$ is naturally included in $(X^* \overset{\alpha^*}{\otimes} Y^*)^* = \mathcal{B}^\alpha(X^*, Y^*)$ — this is exactly what we've shown above; this natural inclusion

$$X \overset{\alpha}{\otimes} Y \hookrightarrow \mathcal{B}^\alpha(X^*, Y^*)$$

has norm at most one.

When is this map an isometric isomorphism? We set forth two conditions of a fairly general nature that ensure this map is an isometry.

$\mathcal{F}(X; X)$ denotes the space of finite rank bounded operators on X.

We say that the Banach space X is *metrically accessible* or has the "*metric approximation property*" if for any $E \in \mathcal{F}(X)$ and any $\epsilon > 0$ there is a finite rank operator from X to X with norm $\leq 1 + \varepsilon$, which agrees with the identity on E.

The $\otimes$-norm α is a *metrically accessible tensor norm* if the inclusion

$$X \overset{\alpha}{\otimes} Y \hookrightarrow \mathcal{B}^\alpha(X^*, Y^*)$$

is an isometry provided one of X, Y is finite dimensional.

Proposition 1.3.4. *The canonical inclusion $X \overset{\alpha}{\otimes} Y \hookrightarrow \mathcal{B}^\alpha(X^*, Y^*)$ is an isometric injection in case*

(a) *both X and Y are metrically accessible, or*
(b) *α is metrically accessible and either X or Y is metrically accessible.*

Proof. (a) Suppose $u = \sum_{i \leq n} x_i \otimes y_i \in X \otimes Y$. Let $\epsilon > 0$ be given. Since X and Y are metrically accessible we can find $v_X \in \mathcal{F}(X; X)$ and $v_Y \in \mathcal{F}(Y; Y)$ such that

$$\|v_X\|, \|v_Y\| \leq \sqrt{1 + \epsilon},$$

$$v_X\, |_{\mathrm{span}\{x_1,\dots,x_n\}} = \mathrm{id}_{\mathrm{span}\{x_1,\dots,x_n\}} \quad \text{and} \quad v_Y\, |_{\mathrm{span}\{y_1,\dots,y_n\}} = \mathrm{id}_{\mathrm{span}\{y_1,\dots,y_n\}}.$$

Let $E = v_X(X)$ and $F = v_Y(Y)$. Of course, $u \in E \overset{\alpha}{\otimes} F$, so there is a

$$\varphi \in (E \overset{\alpha}{\otimes} F)^* (= \mathcal{B}^{\alpha^*}(E, F)) = E^* \overset{\alpha^*}{\otimes} F^*$$

such that

$$|\varphi|_{E^* \overset{\alpha^*}{\otimes} F^*} = 1 \quad \text{and} \quad \varphi(u) = |u|_{E \overset{\alpha}{\otimes} F}.$$

Let $w_X : X \to E$ be v_X's astriction and $w_Y : Y \to F$ be v_Y's astriction; of course, $w_X^* : E^* \to X^*$ and $w_Y^* : F^* \to Y^*$. Ready to compute?

$$\begin{aligned}
|u|_{X \overset{\alpha}{\otimes} Y} &\leq |u|_{E \overset{\alpha}{\otimes} F} \\
&= |\varphi(u)| \\
&= |\varphi((w_X \otimes w_Y)(u))| \\
&= |(w_X^* \otimes w_Y^*)(\varphi)(u)| \\
&\leq \|(w_X^* \otimes w_Y^*)(\varphi)\|_{X^* \overset{\alpha^*}{\otimes} Y^*} \|u\|_{(X^* \overset{\alpha^*}{\otimes} Y^*)^*} \\
&\leq \|w_X^*\| \|w_Y^*\| |\varphi|_{E^* \overset{\alpha^*}{\otimes} F^*} \|u\|_\alpha \\
&= \|w_X\| \|w_Y\| \|u\|_\alpha \\
&= \|v_X\| \|v_Y\| \|u\|_\alpha \\
&\leq (1 + \epsilon) \|u\|_\alpha.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $|u|_{X\overset{\alpha}{\otimes}Y} \leq \|u\|_\alpha$ and so the canonical inclusion of $(X \otimes Y, \alpha)$ into $\mathcal{B}^\alpha(X^*, Y^*)$ is an isometry which a fortiori extends to an isometric inclusion of $X \overset{\alpha}{\otimes} Y$ into $\mathcal{B}^\alpha(X^*, Y^*)$.

(b) Suppose X is metrically accessible; the proof in case Y is guilty of metric accessibility follows a similar pattern of deceit. Take $u \in X \otimes Y$ with $\alpha(u) > 1$. Write $u = \sum_{i \leq n} x_i \otimes y_i$ and let $\epsilon > 0$ be $< \alpha(u) - 1$. Since X is metrically accessible, there is a $v_X \in \mathcal{F}(X; X)$ with $\|v_X\| \leq 1$, and such that $\|v_X x_i - x_i\|$ is small enough for $i = 1, \ldots, n$. Let E be the linear span of $v_X(X) \cup \{x_1, \ldots, x_n\}$. Then $u \in E \overset{\alpha}{\otimes} Y$ and

$$\begin{aligned}
|u - (v_X \otimes id_Y)(u)|_{E\overset{\alpha}{\otimes}Y} &= |\sum_{i \leq n} \Big[x_i - v_X(x_i)\Big] \otimes y_i|_{E\overset{\alpha}{\otimes}Y} \\
&\leq \sum_{i \leq n} \|x_i - v_X(x_i)\| \|y_i\| \\
&< \epsilon < \alpha(u) - 1;
\end{aligned}$$

that's how small we want $\|x_i - v_X(x_i)\|$ to be!

But $|u|_{E\overset{\alpha}{\otimes}Y} \geq \alpha(u) > 1$, so

$$\begin{aligned}
|(v_X \otimes id_Y)(u)|_{E\overset{\alpha}{\otimes}Y} &\geq |u|_{E\overset{\alpha}{\otimes}Y} - |u - (v_X \otimes id_Y)(u)|_{E\overset{\alpha}{\otimes}Y} \\
&> |u|_{E\overset{\alpha}{\otimes}Y} - (\alpha(u) - 1) \\
&= ||u|_{E\overset{\alpha}{\otimes}Y} - \alpha(u)| + 1 \\
&\geq 1.
\end{aligned}$$

Since α is metrically accessible and E is finite dimensional, the inclusion

$$E \overset{\alpha}{\otimes} Y \hookrightarrow \mathcal{B}^\alpha(E^*, Y^*) = (E^* \overset{\alpha^*}{\otimes} Y^*)^*$$

is an isometric embedding. By considering $(v_X \otimes id_Y)(u)$ as a bounded linear functional on $E^* \overset{\alpha^*}{\otimes} Y^*$, we can find $\varphi \in E^* \otimes Y^*$ such that

$$|\varphi|_{E^*\overset{\alpha^*}{\otimes}Y^*} = 1 \qquad \text{and} \qquad |(v_X \otimes id_Y)(u)(\varphi)| > 1.$$

To be quite careful let w_X be v_X's astriction and consider

$$\psi = (w_X^* \otimes id_Y^*)(\varphi) \in X^* \overset{\alpha^*}{\otimes} Y^*;$$

notice that

$$\begin{aligned}
|\psi|_{X^*\overset{\alpha^*}{\otimes}Y^*} &\leq \|w_X^*\| \|id_Y^*\| |\varphi|_{E^*\overset{\alpha^*}{\otimes}Y^*} \\
&\leq 1.
\end{aligned}$$

Writing φ in the form $\sum_{j \leq m} e_j^* \otimes y_j^*$ we can compute $|u(\psi)|$:

$$\begin{aligned}
|u(\psi)| &= |(\sum_{i \leq n} x_i \otimes y_i)\Big[(w_X^* \otimes id_Y^*)(\sum_{j \leq m} e_j^* \otimes y_j^*)\Big]| \\
&= |(\sum_{i \leq n} x_i \otimes y_i)(\sum_{j \leq m} w_X^*(e_j^*) \otimes y_j^*)| \\
&= |\sum_{i \leq n} \sum_{j \leq m} w_X^*(e_j^*)(x_i) y_j^*(y_i)|
\end{aligned}$$

$$\begin{aligned}&= |\sum_{i\leq n}\sum_{j\leq m} e_j^*(v_X(x_i))y_j^*(y_i)| \\ &= |(v_X \otimes id_Y)(\sum_{i\leq n} x_i \otimes y_i)(\sum_{j\leq m} e_j^* \otimes y_j^*)| \\ &= |(v_X \otimes id_Y)(u)(\varphi)| > 1.\end{aligned}$$

We have found that u, viewed as a member of $(X^* \overset{\alpha^*}{\otimes} Y^*)^* = \mathcal{B}^\alpha(X^*, Y^*)$, has a value at $\psi \in B_{X^* \overset{\alpha^*}{\otimes} Y^*}$ exceeding 1. Therefore, u, viewed as a member of $(X^* \overset{\alpha^*}{\otimes} Y^*)^* = \mathcal{B}^\alpha(X^*, Y^*)$, has norm > 1, too. This says that if $|u|_{(X\otimes Y,\alpha)} > 1$, then $\|u\|_\alpha > 1$. But $|u|_{(X\otimes Y,\alpha)} \leq 1$ implies $\|u\|_\alpha \leq 1$. It follows that the natural inclusion of $(X \otimes Y, \alpha)$ into $\mathcal{B}^\alpha(X^*, Y^*)$ is an isometry and, as before, extends to an isometric inclusion of $X \overset{\alpha}{\otimes} Y$ into $\mathcal{B}^\alpha(X^*, Y^*)$. $\square$

Examples: Our old standbys $\wedge$ *and* $\vee$ *are metrically accessible.* For $\vee$ this is just

$$X \overset{\vee}{\otimes} Y \hookrightarrow \mathcal{B}(X^*, Y^*) = (X^* \overset{\wedge}{\otimes} Y^*)^*$$

with the first inclusion an isometry by the definition of $\vee$ and the equality is a consequence of the universal mapping property enjoyed by $\wedge$.

The situation for $\wedge$ is a bit touchier. Suppose $E \in \mathcal{F}$ and Y is any Banach space. Then the canonical inclusion of $E^* \overset{\vee}{\otimes} Y^*$ into $\mathcal{B}(E, Y)$ is isometric because for $u^* \in E^* \otimes Y^*$ we have

$$\begin{aligned}|u^*|_\vee &= \sup\{|(e^{**} \otimes y^{**})(u^*)| : e^{**} \in B_{E^{**}}, y^{**} \in B_{Y^{**}}\} \\ &= \sup\{|(e \otimes y)(u^*)| : e \in B_E, y \in B_Y\},\end{aligned}$$

thanks to Goldstine's theorem. Of course, this last quantity is just $\|u^*\|_{\mathcal{B}(E,Y)}$. Actually, $E^* \overset{\vee}{\otimes} Y^* = \mathcal{B}(E, Y)$! Indeed, if we take $(e_i, e_i^*)_{i\leq n}$, a basis (with appropriate coefficient functionals) for E, and, let $\varphi \in \mathcal{B}(E, Y)$, then for any $e \in E$ and $y \in Y$ we have

$$\varphi(e, y) = \varphi(\sum_{i\leq n} e_i^*(e)e_i, y) = \sum_{i\leq n} e_i^*(e)\varphi(e_i, y) = (\sum_{i\leq n} e_i^* \otimes \varphi_{e_i})(e \otimes y),$$

where $\varphi_{e_i} \in Y^*$ is defined, as you might expect, by $\varphi_{e_i}(y) = \varphi(e_i, y)$, and so φ's action is the same as that of $\sum_{i\leq n} e_i^* \otimes \varphi_{e_i}$, a member of $E^* \otimes Y^*$. This in hand, we now see that all relations in the following are isometries:

$$E \overset{\wedge}{\otimes} Y \hookrightarrow (E \overset{\wedge}{\otimes} Y)^{**} = \mathcal{B}(E, Y)^* = (E^* \overset{\vee}{\otimes} Y^*)^*,$$

and this is just the metric accessibility of $\wedge$.

One important consequence of $\wedge$'s metric accessibility is this:

Proposition 1.3.5. *If X or Y is metrically accessible, then the canonical inclusion of $X \overset{\wedge}{\otimes} Y$ into $\mathcal{B}^\wedge(X^*, Y^*)$ is an isometry.*

Here we see an important point of demarcation between $\vee$ and $\wedge$. $\vee$ is *so* accessible that it matters not what the spaces X and Y are, $X \overset{\vee}{\otimes} Y$ is canonically isometric to a subspace of $(X^* \overset{\wedge}{\otimes} Y^*)^*$; for $\wedge$ the isometric character of the canonical inclusion of $X \overset{\wedge}{\otimes} Y$ in $(X^* \overset{\vee}{\otimes} Y^*)^*$ is *essentially* dependent on either X or Y being metrically accessible.

It might not be a bad idea to understand the notion of a Banach space's metric accessibility a bit better; here's a first pass at such an understanding.

The rest of the notes of this section are from Grothendieck (1955a).

THEOREM 1.3.6 (Grothendieck (1955a), Proposition 39, p. 179). *Let X be a Banach space. Then the following are equivalent statements concerning X:*

(MA_1) *Given a compact subset K of X and an $\varepsilon > 0$, there is a finite rank operator $u_0 : X \to X$ such that $\|u_0\| \leq 1$ and $\|x - u_0 x\| \leq \varepsilon$ for all $x \in K$.*

(MA_2) *Every bounded linear operator $u : X \to X$ is in the closure of the subset $\|u\| B_{\mathcal{F}(X;X)}$ of $\mathcal{F}(X;X)$ of bounded linear operators having finite rank, relative to the topology of uniform convergence on norm compact subsets of X.*

(MA_3) *For every Banach space Y, every bounded linear operator $v : X \to Y$ is in the closure of the subset $\|v\| B_{\mathcal{F}(X;Y)}$ of bounded linear operators from X to Y having finite rank, relative to the topology of uniform convergence on norm compact subsets of X.*

(MA_4) *For every Banach space Z, every bounded linear operator $w : Z \to X$ is in the closure of the subset $\|w\| B_{\mathcal{F}(Z;X)}$ of the set $\mathcal{F}(Z;X)$ of bounded linear operators from Z to X having finite rank, relative to the topology of uniform convergence on norm compact subsets of Z.*

(MA_5) *X is* metrically accessible.

As an aside: In the Résumé it is condition (MA_1) that is given as the definition of metric accessibility (that is, X has the metric approximation property).

PROOF. (MA_1) *implies* (MA_2). Let $u : X \to X$ be a bounded linear operator, K a compact subset of X, and $\epsilon > 0$ given. Choose $\delta > 0$ so that $\|ux\| \leq \epsilon$ whenever $\|x\| \leq \delta$. By (MA_1) there is a finite rank linear operator $u_0 : X \to X$ with $\|u_0\| \leq 1$ such that for any $x \in K$, $\|x - u_0 x\| \leq \delta$. It follows that for any $x \in K$,

$$\|ux - uu_0 x\| = \|u(x - u_0 x)\| \leq \epsilon \quad \text{and} \quad \|uu_0\| \leq \|u\|\|u_0\| \leq \|u\|.$$

uu_0 is a finite rank operator.

(MA_2) plainly *implies* (MA_1) but (MA_2) also *implies* (MA_3). Let Y be any Banach space and let $v : X \to Y$ be a bounded linear operator. Define the operator $V : \mathcal{L}(X;X) \to \mathcal{L}(X;Y)$ by $V(u) = v \circ u$. Plainly, V is a bounded linear operator of norm $\|v\|$; V is also continuous when both $\mathcal{L}(X;X)$ and $\mathcal{L}(X;Y)$ are equipped with the locally convex linear topologies of uniform convergence on compact subsets of X. What's more, V takes finite rank operators to finite rank operators. So if $K \subseteq X$ is compact and we approximate id_X uniformly on K by a finite rank operator u of norm ≤ 1, $V(u)$ will approximate $V(id_X) = v$ uniformly on $K, V(u)$ is a finite rank operator and $\|V(u)\| \leq \|V\|\|u\| \leq \|v\|$. Okay?

(MA_3) plainly *implies* (MA_2). That (MA_2) *implies* (MA_4) is a more or less obvious modification of the pattern used for (MA_2) implies (MA_3) and, again, (MA_4) plainly implies (MA_2).

(MA_1) *implies* (MA_5). If $E \in \mathcal{F}(X)$, then there is a projection $p : X \to X$ so that $p(X) = E$, that is, there is a bounded linear operator $p : X \to X$ such that $p |_E = id_E$; this follows from the Hahn-Banach theorem. B_E is compact so there is a $u \in \mathcal{F}(X;X)$ such that $\|u\| \leq 1$ and $\|x - u(x)\|$ is "small" as long as $x \in B_E$. Look at $v = id_X - p + up : v |_E = u |_E$ and $\|v - id_X\| = \|p - up\| \leq \|p\|\|(id - u) |_{pX} \|$, a quantity that is < 1 if we choose "small" to be the right thing. This says that v^{-1}

exists and has norm close to 1; since $v\mid_E = u\mid_E$ the operator $w = v^{-1}\circ u$ has finite rank and is bounded, linear, acts like the identity on E, and has $\|w\| \leq \|v^{-1}\|\|u\|$ a number close to 1.

(MA$_5$) *implies* (MA$_1$). Note that if K is a compact subset of X, then there are points $x_1, \dots, x_n$ in K so that each element of K is close to one of the x_i's. Let E be the linear span of $x_1, \dots, x_n$ and pick $w \in \mathcal{F}(X;X)$ so that $w\mid_E = id\mid_E$ and $\|w\|$ is not much bigger than 1. Let $u = \frac{w}{\|w\|}$: u is a finite rank operator which displaces members of K by just a little bit and $\|u\| \leq 1$. $\square$

Some examples of Banach spaces that are metrically accessible are clearly called for.

PROPOSITION 1.3.7. *If $1 \leq p < \infty$, then $L^p(\mu)$ is metrically accessible.*

PROOF. In fact, we know that if $f_1, \dots, f_n \in L^p(\mu)$ and $\epsilon > 0$ is given then we can find a measurable set E so that $\mu(E) < \infty$, each of the f_i's is bounded on E and in fact $\int_{E^c} |f_i|^p \leq (\frac{\epsilon}{2})^p$, say. If $\mu(E) = 0$, we set $u_\epsilon = 0$; otherwise, decompose E into disjoint μ-measurable sets $E_1, \dots, E_{k(\epsilon)}$ so selected as to ensure none of the f_i's vary more than $\epsilon/2$ on any of the E_j's, and define u_ϵ by

$$u_\epsilon f = \sum_{j \leq k(\epsilon)} \frac{\int_{E_j} f d\mu}{\mu(E_j)} \chi_{E_j}.$$

Whatever E's disposition, each $u_\epsilon : L^p(\mu) \to L^p(\mu)$ is a finite rank linear operator with $\|u_\epsilon\| \leq 1$ satisfying $\|f_i - u_\epsilon f_i\|_p \leq \epsilon$ for $i = 1, \dots, n$. $L^p(\mu)$'s metric accessibility follows easily from this by a simple total boundedness argument. $\square$

PROPOSITION 1.3.8. *If K is a compact Hausdorff space, then $C(K)$ is metrically accessible.*

PROOF. Again, let $f_1, \dots, f_n \in C(K)$ and $\epsilon > 0$ be given. Define $F : K \to \ell_n^\infty$ by $F(k) = (f_1(k), \dots, f_n(k))$; F is continuous and so $F(K)$ is compact; therefore, there exist points $k_1, \dots, k_m \in K$ such that for any $k \in K$ there is a $j : 1 \leq j \leq m$ for which $|f_i(k) - f_i(k_j)| \leq \epsilon/2$ for $i = 1, \dots, n$. Let

$$U_j = \{k : |f_i(k) - f_i(k_j)| < \epsilon, i = 1, \dots, n\};$$

$U_1, \dots, U_m$ constitute an open cover of K and so we can find a continuous partition of unity $g_1, \dots, g_m$ subordinate to $U_1, \dots, U_m$. Define $u_\epsilon : C(K) \to C(K)$ by

$$(u_\epsilon f)(k) = \sum_{j \leq m} g_j(k) f(k_j).$$

It is easy to see that each u_ϵ is a finite rank linear operator with $\|u_\epsilon\| \leq 1$ for which $\|f_i - u_\epsilon f_i\| < \epsilon$ for $i = 1, \dots, n$. $\square$

A condition essentially weaker than that of metric accessibility is accessibility: A Banach space X is *accessible* or has the "*approximation property*" if given a compact set K in X and an $\epsilon > 0$, then there is a finite rank bounded linear operator $u : X \to X$ such that for any $x \in K$, $\|x - ux\| \leq \epsilon$. To prove a characterization of this condition, we follow the style of the proof employed in the previous theorem. We first prove some very useful lemmas of Grothendieck, the first of which will also find use in the next section:

LEMMA 1.3.9. *If K is a compact subset of the Banach space X, then there is a norm null sequence (x_n) in X such that $K \subseteq \overline{co}\{x_n\}$.*

PROOF. K compact implies $2K$ is compact. So $2K$ has a finite $\frac{1}{4}$-net and hence there are $x_1, \dots, x_{n(1)} \in 2K$ such that each point of $2K$ is within $\frac{1}{4}$ of an x_i. Each of the sets

$$2K \cap \left\{x : \|x - x_1\| \le \frac{1}{4}\right\}, \dots, 2K \cap \left\{x : \|x - x_{n(1)}\| \le \frac{1}{4}\right\}$$

are compact. So

$$\left[2K \cap \left\{x : \|x - x_1\| \le \frac{1}{4}\right\}\right] - x_1, \dots, \left[2K \cap \left\{x : \|x - x_{n(1)}\| \le \frac{1}{4}\right\}\right] - x_{n(1)}$$

are compact and

$$K_2 = \bigcup_{i=1}^{n(1)} \left(\left[2K \cap \left\{x : \|x - x_i\| \le \frac{1}{4}\right\}\right] - x_i\right)$$

is compact.

Again, K_2 compact implies $2K_2$ is compact. So $2K_2$ has a finite $\frac{1}{16}$ net and hence there are $x_{n(1)+1}, \dots, x_{n(2)} \in 2K_2$ such that each point of $2K_2$ is within $\frac{1}{16}$ of an x_i. Each of the sets

$$2K_2 \cap \left\{x : \|x - x_{n(1)+1}\| \le \frac{1}{16}\right\}, \dots, 2K_2 \cap \left\{x : \|x - x_{n(2)}\| \le \frac{1}{16}\right\}$$

are compact. So

$$\left[2K_2 \cap \left\{x : \|x - x_{n(1)+1}\| \le \frac{1}{16}\right\}\right] - x_{n(1)+1}, \dots$$
$$\dots, \left[2K_2 \cap \left\{x : \|x - x_{n(2)}\| \le \frac{1}{16}\right\}\right] - x_{n(2)}$$

are compact and

$$K_3 = \bigcup_{i=n(1)+1}^{n(2)} \left(\left[2K_2 \cap \left\{x : \|x - x_i\| \le \frac{1}{16}\right\}\right] - x_i\right)$$

is compact.

Continue in this way; the result is a sequence of (K_n) of compact sets such that

$$\lim_n \sup_{x \in K_n} \|x\| = 0.$$

Further, if $x \in K$, then $2x \in 2K$ and so there is an $i(1) : 1 \le i(1) \le n(1)$ such that $2x - x_{i(1)} \in K_2$. But $2x - x_{i(1)} \in K_2$ implies $2(2x - x_{i(1)})) \in 2K_2$ and so there is an $i(2) : n(1) < i(2) \le n(2)$ and $2(2x - x_{i(1)}) - x_{i(2)} \in K_3$. Again, $4x - 2x_{i(1)} - x_{i(2)} \in K_3$ implies $2(4x - 2x_{i(1)} - x_{i(2)}) \in 2K_3$, so there is an $i(3) : n(2) < i(3) \le n(3)$ and $2(4x - 2x_{i(1)} - x_{i(2)}) - x_{i(3)} \in K_4$. Continue. On suitable rearrangements we find for any $n > 1$ that

$$x - \frac{x_{i(1)}}{2} - \frac{x_{i(2)}}{4} - \cdots - \frac{x_{i(n)}}{2^n} \in \frac{1}{2^n} K_{n+1}.$$

It follows that if we let $x_0 = 0$, then $x \in \overline{co}\{x_n : n \ge 0\}$ where $\|x_n\| \to 0$. $\square$

We rush to recall that an old result of Mazur says that both the closed convex hull and the closed absolutely convex hull of a (relatively) norm compact subset of a Banach space are compact.

LEMMA 1.3.10. *Let K_0 be a compact set in a Banach space X. Then there exists an absolutely convex compact set $K \subseteq X$, with K_0 a compact set in X_K. Here X_K is the span of K with K as the closed unit ball.*

PROOF. From Lemma 1.3.9 it follows that there is a null sequence (x_n) in X such that $K_0 \subseteq \overline{\text{abs conv hull}\{x_n\}}^{\|\cdot\|}$. Put $y_n = \frac{1}{\sqrt{\|x_n\|}} x_n$ and let

$$K = \overline{\text{abs conv hull}\{y_n\}}^{\|\cdot\|}.$$

Since (y_n) is again a null sequence, by Mazur's theorem K is compact in X by Lemma 1.3.9.

It follows that $\|x_n\|_{X_K} = \left\|\sqrt{\|x_n\|} y_n\right\|_{X_K} \leq \sqrt{\|x_n\|}$ and so (x_n) is a null sequence in X_K. Hence $\overline{\text{abs conv hull}\{x_n\}}^{\|\cdot\|_{X_K}}$ is a compact set in X_K, thanks again to Mazur. However, for relatively compact sets in X_K the $\|\cdot\|$-closure and the $\|\cdot\|_{X_K}$-closure coincide. So K_0 is compact in X_K. $\square$

Now, we're ready to prove the promised characterization:

THEOREM 1.3.11. *The following statements about a Banach space X are equivalent:*

(A) *X is accessible.*

(A_1) *$\mathcal{L}(X;X)$ is the closure of $\mathcal{F}(X;X)$ relative to the topology of uniform convergence on (relatively) norm-compact subsets in X.*

(A_2) *For each Banach space Y, $\mathcal{L}(X;Y)$ is the closure of $\mathcal{F}(X;Y)$, relative to the topology of uniform convergence on (relatively) norm-compact subsets in X.*

(A_3) *For every Banach space Z, $\mathcal{L}(Z;X)$ is the closure of $\mathcal{F}(Z;X)$, relative to the topology of uniform convergence on (relatively) norm-compact subsets in Z.*

(A_4) *For every Banach space Y, $Y \overset{\vee}{\otimes} X$ can be identified with the space of weak*-to weak-continuous, compact linear operators from Y^* to X.*

(A_5) *For every Banach space Z, $Z^* \otimes X$ is dense in the space $K(Z;X)$ of compact linear operators from Z to X.*

PROOF. (A) *implies* (A_1): Suppose X is accessible. Let $v : X \to X$ be a continuous linear operator, K a compact subset of X, and $\varepsilon > 0$ given. Choose $\delta > 0$ so that if $\|x\| \leq \delta$, then $\|v(x)\| \leq \varepsilon$. Since X is accessible, there is a finite rank bounded linear operator $u : X \to X$, say $u = \sum_{i \leq n} x_i^* \otimes x_i$, such that for any $x \in K$, $\|x - ux\| \leq \delta$. This being done, should $x \in K$, then

$$\begin{aligned}
\|v(x) - \Big(\sum_{i \leq n} x_i^* \otimes v(x_i)\Big)(x)\| &= \|v(x) - \sum_{i \leq n} x_i^*(x) v(x_i)\| \\
&= \|v(x - \sum_{i \leq n} x_i^*(x) x_i)\| \\
&= \|v(x - u(x))\| \leq \varepsilon.
\end{aligned}$$

(A_1) follows from (A).

If we suppose (A_1) and apply it to approximate the identity operator $id_X \in \mathcal{L}(X;X)$ by members of $\mathcal{F}(X;X)$, then (A) soon follows.

Obviously (A_1) is a special case of (A_2).

(A_1) *implies* (A_2). Suppose $\mathcal{F}(X;X)$ is dense in $\mathcal{L}(X;X)$ in the topology of uniform convergence on norm-compact subsets of X. Let Y be a Banach space and $u : X \to Y$ be a bounded linear operator. Define $U : \mathcal{L}(X;X) \to \mathcal{L}(X;Y)$ by $U(v) = u \circ v$. Notice that U is a continuous linear operator from $\mathcal{L}(X;X)$ to $\mathcal{L}(X;Y)$ even in case each is equipped with the topology of uniform convergence on norm-compact subsets of X; moreover, $U(id_X) = u$ and U takes $\mathcal{F}(X;X)$ into $\mathcal{F}(X;Y)$. We can approximate id_X, uniformly on compacta, by members of $\mathcal{F}(X;X)$ so U's continuity assures us that we can approximate $u = U(id_X)$, uniformly on compacta, by members of $U(\mathcal{F}(X,X)) \subseteq \mathcal{F}(X,Y)$. So, ($A_2$) follows from ($A_1$).

The equivalence of (A_3) with (A_1) is proved in a manner similar to that of (A_2) with (A_1); one close look at the operator $W : \mathcal{L}(X;X) \to \mathcal{L}(Z;X)$, induced by a $w \in \mathcal{L}(Z;X)$ via the formula $W(v) = v \circ w$, should tell enough to see why.

To see how (A_4) follows from (A), suppose X is accessible and let $u : Y^* \to X$ be a compact linear operator which is weak*-weak continuous. $u(B_{Y^*})$ is a compact subset of X and X is accessible, so there is a $w \in \mathcal{F}(X;X)$ which is as close as you please to id_X on $u(B_{Y^*})$. Naturally $w \circ u$ is weak*-weak continuous and approximates u, as close as you please, on B_{Y^*}, that is, $\|u - w \circ u\|$ is small. But now $w \circ u$ is a weak*-weak continuous finite rank bounded linear operator from Y^* to X; as such $w \circ u$ can be identified with a member of $Y \otimes X$. We've shown that $Y \otimes X$ is dense in the weak*-weak compact linear operators from Y^* to X and that's the content of (A_4).

Now suppose (A_4) is in effect and, with an eye on (A_5), let $w : Z \to X$ be a compact linear operator. Naturally, w^{**} takes Z^{**} to X in a compact, weak*-weak continuous linear manner. Since (A_4) is assumed, we can apply it to w^{**} and the result is that w^{**} is (identifiable with) a member of $Z^* \overset{\vee}{\otimes} X$. It follows that w is in the norm closure of the norm closed set $Z^* \overset{\vee}{\otimes} X$, as well. This is just (A_5).

Finally suppose (A_5) holds. To establish (A), let K_0 be a compact subset of X and $\varepsilon > 0$. Then there is a compact subset K of X which is absolutely convex and such that K_0 is compact in X_K, which is nothing else than the linear span of K with K as the closed unit ball. Look at $u : X_K \to X$, the formal identity inclusion; u is a compact linear operator. (A_5) applies to u and puts u forth as a member of $(X_K)^* \overset{\vee}{\otimes} X$. Now we play with topologies: $u^* : X^* \to (X_K)^*$ has a range which is weak* dense since u is one-to-one; hence u^*X^* is dense in $(X_K)^*$ in the topology of uniform convergence on norm-compact subsets of X_K. But u, being (identifiable with) a member of $(X_K)^* \overset{\vee}{\otimes} X$, is approximable uniformly on compact subsets of X_K by members of $(X_K)^* \otimes X$. Keeping in mind the fact that members of $u^*(X^*) \otimes X$ are just operators of the form $v \circ u$ ($v \in X^* \otimes X$), it soon follows from K_0's compactness in X_K that there is a $v \in X^* \otimes X$ such that $v \circ u$ approximates u on K_0 as closely as you please. But $v \circ u \in \mathcal{F}(X;X)$ and $v \circ u$ approximates the identity on K_0. This is just (A). $\square$

1.4. Bilinear forms and linear operators of type α

Let α be a $\otimes$-norm and X and Y be Banach spaces.

Since $\alpha(u) \leq |u|_\wedge$ for any $u \in X \otimes Y$, $(X \overset{\alpha}{\otimes} Y)^*$ consists of bilinear continuous functionals on $X \times Y$ that are α-continuous.

Recall that a $\varphi \in \mathcal{B}(X, Y)$ is said to be of type α provided it belongs to $(X \overset{\alpha^*}{\otimes} Y)^*$. The space $\mathcal{B}^\alpha(X, Y)$ of bilinear forms of type α on $X \times Y$ is (as the dual of $X \overset{\alpha^*}{\otimes} Y$) a Banach space equipped with the norm $\|\cdot\|$ defined by $\|\varphi\|_\alpha = \|\varphi\|_{(X \overset{\alpha^*}{\otimes} Y)^*}$.

A bounded linear operator $u : X \to Y$ is said to be of type α or α-integral , if the bilinear map φ_u on $X \times Y^*$ given by

$$\varphi_u(x, y^*) = y^*\,(u(x))$$

is of type α. The space of α-integral linear operators from X to Y is denoted by $\mathcal{L}^\alpha(X; Y)$ and is equipped with the norm

$$\|u\|_\alpha = \|\varphi_u\|_{\mathcal{B}^\alpha(X, Y^*)}.$$

The space $\mathcal{L}^\alpha(X; Y)$ is a Banach space, an isometric isomorph of $\mathcal{B}^\alpha(X, Y^*)$, in this norm.

Recall that $\varphi \in \mathcal{B}(X, Y)$ defines a member of the closed unit ball of $(X \overset{\vee}{\otimes} Y)^* = (X \overset{\wedge^*}{\otimes} Y)^*$ precisely when there is a regular Borel probability measure μ defined on $(B_{X^*}, \text{weak}^*) \times (B_{Y^*}, \text{weak}^*)$ and bounded linear operators $a : X \to L^\infty(\mu), b : Y \to L^\infty(\mu)$, each of norm ≤ 1, such that for $x \in X$ and $y \in Y$,

$$\varphi(x, y) = \int_{B_{X^*} \times B_{Y^*}} ax(x^*, y^*) by(x^*, y^*) d\mu(x^*, y^*).$$

Now suppose $u : X \to Y$ is a bounded linear operator that is $\wedge$-integral, or just integral, for short; then, if $\varphi_u \in \mathcal{B}(X, Y^*)$ is given by

$$\varphi_u(x, y^*) = y^*\,(u(x))\,,$$

φ_u is an *integral bilinear functional*, i.e., $\varphi_u \in \mathcal{B}^\wedge(X, Y^*)$. Therefore, should we suppose that $\|u\|_\wedge = \|\varphi_u\|_{\mathcal{B}^\wedge(X, Y^*)} \leq 1$, there exists a regular Borel probability measure μ on $(B_{X^*}, \text{weak}^*) \times (B_{Y^{**}}, \text{weak}^*)$ and bounded linear operators $a : X \to L^\infty(\mu), b : Y^* \to L^\infty(\mu)$, each of norm ≤ 1, such that for $x \in X$ and $y^* \in Y^*$ we have

$$y^*\,(u(x)) = \varphi_u\,(x, y^*) = \int_{B_{X^*} \times B_{Y^{**}}} ax(x^*, y^{**}) by^*(x^*, y^{**}) d\mu.$$

Denoting by $I_{\infty,1}$ the natural inclusion

$$I_{\infty,1} : L^\infty(\mu) \hookrightarrow L^1(\mu)$$

and by $c, b^* \mid_{L^1(\mu)}$, we see that we have just factored $u : X \to Y$ in the following way ($j_Y : Y \hookrightarrow Y^{**}$ is the natural embedding):

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "a"'] & Y \arrow[r, hook, "j_Y"] & Y^{**} \\
L^\infty(\mu) \arrow[r, hook, "I_{\infty,1}"'] & L^1(\mu) \arrow[ur, "c"'] &
\end{tikzcd}$$

As we will see later, this description of integral operators is central to the metric theory of tensor products.

Here is an example where we construct the measure μ:

THEOREM 1.4.1 (Grothendieck (1953/1956a), Theorem 1, p. 93). *The formal identity inclusion*

$$\ell^1 \hookrightarrow c_o$$

is integral and has integral norm ≤ 1.

PROOF. Our aim is to show that the bilinear form ϕ on $\ell^1 \times \ell^1$ given by

$$\phi(\lambda, \mu) = \sum_n \lambda_n \mu_n$$

is integral. To find a suitable integral representation of ϕ we consider the group G of all sequences $\lambda \in \ell^\infty$ such that $|\lambda_i| = 1$ for all i with coordinatewise multiplication. The product topology (or weak* topology of ℓ_∞ relativized to G) makes G into a compact Abelian topological group, isomorphic as a topological group to $\{s : |s| = 1\}^{\mathbb{N}}$. The mapping

$$\lambda \to \lambda \otimes \lambda$$

is a continuous map from G into $B^\wedge\left(\ell^1, \ell^1\right)$ when the latter space is equipped with the weak* topology it inherits from $B^\wedge\left(\ell^1, \ell^1\right)$ being the dual of $\ell^1 \overset{\vee}{\otimes} \ell^1$; what's more, for any $\lambda \in G$, $\lambda \otimes \lambda \in B_{B^\wedge\left(\ell^1, \ell^1\right)}$. Letting μ be the normalized Haar measure on G, the Gelfand integral

$$\int_G \lambda \otimes \lambda d\mu(\lambda)$$

exists and is a member, call it ϕ', of $B_{B^\wedge(\ell^1, \ell^1)}$. It is plain that for any i, $j \in \mathbb{N}$,

$$\begin{aligned} \phi'(e_i, e_j) &= \int_G \lambda \otimes \lambda(e_i \otimes e_j) d\mu(\lambda) \\ &= \int_G \lambda_i \lambda_j d\mu(\lambda) = \delta_{ij}, \end{aligned}$$

and so $\phi' = \phi$ with

$$\begin{aligned} \|\phi\|_\wedge &= \sup_{\lambda,\, \nu \in B_{\ell^1}} \int_G |\langle \lambda, \gamma \rangle| \, |\langle \nu, \gamma \rangle| d\mu(\gamma) \\ &\leq 1 \end{aligned}$$

where $\langle \lambda, \gamma \rangle = \sum_n \lambda_n \gamma_n$, for $\lambda \in \ell^1, \gamma \in \ell^\infty$. $\square$

For $\alpha = |\cdot|_\vee$, the α-integral operators are just the bounded linear operators, and the bilinear forms of type α just the bounded bilinear forms.

1.4.1. General properties of α-forms. By its very definition, $\mathcal{B}^\alpha(X, Y)$ is the dual of $X \overset{\alpha^*}{\otimes} Y$ and so is a Banach space with $\|\cdot\|_\alpha$ for a norm. Further, as a dual, $\mathcal{B}^\alpha(X, Y)$'s closed unit ball is weak* compact or, what is the same, $B_{\mathcal{B}^\alpha(X,Y)}$ is compact in the topology of pointwise convergence on $X \overset{\alpha^*}{\otimes} Y$. Since $X \overset{\alpha^*}{\otimes} Y$ is just the completion of $X \otimes Y$ relative to the norm α^*, we see that $B_{\mathcal{B}^\alpha(X,Y)}$ is compact in the topology of pointwise convergence on $X \otimes Y$ and what's more, the weak* topology of $B_{\mathcal{B}^\alpha(X,Y)}$ is just the topology of pointwise convergence on $X \otimes Y$; but this is just the topology of pointwise convergence on $X \times Y$! Summarizing we have the following.

LEMMA 1.4.2. *If X and Y are Banach spaces and $(\varphi_i)_{i\in I}$ is a net of elements of $\mathcal{B}^\alpha(X,Y)$ such that $\sup_I \|\varphi_i\|_\alpha < \infty$ and such that for each $x \in X$ and each $y \in Y$ the limit $\lim_{i\in I} \varphi_i(x,y)$ exists, then $\varphi(x,y) \equiv \lim_{i\in I} \varphi_i(x,y)$ defines a member of $\mathcal{B}^\alpha(X,Y)$, too, with*

$$\|\varphi\|_\alpha \le \sup_I \|\varphi_i\|_\alpha.$$

Indeed, by Alaoglu's theorem, the net $(\varphi_i)_{i\in I}$ must have a cluster point χ of norm no more than $\sup_I \|\varphi_i\|_\alpha$ and by the condition $\lim_I \varphi_i(x,y)$ exists, $(\varphi_i)_{i\in I}$ can only have one such cluster point, φ.

PROPOSITION 1.4.3. *Suppose X_1, X_2, Y_1 and Y_2 are Banach spaces, $u : X_1 \to X_2$ and $v : Y_1 \to Y_2$ are bounded linear operators and $\varphi \in \mathcal{B}^\alpha(X_2,Y_2)$. Then*

$$\varphi \circ (u \otimes v) \in \mathcal{B}^\alpha(X_1,Y_1) \quad \textit{and} \quad \|\varphi \circ (u\otimes v)\|_\alpha \le \|\varphi\|_\alpha \|u\|\|v\|.$$

In fact, $u \otimes v$ is a bounded linear operator from $X_1 \overset{\alpha^*}{\otimes} Y_1$ to $X_2 \overset{\alpha^*}{\otimes} Y_2$ having operator norm no more than $\|u\|\|v\|$. It follows that $(u\otimes v)^*$ is a bounded linear operator of norm $\le \|u\|\|v\|$, too, with $(u \otimes v)^*$ taking $(X_2 \overset{\alpha^*}{\otimes} Y_2)^*$ to $(X_1 \overset{\alpha^*}{\otimes} Y_1)^*$. It is then plain that $(u\otimes v)^*(\varphi) \in \mathcal{B}^\alpha(X_1,Y_1)$ with $\|(u\otimes v)^*(\varphi)\|_\alpha \le \|u\|\|v\|\|\varphi\|_\alpha$. Of course, an easy check shows that $(u\otimes v)^*(\varphi)$ is just $\varphi \circ (u \otimes v)$: If $x_1 \in X_1$, and $y_1 \in Y_1$, then

$$\begin{aligned}
(u\otimes v)^*(\varphi)(x_1,y_1) &= \varphi\left((u\otimes v)(x_1 \otimes y_1)\right) \\
&= \varphi\left(u(x_1) \otimes v(y_1)\right) \\
&= \varphi\left(u(x_1), v(y_1)\right) \\
&= \left(\varphi \circ (u\otimes v)\right)(x_1,y_1).
\end{aligned}$$

For u and v the natural inclusions of subspaces into superspaces, the above tells us that

PROPOSITION 1.4.4. *If X and Y are Banach spaces and $\varphi \in \mathcal{B}^\alpha(X,Y)$, then $\varphi|_{X_0\times Y_0}$ is an α-form for any closed linear subspaces X_0 of X and Y_0 of Y with*

$$\|\varphi|_{X_0\times Y_0}\|_\alpha \le \|\varphi\|_\alpha.$$

PROPOSITION 1.4.5. *If $\varphi \in \mathcal{B}^\alpha(X,Y)$, then ${}^t\varphi$, defined on $Y\times X$ by ${}^t\varphi(y,x) = \varphi(x,y)$, belongs to $\mathcal{B}^{t_\alpha}(Y,X)$ with $\|{}^t\varphi\|_{t_\alpha} = \|\varphi\|_\alpha$. In fact, $\varphi \in \mathcal{B}(X,Y)$ is of type α if and only if ${}^t\varphi$ is of type ${}^t\alpha$ with $\|\varphi\|_\alpha = \|{}^t\varphi\|_{t_\alpha}$.*

Suppose $\varphi \in \mathcal{B}(X,Y)$. Then for any $x \in X, \varphi_x(y) = \varphi(x,y)$ determines a member φ_x of Y^*. Take a $y^{**} \in Y^{**}$ and consider for any $x \in X$,

$$\tilde\varphi(x,y^{**}) = y^{**}(\varphi_x) = y^{**}\left(\varphi(x,\cdot)\right).$$

It is easy to see that $\tilde\varphi \in \mathcal{B}(X,Y^{**})$. We call $\tilde\varphi$ the *canonical extension* of φ to $X \times Y^{**}$.

THEOREM 1.4.6 (Grothendieck (1953/1956a), Theorem 4, p. 13). *Let φ be a continuous bilinear form on $X\times Y$ and $\tilde\varphi$ its canonical extension to $X \times Y^{**}$. For φ to be of type α it is necessary and sufficient that $\tilde\varphi$ be of type α. In this case*

$$\|\varphi\|_\alpha = \|\tilde\varphi\|_\alpha.$$

The proof hinges on a most important aspect of the theory of $\otimes$ norms: their finite dimensional or local character; for instance, note the following:

Proposition 1.4.7. *Suppose $\varphi \in \mathcal{B}(X,Y)$ satisfies $\|\varphi\,|_{E\times F}\|_\alpha \le k$ for some $k>0$ and all $E \in \mathcal{F}(X)$ and $F \in \mathcal{F}(Y)$. Then $\varphi \in \mathcal{B}^\alpha(X,Y)$ and $\|\varphi\|_\alpha \le k$.*

Proof. Take $u \in X \otimes Y$ with $\alpha^*(u) < 1$. Choose $E \in \mathcal{F}(X)$ and $F \in \mathcal{F}(Y)$ such that $u \in E \otimes F$ and $|u|_{E \overset{\alpha^*}{\otimes} F} < 1$, too. Then

$$|\varphi(u)| = |\varphi\,|_{E\times F}\,(u)| \le \|\varphi\,|_{E\times F}\,\|_\alpha |u|_{E \overset{\alpha^*}{\otimes} F} < k.$$

It follows that $\varphi \in (X \otimes Y, \alpha^*)^*$ with $\|\varphi\|_{(X\otimes Y,\alpha^*)^*} \le k$. Enough said. $\square$

Remark: At this juncture it is convenient to borrow from Chapter 2. The result we wish to call on depends on further development of the finer theory of integral operators, a development that does not rely on the present commiserations. The faith placed in the future will this time be rewarded. The result in question: *the dual of $F^* \overset{\vee}{\otimes} Y$ is identifiable with $F \overset{\wedge}{\otimes} Y^*$ whenever $F \in \mathcal{F}$;* as a consequence, the second dual of the space $\mathcal{L}(F;Y)$ — which is identifiable with $F^* \overset{\vee}{\otimes} Y$ — is identifiable with $\mathcal{L}(F;Y^{**})$, with the dual of $\mathcal{L}(F;Y)$ being, of course, $F \overset{\wedge}{\otimes} Y^*$. In other words, the following equations are in effect

$$(F^* \overset{\vee}{\otimes} Y)^* = \mathcal{L}(F;Y)^* = F \overset{\wedge}{\otimes} Y^*$$

and

$$\mathcal{L}(F;Y)^{**} = (F \overset{\wedge}{\otimes} Y^*)^* = \mathcal{L}(F;Y^{**})$$

so long as $F \in \mathcal{F}$, with all identifications being as natural as one might expect.

Because E is finite dimensional, $E^* \overset{\alpha}{\otimes} Y^* = \mathcal{B}^\alpha(E,Y)$. Indeed, suppose that $\{e_1,\dots,e_n\}$ is a basis for E and let $e_1^*,\dots,e_n^* \in E^*$ be functionals biorthogonal to $e_1,\dots,e_n$. Take $\varphi \in \mathcal{B}(E,Y)$. For $i = 1,2,\dots,n$ define $y_i^*(y) = \varphi(e_i,y)$ $(y \in Y)$ and notice that for any $x = \sum_{i\le n} x_i e_i \in E$ and any $y \in Y$ we have

$$\begin{aligned}\varphi(x,y) = \varphi(\sum_{i\le n} x_i e_i, y) &= \sum_{i\le n} x_i \varphi(e_i,y)\\ &= \sum_{i\le n} e_i^*(x) y_i^*(y)\\ &= [\sum_{i\le n} (e_i^* \otimes y_i^*)](x,y)\end{aligned}$$

and $\varphi = \sum_{i\le n} e_i^* \otimes y_i^* \in E^* \otimes Y^*$.

On to the proof of Theorem 1.4.6. If $\tilde\varphi$ is of type α, then $\tilde\varphi\,|_{X\times Y} = \varphi$ is of type α, too, and

$$\|\varphi\|_\alpha = \|\tilde\varphi\,|_{X\times Y}\,\|_\alpha \le \|\tilde\varphi\|_\alpha.$$

Suppose that $\varphi \in \mathcal{B}^\alpha(X,Y)$ and let $\varepsilon > 0$ be given as well as $E \in \mathcal{F}(X)$ and $F \in \mathcal{F}(Y^{**})$; we'll show that

$$\|\tilde\varphi\,|_{E\times F}\,\|_\alpha \le \|\varphi\|_\alpha + \varepsilon.$$

We view $\varphi \in \mathcal{B}(X,Y)$ as an operator $u_\varphi : X \to Y^*$.

Okay, E and F are finite dimensional so $E \overset{\alpha^*}{\otimes} F$ is isomorphic to $E \overset{\wedge}{\otimes} F$ making

$$\left\{ v \in E \otimes F : |v|_{E \overset{\alpha^*}{\otimes} F} \le 1 \right\}$$

a norm compact set in $E \overset{\wedge}{\otimes} F$. It follows that

$$\left\{ (id_F \otimes u_\varphi \mid_E)\left({}^t v\right) : v \in B_{E \overset{\alpha^*}{\otimes} F} \right\}$$

is norm compact in $F \overset{\wedge}{\otimes} Y^*$. But as remarked above, $F \overset{\wedge}{\otimes} Y^* \subseteq \left(F^* \overset{\vee}{\otimes} Y\right)^*$ and so any element of $B_{\mathcal{L}(F;Y^{**})}$ can be approximated pointwise on $F \overset{\wedge}{\otimes} Y^*$ by members of $B_{\mathcal{L}(F;Y)}$ with the approximation uniform on compact subsets of $F \overset{\wedge}{\otimes} Y^*$. In particular, the natural inclusion $i_{(F,Y^{**})}$ of F into Y^{**} can be so approximated. On so doing we find a $w \in B_{\mathcal{L}(F;Y)}$ so that

$$\sup_{v \in B_{E \overset{\alpha^*}{\otimes} F}} \mid \left(i_{(F,Y^{**})} - w\right)(id_F \otimes u_\varphi \mid_E)\left({}^t v\right) \mid \leq \varepsilon.$$

But for $v \in E \otimes F$ the mess

$$i_{(F,Y^{**})}\left((id_F \otimes u_\varphi \mid_E)({}^t v)\right)$$

is just $\tilde{\varphi} \mid_{E\times F} (v)(!)$ and the monstrosity

$$w\left((id_F \otimes u_\varphi \mid_E)({}^t v)\right)$$

is nothing else than $(\varphi \circ (id_E \otimes w) \mid_{E\times F})(v)$. It follows that

$$\|\tilde{\varphi} \mid_{E\times F} \|_\alpha = \sup_{v \in B_{E \overset{\alpha^*}{\otimes} F}} |\tilde{\varphi}(v)| \leq \varepsilon + \|\varphi \circ (id_E \otimes w) \mid_{E\times F}\|_\alpha \leq \varepsilon + \|\varphi\|_\alpha.$$

By this we are at last finished with the proof of our theorem. □

By duality we get

Corollary 1.4.8 (Grothendieck (1953/1956a), Corollary 1, p. 13). *$B_{X \overset{\alpha}{\otimes} Y}$ is dense in $B_{X \overset{\alpha}{\otimes} Y^{**}}$ in the topology of pointwise convergence in $\mathcal{B}^{\alpha^*}(X,Y)$.*

Not quite so immediate but of great importance to the development of the calculus of tensor products to be pursued later is the following essential fact.

Corollary 1.4.9 (Grothendieck (1953/1956a), Corollary 2, p. 13). *The canonical inclusions*

$$X \overset{\alpha}{\otimes} Y \hookrightarrow X \overset{\alpha}{\otimes} Y^{**},\ X \overset{\alpha}{\otimes} Y \hookrightarrow X^{**} \overset{\alpha}{\otimes} Y \text{ and } X \overset{\alpha}{\otimes} Y \hookrightarrow X^{**} \overset{\alpha}{\otimes} Y^{**}$$

are isometric.

Proof. Of course, $X \overset{\alpha}{\otimes} Y \hookrightarrow X \overset{\alpha}{\otimes} Y^{**}$ has norm at most 1. Take $u \in X \overset{\alpha}{\otimes} Y$. Pick $\varphi \in \mathcal{B}^{\alpha^*}(X,Y)$ (which is $(X \overset{\alpha}{\otimes} Y)^*$), $\|\varphi\|_{\alpha^*} = 1$ with $\varphi(u) = |u|_{X \overset{\alpha}{\otimes} Y}$. Then $\tilde{\varphi}$, the canonical extension of φ to a member of $\mathcal{B}^{\alpha^*}(X,Y^{**})$, has $\|\tilde{\varphi}\|_{\alpha^*} = 1$, and so

$$|u|_{X \overset{\alpha}{\otimes} Y^{**}} \geq |\tilde{\varphi}(u)| = |\varphi(u)| = |u|_{X \overset{\alpha}{\otimes} Y}.$$

The second claim in the corollary follows from the first and a careful consideration of ${}^t\alpha$ while the third claim is an immediate consequence of the first two. □

Theorem 1.4.6 leads to many interesting conclusions not the least of which concerns metric accessibility of $\otimes$-norms:

Proposition 1.4.10. *If α is a metrically accessible $\otimes$-norm, then so too are ${}^t\alpha$, α^* and $\overset{\vee}{\alpha}$.*

PROOF. Only α^* needs to be commented upon in any detail.

Theorem 1.4.6 tells us that starting with $\varphi \in \mathcal{B}^\alpha(X,Y), \tilde{\varphi} \in \mathcal{B}^\alpha(X,Y^{**})$, the canonical extension of φ, satisfies $\|\tilde{\varphi}\|_\alpha = \|\varphi\|$. Of course, ${}^t\tilde{\varphi} \in \mathcal{B}^{t_\alpha}(Y^{**},X)$ and $\|{}^t\tilde{\varphi}\|_{t_\alpha} = \|\tilde{\varphi}\|_\alpha$. Theorem 1.4.6 comes into play again and the result is $\widetilde{{}^t\tilde{\varphi}} \in \mathcal{B}^{t_\alpha}(Y^{**},X^{**})$ and $\left\|\widetilde{{}^t\tilde{\varphi}}\right\|_{t_\alpha} = \|{}^t\tilde{\varphi}\|_{t_\alpha}$. At last, we find $\tilde{\tilde{\varphi}} ={}^t\, \widetilde{{}^t\tilde{\varphi}} \in \mathcal{B}^\alpha(X^{**},Y^{**})$ with

$$\|\tilde{\tilde{\varphi}}\|_\alpha = \left\|\widetilde{{}^t\tilde{\varphi}}\right\|_{t_\alpha} = \left\|{}^t\tilde{\varphi}\right\|_{t_\alpha} = \|\tilde{\varphi}\|_\alpha = \|\varphi\|$$

and $\tilde{\tilde{\varphi}}\,|_{X\times Y} = \varphi$. The result (and this has nothing to do with accessibility) is that there's a natural inclusion

$$\mathcal{B}^\alpha(X,Y) \hookrightarrow \mathcal{B}^\alpha(X^{**},Y^{**})$$

that's an isometry as well.

Now suppose α is a metrically accessible norm, E is finite dimensional and Y is any old Banach space. Consider the canonical inclusion

$$E^* \overset{\alpha}{\otimes} Y^* \hookrightarrow \mathcal{B}^\alpha(E,Y).$$

Is it an isometry? YES! After all,

$$E^* \overset{\alpha}{\otimes} Y^* \hookrightarrow \mathcal{B}^\alpha(E^{**},Y^{**})$$

is an isometry since α is accessible and so we have the diagram

$$\begin{tikzcd} E^* \overset{\alpha}{\otimes} Y^* \arrow[r, hook] \arrow[rr, bend right, "\text{isometry}"'] & \mathcal{B}^\alpha(E,Y) \arrow[r, hook, "\text{isometry}"'] & \mathcal{B}^\alpha(E^{**},Y^{**}) \end{tikzcd}$$

which leaves $E^* \overset{\alpha}{\otimes} Y^* \hookrightarrow \mathcal{B}^\alpha(E,Y)$ little (no) choice but to be an isometry.

Actually, we know from our Remark preceding the proof of Theorem 1.4.6 that $E^* \overset{\alpha}{\otimes} Y^* = \mathcal{B}^\alpha(E,Y)$.

Now let's consider α^*'s metric accessibility; the test is whether or not the natural inclusion

$$E \overset{\alpha^*}{\otimes} Y \hookrightarrow \mathcal{B}^{\alpha^*}(E^*,Y^*)$$

is an isometry. Well, consider the following sequence of natural isometric inclusions:

$$E \overset{\alpha^*}{\otimes} Y \hookrightarrow (E \overset{\alpha^*}{\otimes} Y)^{**} = \left((E \overset{\alpha^*}{\otimes} Y)^*\right)^* = \mathcal{B}^\alpha(E,Y)^* = (E^* \overset{\alpha}{\otimes} Y^*)^* = \mathcal{B}^{\alpha^*}(E^*,Y^*).$$

Test passed: α^* is accessible if α is. $\square$

COROLLARY 1.4.11. *$u : X \to Y$ is α-integral if and only if $u^* : Y^* \to X^*$ is ${}^t\alpha$-integral, in which case, $\|u\|_\alpha = \|u^*\|_{{}^t\alpha}$.*

Consequently, $u : X \to Y$ is α-integral if and only if $u^{**} : X^{**} \to Y^{**}$ is, with $\|u\|_\alpha = \|u^{**}\|_\alpha$.

PROOF. $u \in \mathcal{L}^\alpha(X;Y)$ precisely when $\varphi_u : X\times Y^* \to \mathbb{K}$ defined by $\varphi_u(x,y^*) = y^*(u(x))$ belongs to $\mathcal{B}^\alpha(X,Y^*)$. This happens precisely when ${}^t\varphi_u \in \mathcal{B}^{{}^t\alpha}(Y^*,X)$. But, again, this is so exactly when $\widetilde{{}^t\varphi_u} \in \mathcal{B}^{{}^t\alpha}(Y^*,X^{**})$. However, a quick evaluation shows that

$$\widetilde{{}^t\varphi_u}(y^*,x^{**}) = x^{**}(u^*y^*) = \varphi_{u^*}(y^*,x^{**}),$$

so ${}^t\tilde{\varphi}_u = \varphi_{u^*} \in \mathcal{B}^{t\alpha}(Y^*, X^{**})$ precisely when $u^* : Y^* \to X^*$ belongs in $\mathcal{L}^{t\alpha}(Y^*; X^*)$. Norms? In fact,

$$\|u\|_\alpha = \|\varphi_u\|_\alpha = \|{}^t\varphi_u\|_{{}^t\alpha} = \|{}^t\tilde{\varphi}_u\|_{{}^t\alpha} = \|\varphi_{u^*}\|_{{}^t\alpha} = \|u^*\|_{{}^t\alpha}.$$

$\square$

1.4.2. General properties of α-integral operators. For emphasis we repeat that a bounded linear operator $u : X \to Y$ is α-integral if the associated bounded bilinear form φ_u, defined by

$$\varphi_u(x, y^*) = y^*(u(x))$$

on $X \times Y^*$ is of type α; the space $\mathcal{L}^\alpha(X; Y)$ is a Banach space when equipped with the norm

$$\|u\|_\alpha = \|\varphi_u\|_\alpha.$$

We immediately have

LEMMA 1.4.12. *If X and Y are Banach spaces and $(u_i)_{i\in I}$ is a net of elements of $\mathcal{L}^\alpha(X; Y)$ such that $\sup_I \|u_i\|_\alpha < \infty$ and such that for each $x \in X$ the weak limit $\lim_{i\in I} u_i(x)$ exists, then $u(x) \equiv \lim_{i\in I} u_i(x)$ defines a member of $\mathcal{L}^\alpha(X; Y)$, too, with*

$$\|u\|_\alpha \le \sup_I \|u_i\|_\alpha.$$

PROOF. Indeed, the corresponding "pointwise" property holds for bilinear forms of type α (cf. Lemma 1.4.2). $\square$

In this section we'll gather some important details about α-integral operators. First, we take care of identifications. Suppose $\varphi \in \mathcal{B}(X, Y)$ and we define $u_\varphi : X \to Y^*$ by $u_\varphi(x)(y) = \varphi(x, y)$. Then u_φ is α-integral if and only if φ is of type α; in this case, $\|u_\varphi\|_\alpha = \|\varphi\|_\alpha$.

To begin, suppose $u_\varphi : X \to Y^*$ is α-integral. Then $\psi_{u_\varphi} \in \mathcal{B}(X, Y^{**})$ defined by

$$\psi_{u_\varphi}(x, y^{**}) = y^{**}(u_\varphi(x))$$

is in $\mathcal{B}^\alpha(X, Y^{**})$. Further, $\psi_{u_\varphi}|_{X\times Y} = \varphi$ and so $\varphi \in \mathcal{B}^\alpha(X, Y)$ with $\|\varphi\|_\alpha \le \|\psi_{u_\varphi}\|_\alpha = \|u_\varphi\|_\alpha$. To finish, suppose $\varphi \in \mathcal{B}^\alpha(X, Y)$. Then $\tilde{\varphi} \in \mathcal{B}^\alpha(X, Y^{**})$ and $\|\varphi\|_\alpha = \|\tilde{\varphi}\|_\alpha$ where $\tilde{\varphi}$ is the canonical extension of φ. But for $u : X \to Y^*$ defined by $u(x)(y) = \varphi(x, y)$ we know that $\varphi_u = \tilde{\varphi}$; $\|u\|_\alpha = \|\varphi_u\|_\alpha = \|\tilde{\varphi}\|_\alpha = \|\varphi\|_\alpha$.

PROPOSITION 1.4.13. *If $u : X \to Y$ is a bounded linear operator and $j : Y \hookrightarrow Y^{**}$ denotes the canonical isometric embedding, then $u \in \mathcal{L}^\alpha(X; Y)$ if and only if $ju \in \mathcal{L}^\alpha(X; Y^{**})$ with $\|u\|_\alpha = \|ju\|_\alpha$.*

PROOF. Let $\varphi_u \in \mathcal{B}(X, Y^*)$ correspond to $u : X \to Y$; then ju is the operator in $\mathcal{L}(X; Y^{**})$ corresponding to φ_u. Notice that:

$$u \in \mathcal{L}^\alpha \iff \varphi_u \in \mathcal{B}^\alpha \iff ju \in \mathcal{L}^\alpha$$

with norms following suit. $\square$

Finally, it is worth noting that

PROPOSITION 1.4.14. *If W, X, Y and Z are Banach spaces and $w : W \to X$, $v : X \to Y$ and $u : Y \to Z$ are bounded linear operators with v α-integral, then uvw is α-integral, too, with*

$$\|uvw\|_\alpha \le \|u\|\|v\|_\alpha\|w\|.$$

PROOF. Of course, $\varphi_v \in \mathcal{B}^\alpha(X, Y^*)$ so that $\varphi_{uvw} = \varphi_v \circ (w \otimes u^*) \in \mathcal{B}^\alpha(W, Z^*)$ with $\|\varphi_{uvw}\|_\alpha \leq \|\varphi_v\|_\alpha \|w\| \|u^*\| = \|\varphi_v\|_\alpha \|w\| \|u\|$. But this just says that the operator from W to Z^{**} defined by $\varphi_{uvw}(w, z^*)$ is α-integral with α-integral norm $= \|\varphi_{uvw}\|_\alpha$, itself $\leq \|\varphi_v\|_\alpha \|w\| \|u\| = \|v\|_\alpha \|w\| \|u\|$; however, this operator is quickly seen to be nothing else but $j_Z uvw$ and so uvw, itself, is in $\mathcal{L}^\alpha(W; Z)$ with

$$\|uvw\|_\alpha = \|j_Z uvw\|_\alpha \leq \|u\| \|v\|_\alpha \|w\|.$$

$\square$

As in the case of α-forms, α-integrability is determined by behavior on finite dimensional pieces of the domain; more precisely:

PROPOSITION 1.4.15. *If $u : X \to Y$ is a bounded linear operator and for some $k > 0$ we have*

$$\|u \mid_E \|_\alpha < k$$

for each $E \in \mathcal{F}(X)$, then $u \in \mathcal{L}^\alpha(X; Y)$ with $\|u\|_\alpha \leq k$.

PROOF. As usual, φ_u is the bilinear form associated with u. $\varphi_u \in \mathcal{B}(X, Y^*)$. For any $F \in \mathcal{F}(Y^*)$ we see that

$$\|\varphi_u \mid_{E \times F}\|_\alpha \leq \|\varphi_u \mid_{E \times Y^*}\|_\alpha = \|u \mid_E\|_\alpha < k.$$

It follows that $\varphi_u \in \mathcal{B}^\alpha(X, Y^*)$ and $\|\varphi_u\|_\alpha \leq k$; from this we get immediately that $u \in \mathcal{L}^\alpha(X; Y)$ and $\|u\|_\alpha = \|\varphi_u\|_\alpha \leq k$. $\square$

1.4.3. Composition of α-integral and $\overset{\vee}{\alpha}$-integral operators.

THEOREM 1.4.16 (Grothendieck (1953/1956a), Theorem 5, p. 15). *Suppose α is metrically accessible or Y is and let $u : X \to Y$ be an α-integral operator with $v : Y \to Z$ an $\overset{\vee}{\alpha}$-integral operator. Then vu is an integral operator and*

$$\|vu\|_\wedge \leq \|v\|_{\overset{\vee}{\alpha}} \|u\|_\alpha.$$

PROOF. We'll check vu on E's from $\mathcal{F}(X)$. $u \mid_E \in E^* \otimes Y = E^* \overset{\alpha}{\otimes} Y = \mathcal{B}^\alpha(E, Y^*)$; the first equality is due to E^*'s finite dimensionality while the second is due to the accessibility conditions hypothesized along with E^*'s finite dimensionality. Of course, when we write equality above, all the norms are equal, too, so

$$\left| u \mid_E \right|_{E^* \overset{\alpha}{\otimes} Y} = \|\varphi_{u|_E}\|_\alpha = \left\| u \mid_E \right\|_\alpha$$

where $\varphi_{u|_E}$ is the bilinear form in $\mathcal{B}^\alpha(E, Y^*)$ induced by $u \mid_E$. We want to check $\|v \circ u \mid_E \|_\wedge$ and so we have to see how big the induced bilinear form $\varphi_{v \circ u|_E} \in \mathcal{B}^\wedge(E, Z^*) = (E \overset{\vee}{\otimes} Z^*)^*$ gets when evaluated at w's from $E \otimes Z^*$ with $|w|_\vee \leq 1$. Take such a w, say $w = \sum_{i \leq n} e_i \otimes z_i^*$ and view w, as an operator from E^* to Z^*; we soon see that

$$\varphi_{v \circ u|_E}(w) = \sum_{i \leq n} z_i^*\big(v(u(e_i))\big) = \sum_{i \leq n} v^*\big(z_i^*(u \mid_E (e_i))\big) = \varphi_{v^* \circ w}\big(u \mid_E \big).$$

So

$$\begin{aligned}
\left|\varphi_{v \circ u|_E}(w)\right| &= \left|\varphi_{v^* \circ w}(u \mid_E)\right| \\
&\leq \left\|\varphi_{v^* \circ w}\right\|_{(E^* \overset{\alpha}{\otimes} Y)^*} \left| u \mid_E \right|_{E^* \overset{\alpha}{\otimes} Y} \\
&= \left\|\varphi_{v^* \circ w}\right\|_{\mathcal{B}^{\alpha^*}(E^*, Y)} \left\| u \mid_E \right\|_{\mathcal{L}^\alpha(E; Y)}
\end{aligned}$$

$$
\begin{aligned}
&= \left\|v^* \circ w\right\|_{\mathcal{L}^{\alpha^*}(E^*;Y^*)} \left\|u\mid_E\right\|_{\mathcal{L}^{\alpha}(E;Y)} \\
&\leq \left\|v^*\right\|_{\mathcal{L}^{\alpha^*}(Z^*;Y^*)} \left\|w\right\|_{\vee} \left\|u\mid_E\right\|_{\alpha} \\
&\leq \left\|v\right\|_{{}^t\alpha^*} \left\|u\mid_E\right\|_{\alpha} \\
&= \left\|v\right\|_{\overset{\vee}{\alpha}} \left\|u\mid_E\right\|_{\alpha}.
\end{aligned}
$$

From this we see that

$$\left\|v \circ u\mid_E\right\|_{\wedge} = \left\|\varphi_{v\circ u|_E}\right\|_{\wedge} \leq \left\|v\right\|_{\overset{\vee}{\alpha}} \left\|u\mid_E\right\|_{\alpha}$$

and soon thereafter that $v \circ u \in \mathcal{L}^{\wedge}(X;Z)$ and

$$\|v \circ u\|_{\wedge} \leq \|v\|_{\overset{\vee}{\alpha}} \|u\|_{\alpha}.$$

□

The results of Theorem 1.4.16 are *tight* in the following sense:

PROPOSITION 1.4.17. *If α or Y is metrically accessible and $u : X \to Y$ is a bounded linear operator such that $v \circ u$ is integral whenever $v : Y \to Z$ is $\overset{\vee}{\alpha}$-integral, then u is, in fact, α-integral.*

PROOF. Suppose u is as advertised. Then given a Banach space Z there is a $k_Z > 0$ so that $k_Z < \infty$ and

$$(*) \qquad \|v \circ u\|_{\wedge} \leq k_Z \|v\|_{\overset{\vee}{\alpha}}$$

holds for any $\overset{\vee}{\alpha}$-integral linear operator $v : Y \to Z$; this follows from the fact that the naturally occurring operator $U : \mathcal{L}^{\overset{\vee}{\alpha}}(Y;Z) \to \mathcal{L}^{\wedge}(X;Z)$ given by $Uv = v \circ u$ is linear and has a closed graph. Furthermore, there is a $k > 0$ so that $k_Z \leq k(< \infty)$ for all Z's. If not, then there would exist Banach spaces Z_n such that the *best possible* k_{Z_n}'s in $(*)$ tend to ∞. Choose $v_n \in \mathcal{L}^{\overset{\vee}{\alpha}}(Y;Z_n)$ so that $\|v_n\|_{\overset{\vee}{\alpha}} < 1$ yet $\|v_n \circ u\|_{\wedge} > \frac{k_{Z_n}}{2}$. Let $Z = (\sum_n Z_n)_{\ell^\infty}$. Because each Z_n is norm-one complemented in Z,

$$\|v_n : Y \to Z\|_{\overset{\vee}{\alpha}} = \|v_n : Y \to Z_n\|_{\overset{\vee}{\alpha}} < 1$$

and

$$\frac{k_{Z_n}}{2} \leq \|v_n \circ u : X \to Z_n\|_{\wedge} = \|v_n \circ u : X \to Z\|_{\wedge} \leq k_Z.$$

Since $k_{Z_n} \nearrow \infty$, this is impossible.

Okay, now take $E \in \mathcal{F}(X)$. Any $v \in \mathcal{L}^{\overset{\vee}{\alpha}}(Y;E)$ produces $v \circ u\mid_E \in \mathcal{L}^{\wedge}(E;E)$ or equivalently $\varphi_{v\circ u|_E} \in \mathcal{B}^{\wedge}(E,E^*)$; moreover,

$$\left\|v \circ u\mid_E\right\|_{\mathcal{L}^{\wedge}(E;E)} = \left\|\varphi_{v\circ u|_E}\right\|_{\wedge} \leq k\left\|v\right\|_{\overset{\vee}{\alpha}}.$$

Define $\Phi : \mathcal{B}^{\overset{\vee}{\alpha}}(Y,E^*) \to \mathcal{B}^{\wedge}(E,E^*)$ by taking $\psi \in \mathcal{B}^{\overset{\vee}{\alpha}}(Y,E^*)$, inducing the operator $v_\psi \in \mathcal{L}^{\overset{\vee}{\alpha}}(Y;E^{**}) = \mathcal{L}^{\overset{\vee}{\alpha}}(Y;E)$ — E is finite dimensional, remember — and define $\Phi(\psi) = \varphi_{v_\psi} \circ u\mid_E$. Naturally, $\|\Phi\| \leq k$. A picture follows (commutatively drawn, of course):

$$
\begin{tikzcd}
\mathcal{B}^{\wedge}(E,E^*)^* \arrow[r, "\Phi^*"] & \mathcal{B}^{\overset{\vee}{\alpha}}(Y,E^*)^* \\
E \overset{\vee}{\otimes} E^* \arrow[u] & Y \overset{t\alpha}{\otimes} E^* \arrow[u] \\
E^* \overset{\vee}{\otimes} E \arrow[u, "{}^t(.)"] \arrow[r, "id_{E^*} \otimes u"'] & E^* \overset{\alpha}{\otimes} Y \arrow[u, "{}^t(.)"']
\end{tikzcd}
$$

Let's compute. By our accessibility assumptions, and the finite dimensionality of E,

$$E^* \otimes Y = E^* \overset{\alpha}{\otimes} Y = \mathcal{B}^{\alpha}(E, Y^*)$$

and

$$\left| u \mid_E \right|_{E^* \overset{\alpha}{\otimes} Y} = \left\| \varphi_{u|_E} \right\|_{\mathcal{B}^{\alpha}(E,Y^*)} = \left\| u \mid_E \right\|_{\mathcal{L}^{\alpha}(E;Y)} .$$

Start with $id_E : E \to E$ and realize id_E as an $\varepsilon \in E^* \otimes E = E^* \overset{\vee}{\otimes} E$; $(id_{E^*} \otimes u \mid_E)(\varepsilon)$ is just $u \mid_E$. In our diagram all vertical arrows are isometries so if we start with ε and proceed up the left side then across the top, the resulting member of $\mathcal{B}^{\overset{\vee}{\alpha}}(Y, E^*)^*$ will have norm $\leq k$ — after all $\|\varepsilon\|_{\vee} = 1$; on the other hand, if we go across the bottom to $u \mid_E$, then, thanks to the right side's isometric proclivity, we get a member of $E^* \overset{\alpha}{\otimes} Y$ whose norm is $\leq k$, too. In other words,

$$\left\| u \mid_E \right\|_{\mathcal{L}^{\alpha}(E;Y)} = \left| u \mid_E \right|_{E^* \overset{\alpha}{\otimes} Y} \leq k.$$

Enough said. $\square$

1.4.4. Accessibility and metric accessibility (continued). Now we return to accessibility and some further equivalences thereof. We continue with the numbering as in Theorem 1.3.11 (p. 39).

THEOREM 1.4.18. *Each of the following conditions is both necessary and sufficient for a Banach space X to be accessible.*

(A_6) *The natural injection of $X^* \overset{\wedge}{\otimes} X$ into $\mathcal{L}(X;X)$ is one-to-one.*
(A_7) *Given $(x_n^*) \subseteq X^*$ and $(x_n) \subseteq X$ such that $\sum_n \|x_n^*\| \|x_n\| < \infty$ and $\sum_n x_n^*(x) x_n = 0$ for all $x \in X$ it follows that $\sum_n x_n^*(x_n) = 0$.*

The proof turns on a better understanding of the topology of uniform convergence on compacta. Lemma 1.3.9 finds application in the present context by helping to describe those linear functionals on $\mathcal{L}(X;Y)$ that are continuous with respect to the topology of uniform convergence on compacta:

PROPOSITION 1.4.19. *The linear functional ℓ on $\mathcal{L}(X;Y)$ is continuous on $\mathcal{L}(X;Y)$ relative to the topology of uniform convergence on compact subsets of X precisely when there exist sequences $(y_n^*) \subseteq Y^*; (x_n) \subseteq X$ such that $\sum_n \|y_n^*\| \|x_n\| < \infty$ and $\ell(u) = \sum_n y_n^*(u(x_n))$ for all $u \in \mathcal{L}(X;Y)$.*

PROOF. It is easy to see that ℓ's of the prescribed fashion are continuous on $\mathcal{L}(X;Y)$ when it is equipped with the topology of uniform convergence on compacta;

in fact, a slight adjustment in the y_n^*'s and x_n's will allow us to assume $\sum_n \|y_n^*\| < \infty$ and $\lim_n \|x_n\| = 0$. This having been done, note that

$$|\ell(u)| = |\sum_n y_n^*(u(x_n))| \leq (\sum_n \|y_n^*\|)\|u\|_{\{x_n\}},$$

where $\|u\|_K = \sup_{x\in K} \|ux\|$.

On the other hand, for a linear functional ℓ on $\mathcal{L}(X;Y)$ to be continuous with respect to the topology of uniform convergence on compact subsets of X entails the existence of some compact set $K \subseteq X$ such that for any $u \in \mathcal{L}(X;Y)$,

$$|\ell(u)| \leq \|u\|_K = \sup_{x\in K} \|ux\|.$$

By Lemma 1.3.9, there is a norm null sequence (x_n) in X such that $K \subseteq \overline{co}\{x_n : n \geq 0\}$; in terms of ℓ this tells us that for any $u \in \mathcal{L}(X;Y)$ we have $|\ell(u)| \leq \sup_n \|ux_n\|$. Look at $Z \subseteq c_o(Y)$ given by

$$Z = \{(ux_n) : u \in \mathcal{L}(X;Y)\};$$

Z is a linear space and if we define $\hat{\ell}$ on Z by $\hat{\ell}((ux_n)) = \ell(u)$, then $\hat{\ell}$ is continuous on Z, viewed as a subspace of $c_o(Y)$. The Hahn-Banach theorem tells us we can extend $\hat{\ell}$ to a member L of $c_o(Y)^*$ without increasing norm even; naturally, L can be realized as a member of $\ell^1(Y^*) = c_o(Y)^*$ and so L's value at $(y_n) \in c_o(Y)$ is given by $L(y_n) = \sum_n y_n^*(y_n)$ for some $(y_n^*) \in \ell^1(Y^*)$. Back to ℓ: $\ell(u) = \hat{\ell}((ux_n)) = L((ux_n)) = \sum_n y_n^*(u(x_n))$. Okay? $\square$

Now we'll get on with the proof of our alternative descriptions of accessibility.

Proof of Theorem 1.4.18. Suppose X is not accessible. Then id_X is not in the closure of $\mathcal{F}(X;X)$ relative to the (locally convex linear) topology of uniform convergence on compact subsets of X; it follows that there is a linear functional ℓ on $\mathcal{L}(X;X)$ that is continuous with respect to the topology of uniform convergence on compacta, vanishes on $\mathcal{F}(X;X)$, but takes value 1 at id_X. ℓ must be of the form

$$\ell(u) = \sum_n x_n^*(u(x_n))$$

for some judiciously chosen sequences $(x_n^*) \subseteq X^*, (x_n) \subseteq X$ with

$$\sum_n \|x_n^*\|\|x_n\| < \infty.$$

Take any $x \in X$ and $x^* \in X^*$:

$$0 = \ell(x^* \otimes x) = \sum_n x_n^*(x)x^*(x_n) = x^*(\sum_n x_n^*(x)x_n);$$

therefore, $\displaystyle\sum_n x_n^*(x)x_n = 0$ for all $x \in X$. Yet

$$1 = \ell(id_X) = \sum_n x_n^*(x_n).$$

X does not satisfy (A_7).

Suppose X is accessible. Then id_X is in the closure of $\mathcal{F}(X;X)$ relative to the topology of uniform convergence on compact subsets of X. If we take $(x_n^*) \subseteq X^*$ and $(x_n) \subseteq X$ such that $\sum_n \|x_n^*\|\|x_n\| < \infty$ and $\sum_n x_n^*(x)x_n = 0$ for all $x \in X$, then for any $x^* \in X^*$ we have $x^*(\sum_n x_n^*(x)x_n) = 0$; it follows that $\sum_n x_n^* \otimes x_n$ defines a linear functional on $\mathcal{L}(X;X)$ that vanishes at each $x^* \otimes x$ hence on $\mathcal{F}(X;X)$ and therefore everywhere. Evaluating $\sum_n x_n^* \otimes x_n$ at id_X gives $0 = (\sum_n x_n^* \otimes x_n)(id_X) = \sum_n x_n^*(x_n)$. This is ($A_7$).

We've shown that accessibility of X is equivalent to condition (A_7). Now let's take aim at (A_6)'s equivalence with accessibility.

Suppose the inclusion of $X^* \overset{\wedge}{\otimes} X$ into $\mathcal{L}(X;X)$ is $1-1$. Let $(x_n^*) \subseteq X^*$ and $(x_n) \subseteq X$ satisfy $\sum_n \|x_n^*\|\|x_n\| < \infty$ as well as $\sum_n x_n^*(x)x_n = 0$ for all $x \in X$. Then $\sum_n x_n^* \otimes x_n \in X^* \overset{\wedge}{\otimes} X$ and, as an operator, $\sum_n x_n^* \otimes x_n$ is 0; it follows that $\sum_n x_n^* \otimes x_n = 0$ as a member of $X^* \overset{\wedge}{\otimes} X$ and so $\varphi(\sum_n x_n^* \otimes x_n) = 0$ for every $\varphi \in B(X^*, X) = (X^* \overset{\wedge}{\otimes} X)^*$. Try $\varphi = tr$: $tr(x^*, x) = x^*(x)$; then

$$0 = tr(\sum_n x_n^* \otimes x_n) = \sum_n tr(x_n^* \otimes x_n) = \sum_n x_n^*(x_n).$$

Finally, we suppose (A_7) holds, $u \in X^* \overset{\wedge}{\otimes} X$ and the operator $\tilde{u} : X \to X$ induced by u is the 0 operator. Since $u \in X^* \overset{\wedge}{\otimes} X$, we can find $(x_n^*) \in \ell^1(X^*)$, $(x_n) \in c_0(X)$ such that $u = \sum_n x_n^* \otimes x_n$; of course, $\tilde{u}(x) = \sum_n x_n^*(x)x_n = 0$ for all $x \in X$. Now, if $\varepsilon > 0, (x_n)$ is norm null so there exists a $w \in \mathcal{F}(X;X)$ so that $\|wx_n - x_n\| < \varepsilon/(1 + \sum_n \|x_n^*\|)$, for all n; after all, (A_7) does imply X is accessible! It follows that

$$|\sum_n x_n^* \otimes x_n - \sum_n x_n^* \otimes wx_n|_\wedge \leq \varepsilon \sum_n \|x_n^*\|.$$

But here's the catch:

$$0 = w(\tilde{u}(x)) = w(\sum_n x_n^*(x)x_n) = \sum_n x_n^*(x)w(x_n) = (\sum_n x_n^* \otimes w(x_n))(x)$$

and so $\sum_n x_n^* \otimes wx_n$ is zero as a member of $\mathcal{L}(X;X)$. But $\sum_n x_n^* \otimes wx_n$, despite its deceiving looks, is a member of $\mathcal{F}(X;X) = X^* \otimes X$ and there's no subtlety about this: the range of $\sum_n x_n^* \otimes wx_n$ is contained in the range of w, a finite dimensional subspace of X. In other words,

$$\begin{aligned}
|\sum_n x_n^* \otimes x_n|_\wedge &= |\sum_n x_n^* \otimes x_n - \sum_n x_n^* \otimes wx_n + \sum_n x_n^* \otimes wx_n|_\wedge \\
&= |\sum_n x_n^* \otimes x_n - \sum_n x_n^* \otimes wx_n|_\wedge \\
&\leq \varepsilon \sum_n \|x_n^*\|.
\end{aligned}$$

$\varepsilon > 0$ is arbitrary, so we conclude that $|\sum_n x_n^* \otimes x_n|_\wedge = 0$ or $u = 0$ in $X^* \overset{\wedge}{\otimes} X$. $\square$

Having worked so hard to understand accessibility let's derive a couple of useful consequences. The first consequence to be drawn comes directly from condition (A_7).

Corollary 1.4.20. *If X^* is accessible, then so is X.*

Proof. We test X for the symptoms described in (A_7): Take $(x_n^*) \subseteq X^*$ and $(x_n) \subseteq X$ such that $\sum_n \|x_n^*\|\|x_n\| < \infty$ and $\sum_n x_n^*(x)x_n = 0$ for each $x \in X$. Then

$\sum_n \|j_X x_n\|\|x_n^*\| < \infty$ and for any $x \in X$ and $x^* \in X^*$,

$$\sum_n (j_X x_n)(x^*)x_n^*(x) = \sum_n x^*(x_n)x_n^*(x) = x^*(\sum_n x_n^*(x)x_n) = 0.$$

By (A_7) (for X^*),

$$0 = \sum_n j_X(x_n)(x_n^*) = \sum_n x_n^*(x_n).$$

□

Concerning our interpretations of members of $X^* \overset{\vee}{\otimes} Y$ as compact linear operators we have the following.

THEOREM 1.4.21. *If either X^* or Y is accessible, then $X^* \overset{\vee}{\otimes} Y$ coincides with $\mathcal{K}(X;Y)$.*

PROOF. The situation when Y is accessible is already proved in Theorem 1.3.11 ((A)'s equivalence with (A_5)).

Now suppose X^* is accessible, let $u : X \to Y$ be a compact linear operator and suppose $\varepsilon > 0$ has been given. Now $u^* : Y^* \to X^*$ is a compact linear operator thanks to Schauder's theorem and so there is a finite rank bounded linear operator $v : X^* \to X^*$ such that $\|x^* - vx^*\| \leq \varepsilon$ for all $x^* \in \overline{u^* B_{Y^*}}$, thanks to X^*'s accessibility. Again, $\|u^* - vu^*\| \leq \varepsilon$ and vu^* is a bounded linear operator with finite dimensional range; it follows that $u^{**}v^* = (vu^*)^*$ is a finite rank bounded linear operator and

$$\|u^{**} - u^{**}v^*\| = \|u^* - vu^*\| \leq \varepsilon.$$

From this it follows easily that $\|u - u^{**}v^* \,|_X\,\| \leq \varepsilon$. □

It is worth mentioning now that the above actually characterizes the accessibility of X^*, that is, one can show that X^* *is accessible if and only if for every Banach space Y, $X^* \overset{\vee}{\otimes} Y = \mathcal{K}(X;Y)$.*

There are conditions similar to those formulated in (A_6) and (A_7) that characterize metric accessibility, and such conditions are the objects of attention in the next result.

THEOREM 1.4.22. *Each of the following statements is equivalent to the statement that the Banach space X is metrically accessible.*

(MA_6) *The natural inclusion of $X \overset{\wedge}{\otimes} X^*$ into $\mathcal{B}^\wedge(X^*, X)$ is an isometry.*

(MA_7) *For each Banach space Y, the natural inclusion of $X \overset{\wedge}{\otimes} Y$ into $\mathcal{B}^\wedge(X^*, Y^*)$ is an isometry.*

PROOF. We've already noted that (MA_7) is a consequence of the metric accessibility of X and $\wedge$ in Proposition 1.3.5; for (MA_6) we observe the following

$$\begin{tikzcd} X \overset{\wedge}{\otimes} X^* \arrow[r, hook] \arrow[rr, hook, bend right] & \mathcal{B}^\wedge(X^*, X) \arrow[r, hook] & \mathcal{B}^\wedge(X^*, X^{**}) \end{tikzcd}$$

Theorem 1.4.6 tells us $\mathcal{B}^\wedge(X^*, X) \hookrightarrow \mathcal{B}^\wedge(X^*, X^{**})$ is an isometry while (MA_7) ensures us that $X \overset{\wedge}{\otimes} X^* \hookrightarrow \mathcal{B}^\wedge(X^*, X^{**})$ is an isometry; little choice is left for $X \overset{\wedge}{\otimes} X^* \hookrightarrow \mathcal{B}^\wedge(X^*, X)$ to be but an isometry.

The dual of $X \overset{\wedge}{\otimes} X^*$ is $\mathcal{B}(X,X^*) = \mathcal{L}(X;X^{**})$ and the dual of $X^* \overset{\vee}{\otimes} X$ is $\mathcal{B}^\wedge(X^*,X)$. If ($\mathrm{MA}_6$) is assumed, then $B_{X^*\overset{\vee}{\otimes}X}$ must be dense in $B_{\mathcal{L}(X;X^{**})}$ in the topology of pointwise convergence (by duality); hence, $B_{X^*\overset{\vee}{\otimes}X}$ is dense in $B_{\mathcal{L}(X;X)}$ in that same topology. But this implies $B_{X^*\overset{\vee}{\otimes}X}$ is dense in $B_{\mathcal{L}(X;X)}$ in the topology of uniform convergence on norm compact subsets of X, thanks to the total boundedness of such sets in Banach spaces. This, though, just says X is metrically accessible. $\square$

As in the case of accessibility we have metric accessibility passing from X^* to X.

Corollary 1.4.23. *If X^* is metrically accessible, then so is X.*

Proof. Look at the following picture:

$$\begin{tikzcd}
X \overset{\wedge}{\otimes} X^* \arrow[r] \arrow[d] & \mathcal{B}^\wedge(X^*,X) \arrow[d] \\
X^{**} \overset{\wedge}{\otimes} X^* \arrow[r] & \mathcal{B}^\wedge(X^*,X^{**})
\end{tikzcd}$$

Vertical maps are isometries thanks to Theorem 1.4.6. The bottom of it all is an isometry because X^* is metrically accessible so (MA_6) applies for X^*. This forces the top to be an isometry. All maps are the natural ones, naturally. $\square$

1.5. α-nuclear forms and operators

Let α be a $\otimes$-norm and suppose that X and Y are Banach spaces. Consider the natural map $\nu : X^* \overset{\alpha}{\otimes} Y^* \hookrightarrow \mathcal{B}^\alpha(X,Y)$. Thanks to the diagram

$$\begin{tikzcd}
X^* \overset{\alpha}{\otimes} Y^* \arrow[r, hook, "\nu"] \arrow[rr, bend right, "\text{norm} \le 1"'] & \mathcal{B}^\alpha(X,Y) \arrow[r, hook, "\text{isometry}"'] & \mathcal{B}^\alpha(X^{**},Y^{**})
\end{tikzcd}$$

we see that ν has norm at most 1. Any $\varphi \in \mathcal{B}^\alpha(X,Y)$ that is in ν's range is called *α-nuclear*; we equip the space $\mathcal{B}_\alpha(X,Y)$ of all α-nuclear bilinear forms with the norm induced by $(X^* \overset{\alpha}{\otimes} Y^*)/\ker(\nu)$. It follows that the α-nuclear norm $N_\alpha(\varphi)$ of $\varphi \in \mathcal{B}_\alpha(X,Y)$ is given by

$$N_\alpha(\varphi) = \inf\left\{ |u^*|_{X^*\overset{\alpha}{\otimes}Y^*} : \nu(u^*) = \varphi \right\}.$$

Of course if $\varphi \in \mathcal{B}_\alpha(X,Y)$, then $\|\varphi\|_\vee \le \|\varphi\|_\alpha \le N_\alpha(\varphi)$.

Again, consider the natural map $\tilde{\nu} : X^* \overset{\alpha}{\otimes} Y \hookrightarrow \mathcal{L}^\alpha(X;Y)$; a look at the diagram

$$\begin{tikzcd}
X^* \overset{\alpha}{\otimes} Y \arrow[r, hook, "\tilde{\nu}"] \arrow[rrr, bend right, "\text{norm at most 1}"'] & \mathcal{L}^\alpha(X;Y) \arrow[r, hook, "\text{isometry}"'] & \mathcal{B}^\alpha(X,Y^*) \arrow[r, hook, "\text{isometry}"'] & \mathcal{B}^\alpha(X^{**},Y^*)
\end{tikzcd}$$

tells us that the map $\tilde{\nu}$ has norm at most 1. Any $u \in \mathcal{L}^\alpha(X;Y)$ that is in $\tilde{\nu}$'s range is called an *α-nuclear operator* and we endow the space $\mathcal{L}_\alpha(X;Y)$ of all α-nuclear operators with the norm induced by $\tilde{\nu}$ from $X^* \overset{\alpha}{\otimes} Y/\ker(\tilde{\nu})$. So if $u \in \mathcal{L}_\alpha(X;Y)$, then the α-nuclear norm $N_\alpha(u)$ of u is given by

$$N_\alpha(u) = \inf\left\{|v|_{X^* \overset{\alpha}{\otimes} Y} : \tilde{\nu}(v) = u\right\}.$$

Both $(\mathcal{B}_\alpha(X,Y), N_\alpha)$ *and* $(\mathcal{L}_\alpha(X;Y), N_\alpha)$ *are Banach spaces.*

PROPOSITION 1.5.1. *Let* $\varphi \in \mathcal{B}(X,Y)$ *and define* $u_\varphi : X \to Y^*$ *by* $u_\varphi(x)(y) = \varphi(x,y)$. *Then* $\varphi \in \mathcal{B}_\alpha(X;Y)$ *if and only if* $u_\varphi \in \mathcal{L}_\alpha(X;Y^*)$ *with* $N_\alpha(\varphi) = N_\alpha(u_\varphi)$.

PROOF. The key to why this is so is the following commutative diagram:

$$\begin{tikzcd}
 & \mathcal{B}^\alpha(X,Y) \arrow[dd, leftrightarrow, "\substack{\text{natural}\\ \text{isometry}}"] & \varphi \arrow[dd, leftrightarrow] \\
X^* \overset{\alpha}{\otimes} Y^* \arrow[ur, "\nu"] \arrow[dr, "\tilde{\nu}"'] & & \\
 & \mathcal{L}^\alpha(X;Y^*) & u_\varphi
\end{tikzcd}$$

After all, φ is in ν's range precisely when u_φ is in $\tilde{\nu}$'s range and $\ker\nu = \ker\tilde{\nu}$. $\square$

All is not completely analogous to the situation encountered with α-integral forms and α-integral operators. We did *not* define $u : X \to Y$'s membership in $\mathcal{L}_\alpha(X;Y)$ by demanding that φ_u, the bilinear functional on $X \times Y^*$ induced by u via $\varphi_u(x,y^*) = y^*u(x)$, be in $\mathcal{B}_\alpha(X,Y^*)$ and generally this is *not* the case. The best that can be said is just this:

PROPOSITION 1.5.2. *If* $u : X \to Y$ *is an* α*-nuclear operator, then* φ_u *is an* α*-nuclear form on* $X \times Y^*$ *with* $N_\alpha(\varphi_u) \leq N_\alpha(u)$*; in turn, if* φ_u *is an* α*-nuclear form on* $X \times Y^*$*, then* $j_Y \circ u : X \to Y^{**}$ *is an* α*-nuclear operator with* $N_\alpha(j_Y \circ u) \leq N_\alpha(\varphi_u)$.

PROOF. To see why this is so, assume $u : X \to Y$ is an α-nuclear operator and picture this:

$$\begin{tikzcd}
X^* \overset{\alpha}{\otimes} Y \arrow[r, hook, "id_{X^*} \otimes j_Y"] \arrow[d, "\tilde{\nu}"'] & X^* \overset{\alpha}{\otimes} Y^{**} \arrow[d, "\nu"] \\
\mathcal{L}^\alpha(X;Y) \arrow[r] & \mathcal{B}^\alpha(X,Y^*) \\[-20pt]
u \arrow[r, mapsto] & \varphi_u
\end{tikzcd}$$

There must be $\tilde{u} \in X^* \overset{\alpha}{\otimes} Y$ so that

$$\tilde{\nu}\tilde{u} = u.$$

But then $\varphi_u = \nu\left((id_{X^*} \otimes j_Y)(\tilde{u})\right)$ so that $\varphi_u \in \mathcal{B}_\alpha(X,Y^*)$ and

$$N_\alpha(\varphi_u) \leq \left|(id_{X^*} \otimes j_Y)(\tilde{u})\right|_{X^* \overset{\alpha}{\otimes} Y^{**}} \leq |\tilde{u}|_{X^* \overset{\alpha}{\otimes} Y}$$

from which it follows that

$$N_\alpha(\varphi_u) \leq N_\alpha(u)$$

since

$$N_\alpha(u) = \inf\left\{|\tilde{u}|_{X^* \overset{\alpha}{\otimes} Y} : \tilde{\nu}\tilde{u} = u\right\}.$$

The second assertion, regarding $\varphi_u \in \mathcal{B}_\alpha(X, Y^*)$ implying $j_Y \circ u \in \mathcal{L}_\alpha(X; Y^{**})$ is an immediate consequence of the fact that $j_Y \circ u$ is just the operator induced by φ_u and we saw that starting from an α-nuclear bilinear form, one is always led to an α-nuclear linear operator of the same norm. $\square$

Regarding compositions we have:

Proposition 1.5.3. *If $\varphi \in \mathcal{B}_\alpha(X, Y), u \in \mathcal{L}(W; X)$ and $v \in \mathcal{L}(Z; Y)$, then*

$$\varphi \circ (u \otimes v) \in \mathcal{B}_\alpha(W, Z) \text{ with } N_\alpha\left(\varphi \circ (u \otimes v)\right) \leq \|u\|\|v\| N_\alpha(\varphi).$$

Similarly, if $u : W \to X, v : X \to Y$ and $w : Y \to Z$ are bounded linear operators with $v \in \mathcal{L}_\alpha(X; Y)$, then $wvu \in \mathcal{L}_\alpha(W; Z)$ with $N_\alpha(wvu) \leq \|w\| N_\alpha(v)\|u\|$.

Proof. Let's look at the first of these assertions with the following commutative diagram as our guide:

$$\begin{tikzcd}
\varphi \arrow[r, mapsto] & \varphi \circ (u \otimes v) \\
\mathcal{B}^\alpha(X, Y) \arrow[r] & \mathcal{B}^\alpha(W, Z) \\
X^* \overset{\alpha}{\otimes} Y^* \arrow[u, "\nu"] \arrow[r, "u^* \otimes v^*"'] & W^* \overset{\alpha}{\otimes} Z^* \arrow[u, "\nu"]
\end{tikzcd}$$

If you start with $\varphi \in \mathcal{B}_\alpha(X, Y)$, then there must be a $\tilde{u} \in X^* \overset{\alpha}{\otimes} Y^*$ so that $\nu\tilde{u} = \varphi$. But then $\varphi \circ (u \otimes v)$ must be $\nu\left((u^* \otimes v^*)(\tilde{u})\right)$ and so $\varphi \circ (u \otimes v) \in \mathcal{B}_\alpha(W, Z)$ with

$$N_\alpha\left(\varphi \circ (u \otimes v)\right) \leq \left|(u^* \otimes v^*)(\tilde{u})\right|_{W^* \overset{\alpha}{\otimes} Z^*} \leq \|u^*\|\|v^*\||\tilde{u}|_{X^* \overset{\alpha}{\otimes} Y^*}$$

which, since $\tilde{u}$ is arbitrarily chosen to satisfy $\nu(\tilde{u}) = \varphi$, gives us

$$N_\alpha\left(\varphi \circ (u \otimes v)\right) \leq \|u^*\|\|v^*\| N_\alpha(\varphi) \ = \|u\|\|v\| N_\alpha(\varphi).$$

Similar reasoning based (if you please) on the commutative diagram

$$\begin{tikzcd}
a \arrow[r, mapsto] & wau \\
\mathcal{L}^\alpha(X; Y) \arrow[r] & \mathcal{L}^\alpha(W; Z) \\
X^* \overset{\alpha}{\otimes} Y \arrow[u, "\tilde{\nu}"] \arrow[r, "u^* \otimes w"'] & W^* \overset{\alpha}{\otimes} Z \arrow[u, "\tilde{\nu}"']
\end{tikzcd}$$

will lead to a proof of the second assertion. $\square$

Accessibility soon plays an important role in the study of α-nuclearity. For instance:

Proposition 1.5.4. *If either X^* or Y is accessible, then the natural inclusion $X^* \overset{\alpha}{\otimes} Y \hookrightarrow \mathcal{L}(X; Y)$ is $1-1$.*

Of course, we have already seen this to be so for $\alpha = \wedge$ and we'll have recourse to call upon this in the proof of the more general result just claimed.

Proof. Let's denote by $\tilde{u}$ the operator from X to Y determined by $u \in X^* \overset{\alpha}{\otimes} Y$. Suppose that $\tilde{u} = 0$. We want to show that $u = 0$ in $X^* \overset{\alpha}{\otimes} Y$. There is but one approach available to us: duality. We plan to show that given any member φ of $(X^* \overset{\alpha}{\otimes} Y)^* = \mathcal{B}^{\alpha^*}(X^*, Y)$, then $\varphi(u) = 0$.

So take such a $\varphi \in \mathcal{B}^{\alpha^*}(X^*, Y)$; φ induces an operator $v_\varphi \in \mathcal{L}^{\alpha^*}(X^*; Y^*)$ and so a natural map is presented to us:

$$\Phi : \mathcal{B}(Y^*, Y) \to \mathcal{B}^{\alpha^*}(X^*, Y) \quad \text{given by} \quad \Phi(\psi) = \psi \circ (v_\varphi \otimes id_Y).$$

Φ is a bounded linear operator for which $\|\Phi\| \leq \|v_\varphi\|_{\alpha^*} = \|\varphi\|_{\alpha^*}$. Notice that the following diagram is commutative:

$$\begin{tikzcd}
X^* \otimes Y \arrow[r, "v_\varphi \otimes id_Y"] \arrow[d, "\text{inclusion}"'] & Y^* \overset{\wedge}{\otimes} Y \arrow[d, "\text{inclusion}"] \\
(X^* \overset{\alpha}{\otimes} Y)^{**} \arrow[d, equal] & (Y^* \overset{\wedge}{\otimes} Y)^{**} \arrow[d, equal] \\
\mathcal{B}^{\alpha^*}(X^*, Y)^* \arrow[r, "\Phi^*"'] & \mathcal{B}(Y^*, Y)^*
\end{tikzcd}$$

From this we see that $v_\varphi \otimes id_Y$ is a bounded linear operator from $(X^* \otimes Y, \alpha)$ to $(Y^* \otimes Y, \wedge)$(!) with

$$\|v_\varphi \otimes id_Y\|_{\mathcal{L}((X^* \otimes Y, \alpha);(Y^* \otimes Y, \wedge))} \leq \|\varphi\|_{\alpha^*};$$

so that $v_\varphi \otimes id_Y$ extends naturally to an operator taking $X^* \overset{\alpha}{\otimes} Y$ into $Y^* \overset{\wedge}{\otimes} Y$, an operator having norm $\leq \|\varphi\|_{\alpha^*}$. It is easy to see that for $w \in X^* \overset{\alpha}{\otimes} Y$,

$$\varphi(w) = tr\left((v_\varphi \otimes id_Y)(w)\right)!$$

But $(v_\varphi \otimes id_Y)(w)$, as an operator on Y is just $\tilde{w} \circ v_\varphi^* \mid_Y$ and so if $u \in X^* \overset{\alpha}{\otimes} Y$ is such that $\tilde{u} = 0$, then $\tilde{u} \circ v_\varphi^* \mid_Y = 0$ so that $(\widetilde{v_\varphi \otimes id_Y})(u) = 0$, where

$$(v_\varphi \otimes id_Y)(u) \in Y^* \overset{\wedge}{\otimes} Y.$$

Supposing Y is accessible, tells us that $(v_\varphi \otimes id_Y)(u) = 0$ in $Y^* \overset{\wedge}{\otimes} Y$, forcing $\varphi(u) = 0$.

If X^* is accessible, then one travels a similar path. Again, let $u \to \tilde{u}$ denote the interpretation of $u \in X^* \overset{\alpha}{\otimes} Y$ as an operator $\tilde{u}$ from X to Y and suppose $\tilde{u} = 0$. Take $\varphi \in \left(X^* \overset{\alpha}{\otimes} Y\right)^* = \mathcal{B}^{\alpha^*}(X^*, Y)$. On the one hand, φ induces an operator $v_\varphi \in \mathcal{L}^{\alpha^*}(X^*; Y^*)$ and, on the other hand, ${}^t\varphi \in \mathcal{B}^{\overset{\vee}{\alpha}}(Y, X^*)$ induces an operator $w_{{}^t\varphi} \in \mathcal{L}^{\overset{\vee}{\alpha}}(Y; X^{**})$; it is easy to verify that $w_{{}^t\varphi} = v_\varphi^* \mid_Y$. As in the first paragraphs, $\left(w_{{}^t\varphi} \otimes id_{X^*}\right)({}^t u)$ finds itself in $X^{**} \overset{\wedge}{\otimes} X^*$ and

$$tr\left((w_{{}^t\varphi} \otimes id_{X^*})({}^t u)\right) = \varphi(u).$$

But $\tilde{u} = 0$ and $\left(\widetilde{w_{{}^t\varphi} \otimes id_{X^*}}\right)({}^t u) : X^* \to X^*$ is just $\tilde{u}^* \circ v_\varphi$, so $\left(w_{{}^t\varphi} \otimes id_{X^*}\right)({}^t u)$ must be zero as a member of $X^{**} \overset{\wedge}{\otimes} X^*$ — thanks to X^* being accessible. $\varphi(u) = 0$, too, and again we deduce from duality that $u = 0$ as a member of $X^* \overset{\alpha}{\otimes} Y$. $\square$

Of course, a consequence of the injective behavior of the inclusion of $X^* \overset{\alpha}{\otimes} Y$ into $\mathcal{L}(X; Y)$ when either X^* or Y is accessible concerns α-nuclear operators:

PROPOSITION 1.5.5. *If either X^* or Y is accessible, then $X^* \overset{\alpha}{\otimes} Y \hookrightarrow \mathcal{L}^\alpha(X;Y)$ is one-to-one and $X^* \overset{\alpha}{\otimes} Y = \mathcal{L}_\alpha(X;Y)$ isometrically.*

After all, the kernel of $\tilde{\nu}$ is but a single point, 0, and so the norm N_α is just the norm α in $X^* \overset{\alpha}{\otimes} Y$.

Generally:

PROPOSITION 1.5.6. *If $u : X \to Y$ is α-nuclear, then $u^* : Y^* \to X^*$ is ${}^t\alpha$-nuclear.*

In fact, for u to be α-nuclear there must be a $v_u \in X^* \overset{\alpha}{\otimes} Y$ so that $u = \tilde{v}_u$ in the notation employed above. But $v_u \in X^* \overset{\alpha}{\otimes} Y$ ensures ${}^t(v_u) \in Y \overset{t_\alpha}{\otimes} X^*$ and so $\widetilde{{}^t(v_u)} : Y^* \to X^*$ is ${}^t\alpha$-nuclear. It is true that $\widetilde{({}^t v_u)} = u^*$, of course. *Notice that* $N_{t_\alpha}(u^*) \leq N_\alpha(u)$.

If $u \in \mathcal{L}(X;Y)$ and $u^* \in \mathcal{L}_{t_\alpha}(Y^*;X^*)$ then $j_Y u \in \mathcal{L}_\alpha(X^{**};Y^{**})$; it would be nice if actually $u \in \mathcal{L}_\alpha(X;Y)$ and this can be deduced *sometimes*. For instance:

PROPOSITION 1.5.7. *If X^* is accessible and $u : X \to Y$ has a ${}^t\alpha$-nuclear adjoint, then u is α-nuclear. Moreover, $N_\alpha(u) = N_{t_\alpha}(u^*)$.*

PROOF. Suppose u^* is ${}^t\alpha$-nuclear; since ${}^t\alpha \geq \vee, u^*$ is a compact linear operator and so u is, too. This tells us that u^* is actually weak*-weak continuous. Now u^* is the operator in $\mathcal{L}^{t_\alpha}(Y^*;X^*)$ associated with some $v \in Y^{**} \overset{t_\alpha}{\otimes} X^*$. Actually, v belongs to $Y \overset{t_\alpha}{\otimes} X^*$, a closed linear subspace of $Y^{**} \overset{t_\alpha}{\otimes} X^*$, thanks to Corollary 1.4.8. In fact, suppose $\varphi \in \left(Y^{**} \overset{t_\alpha}{\otimes} X^*\right)^* = \mathcal{B}^{\overset{\vee}{\alpha}}(Y^{**}, X^*)$ vanishes on $Y \overset{t_\alpha}{\otimes} X^*$ and let $w_\varphi : Y^{**} \to X^{**}$ be the operator induced by φ. Just like in the previous proofs involving the interplay of accessibility and α-nuclearity, $w_\varphi \otimes id_{X^*}$ takes $Y^{**} \overset{t_\alpha}{\otimes} X^*$ into $X^{**} \overset{\wedge}{\otimes} X^*$ in a continuous linear fashion with

$$tr\left((w_\varphi \otimes id_{X^*})(v)\right) = \varphi(v).$$

Now, proceeding by a check against elementary tensors, one soon realizes that

$$(\widetilde{w_\varphi \otimes id}_{X^*})(v)^* = w_\varphi \circ u^{**}$$

where $((\widetilde{w_\varphi \otimes id}_{X^*})(v)$ is the operator from X^* to itself induced by

$$\left(w_\varphi \otimes id_{X^*}\right)(v) \in X^{**} \overset{\wedge}{\otimes} X^*.$$

Since w_φ vanishes on Y and since $u^{**}(X^{**})$ is contained in Y (actually $j_Y Y$, a negligible difference here), $(\widetilde{w_\varphi \otimes id}_{X^*})(v)^* = 0$ and so $(\widetilde{w_\varphi \otimes id}_{X^*})(v) = 0$. But X^* is accessible so $\left(w_\varphi \otimes id_{X^*}\right)(v) = 0$ in $X^{**} \overset{\wedge}{\otimes} X^*$ and this soon tells us that $\varphi(v) = 0$. Every $\varphi \in \left(Y^{**} \overset{t_\alpha}{\otimes} X^*\right)^*$ vanishing on $Y \overset{t_\alpha}{\otimes} X^*$ vanishes at v as well: $v \in Y \overset{t_\alpha}{\otimes} X^*$ with $N_\alpha(u^*) = |v|_{Y \overset{t_\alpha}{\otimes} X^*}$ thanks to the one-to-one behavior of the inclusion map $Y^{**} \overset{t_\alpha}{\otimes} X^* \hookrightarrow \mathcal{L}^{t_\alpha}(Y^*;X^*)$. Ah ha! ${}^t v \in X^* \overset{\alpha}{\otimes} Y$ and it is plain that u is the operator defined by ${}^t v$ with $N_\alpha(u) \leq |{}^t v|_{X^* \overset{\alpha}{\otimes} Y} = |v|_{Y \overset{t_\alpha}{\otimes} X^*} = N_{{}^t\alpha}(u^*)$. Enough said. $\square$

Another role well-played by accessibility:

PROPOSITION 1.5.8. *Suppose $u : X \to Y$ is α-nuclear into Y^{**} (or, more precisely suppose $j_Y u : X \to Y^{**}$ is α-nuclear) and X^* is accessible; then u is α-nuclear with*

$$N_\alpha(u) = N_\alpha(j_Y u).$$

PROOF. $j_Y u$ is just $u^{**} |_X$ so this is an immediate consequence of our previous efforts. □

1.6. The Dvoretzky-Rogers theorem, Grothendieck-style

The famous theorem of A. Dvoretzky and C. Rogers was held in appropriately high esteem by Grothendieck, so high that he devoted an entire paper discussing it, extending and sharpening it and applying it to compare $\ell^p \overset{\wedge}{\otimes} X$, ℓ^p_X and $\ell^p \overset{\vee}{\otimes} X$ for $1 < p < \infty$ when X is infinite dimensional.

1.6.1. The fundamental lemma. Now we turn to a discussion of Grothendieck's rendering of the Dvoretzky-Rogers Theorem. We'll use some facts from elementary integral geometry. For those readers not familiar with these results we refer to Appendix B for an introduction to this special part of integral geometry containing all the necessary results with proofs.

LEMMA 1.6.1 (Grothendieck (1953/1956b), Lemma, p. 97). *Let E be an n-dimensional Banach space. There exist $x_1, \dots, x_n \in S_E$ such that if $1 \le r \le n$, then for any $\lambda_1, \dots, \lambda_r \in \mathbb{R}$,*

$$\|\sum_{i=1}^{r} \lambda_i x_i\| \le M_r \|(\lambda_i)\|_{\ell^2_r},$$

where

$$M_r = 1 + \frac{1}{n}(1^2 + 2^2 + \dots + (r-1)^2)^{1/2} \le 1 + \frac{r\sqrt{r}}{n\sqrt{3}}.$$

This is Grothendieck's version of the *Dvoretzky-Rogers Lemma/Theorem*.

PROOF. We start by inscribing inside B_E an ellipsoid of maximum volume. If we apply an invertible linear transformation T to E we may transform this ellipsoid into the closed unit ball of ℓ^2_n; of course, under this transformation B_E is changed too. It is important to note that, because T is invertible, $T(B_E)$ is a closed unit ball of a norm on E and that, in fact, T acts as an isometry from E to $T(E)$ $(= (E, \|\cdot\|_{T(B_E)})$.

This in mind, we may as well assume that the ellipsoid described above *is* $B_{\ell^2_n} = B$.

Here's what we'll do! We're going to find an orthonormal basis $\{u_1, \dots, u_n\}$ and points $\{x_1, \dots, x_n\}$ for which

$$\|u_i\|_E = 1 = \|x_i\|_{\ell^2_n},\ i = 2, \dots, n,$$

with

$$x_i = \sum_{j \le i} a_{i,j} u_j,\ i = 1, \dots, n$$

and

$$a^2_{i,1}, + \dots + a^2_{i,i-1} \le \frac{i-1}{n},\ i = 1, \dots, n.$$

Naturally, the proof will be an induction on i.

Starting with $i = 1$, take $u_1 = x_1$ to be any point of contact of S_E and $S_{\ell_n^2}$; since $B_{\ell_n^2}$ is an ellipsoid of *maximum* volume inscribed in B_E, we can be assured that there is such a point. Extend $\{u_1\}$ to an orthonormal basis $\{u_1, u_2, \ldots, u_n\}$ of ℓ_n^2 in any way you wish; we have achieved the conditions set forth above for $i = 1$.

Now suppose we have found $x_1, \ldots, x_i$ and an orthonormal set $\{u_1, \ldots, u_i\}$ such that for any $j \leq i$,

$$\|x_j\|_E = 1 = \|u_j\|_{\ell_n^2}, \ x_j = \sum_{k \leq j} a_{j,k} u_k$$

and

$$a_{k,1}^2 + \cdots + a_{k,k-1}^2 \leq \frac{k-1}{n}, \ k = 1, \ldots, j.$$

Extend $\{u_1, \ldots, u_i\}$ to an orthonormal basis for ℓ_n^2 as you wish.

Let $\varepsilon > 0$. Look at the ellipsoid C_ε whose members are vectors of the form

$$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$$

where

$$(1+\varepsilon)^{n-i}(a_1^2 + \cdots + a_i^2) + (1 + \varepsilon + \varepsilon^2)^{-i}(a_{i+1}^2 + \cdots + a_n^2) \leq 1.$$

If $S : \ell_n^2 \to \ell_n^2$ is the linear transformation such that

$$Su_k = \begin{cases} \left(\dfrac{1}{\left(\frac{1}{1+\varepsilon}\right)^{\frac{n-i}{2}}} \right) u_k & \text{if } k = 1, \ldots, i, \\[2ex] \left(\dfrac{1}{\left(1+\varepsilon+\varepsilon^2\right)^{\frac{i}{2}}} \right) u_k & \text{if } k = i+1, \ldots, n, \end{cases}$$

then

$$C_\varepsilon = S^{-1}(B).$$

Hence

$$\mathrm{vol}(C_\varepsilon) = \det(S^{-1}) \mathrm{vol}(B),$$

where

$$\det(S^{-1}) = \left(\frac{1}{1+\varepsilon}\right)^{i \cdot \frac{n-i}{2}} (1 + \varepsilon + \varepsilon^2)^{(n-i)\frac{i}{2}} > 1.$$

So C_ε is an ellipsoid, centered at the origin, with $\mathrm{vol} C_\varepsilon > \mathrm{vol} B$. It follows from B's maximality properties that C_ε cannot lie entirely inside B_E. There's a point $p(\varepsilon)$ common to C_ε and S_E. Naturally, $p(\varepsilon) \in S_E$ ensures us that $\|p(\varepsilon)\|_E = 1$. If we suppose

$$p(\varepsilon) = \alpha_1(\varepsilon) u_1 + \cdots + \alpha_n(\varepsilon) u_n,$$

then the fact that $\| \ \|_{\ell_n^2} \geq \| \ \|_E$ (B is inscribed in B_E) tells us

(1) $$\alpha_1(\varepsilon)^2 + \cdots + \alpha_n(\varepsilon)^2 = \|p(\varepsilon)\|_{\ell_n^2}^2 \geq \|p(\varepsilon)\|_E^2 = 1.$$

But $p(\varepsilon)$ is in C_ε, too; so

(2) $$\begin{cases} (1+\varepsilon)^{n-i}\Big(\alpha_1(\varepsilon)^2 + \cdots + \alpha_i(\varepsilon)^2\Big) \\ \qquad + (1+\varepsilon+\varepsilon^2)^{-i}\Big(\alpha_{i+1}(\varepsilon)^2 + \cdots + \alpha_n(\varepsilon)^2\Big) \leq 1 \end{cases}$$

Subtracting the former (1) from the latter (2) reveals

$$(3)\qquad \begin{cases} \Big[(1+\varepsilon)^{n-i}-1\Big]\Big(\alpha_1(\varepsilon)^2+\cdots+\alpha_i(\varepsilon)^2\Big) \\ \qquad\qquad +\Big[(1+\varepsilon+\varepsilon^2)^{-i}-1\Big]\Big(\alpha_{i+1}(\varepsilon)^2+\cdots+\alpha_n(\varepsilon)^2\Big)\le 0 \end{cases}$$

But $p(\varepsilon)\in S_E$, so a judicious choice of $\varepsilon_m \downarrow 0$ provides $x_{i+1}\in S_E$ such that

$$x_{i+1}=\lim_{m\to\infty} p(\varepsilon_m).$$

Express x_{i+1} in terms of $\{u_1,\ldots,u_n\}$:

$$x_{i+1}=a_{i+1,1}u_1+\cdots+a_{i+1,n}u_n;$$

of course, if $1\le k\le n$, then

$$a_{i+1,k}=\lim_{m\to\infty}\alpha_k(\varepsilon_m).$$

It follows from this and the inequality (2) that $x_{i+1}\in S_{\ell_n^2}$ and so x_{i+1} is a point of contact of S_E and $S_{\ell_n^2}$.

A return to the inequality (3) is expected. By the binomial theorem

$$(1+\varepsilon)^{n-i}-1=(n-i)\varepsilon+Q(\varepsilon)\varepsilon^2$$

where $Q(\varepsilon)$ is a polynomial in ε; also,

$$\begin{aligned}(1+\varepsilon+\varepsilon^2)^{-i}-1&=\frac{1}{(1+\varepsilon+\varepsilon^2)^i}\Big[1-(1+\varepsilon+\varepsilon^2)^i\Big]\\ &=\frac{1}{(1+\varepsilon+\varepsilon^2)^i}\Big[(-i)\varepsilon+\tilde{Q}(\varepsilon)\varepsilon^2\Big]\end{aligned}$$

where $\tilde{Q}(\varepsilon)$ is a polynomial in ε. So the inequality (3) may be rewritten in the form:

$$\begin{aligned}&\Big[(n-i)\varepsilon+Q(\varepsilon)\varepsilon^2\Big]\Big(\alpha_1(\varepsilon)^2+\cdots+\alpha_i(\varepsilon)^2\Big)\\ &\qquad\qquad+\Big[\frac{-i\varepsilon+\tilde{Q}(\varepsilon)\varepsilon^2}{(1+\varepsilon+\varepsilon^2)^i}\Big]\Big(\alpha_{i+1}(\varepsilon)^2+\cdots+\alpha_n(\varepsilon)^2\Big)\le 0.\end{aligned}$$

Divide both sides by $\varepsilon>0$ and let ε follow the path blazed by (ε_m); the result is

$$(n-i)\Big(a_{i+1,1}^2+\cdots+a_{i+1,i}^2\Big)+(-i)\Big(a_{i+1,i+1}^2+\cdots+a_{i+1,n}^2\Big)\le 0.$$

Let the dust settle. Look closely and see that

$$\Big(a_{i+1,1}^2+\cdots+a_{i+1,i}^2\Big)\le\frac{i}{n}\Big(a_{i+1,1}^2+\cdots+a_{i+1,n}^2\Big)=\frac{i}{n},$$

which is what we claimed to be able to show. It's good to be king. All that's left is to find a vector u'_{i+1} which when added to $\{u_1,\ldots,u_i\}$ makes $\{u_1,\ldots,u_i,u'_{i+1}\}$ orthonormal and such that $\{u_1,\ldots,u_i,u'_{i+1}\}$ spans the same linear subspace of E that $\{u_1,\ldots,u_i,x_{i+1}\}$ does. Then to keep this machinery greased and ready for the next overhaul, extend the orthonormal set $\{u_1,\ldots,u_i,u'_{i+1}\}$ to an orthonormal basis for E.

Realize that the proof thus far provided linearly independent points $x_1,\ldots,x_n$ in E so that all have norm one in E and, relative to the Hilbertian norm $\|\ \|_2$ induced by the ellipsoid of maximal volume inscribed in B_E, we have that $\|x_i\|_2=1$

for each i as well. What's more, if y_i is the orthogonal projection of x_i onto the subspace generated by $x_1, \dots, x_{i-1}$, then

$$\|y_i\|^2_{\ell^2_n} \leq \frac{i-1}{n}.$$

Put $z_i = x_i - y_i$ and realize that $z_1, \dots, z_n$ are orthogonal and $\|z_i\|_{\ell^2_n} \leq 1$ for each $i = 1, \dots, n$. Now estimate: If $1 \leq r \leq n$ and $(\lambda_1, \dots, \lambda_r) \in \ell^2_r$, then

$$\Big\| \sum_{i=1}^r \lambda_i x_i \Big\| \leq \Big\| \sum_{i=1}^r \lambda_i z_i \Big\| + \Big\| \sum_{i=1}^r \lambda_i y_i \Big\|.$$

Take things "one at a time":

$$\begin{aligned} \Big\| \sum_{i=1}^r \lambda_i z_i \Big\| &\leq \Big\| \sum_{i=1}^r \lambda_i z_i \Big\|_{\ell^2_r} \\ &= \Big(\sum_{i=1}^r \lambda_i^2 \|z_i\|_2^2 \Big)^{1/2} \\ &\leq \Big(\sum_{i=1}^r \lambda_i^2 \Big)^{1/2}, \end{aligned}$$

and

$$\begin{aligned} \Big\| \sum_{i=1}^r \lambda_i y_i \Big\| &\leq \sum_{i=1}^r |\lambda_i| \|y_i\| \\ &\leq \Big(\sum_{i=1}^r |\lambda_i|^2 \Big)^{1/2} \Big(\sum_{i=1}^r \|y_i\|^2 \Big)^{1/2} \\ &\leq \Big(\sum_{i=1}^r |\lambda_i|^2 \Big)^{1/2} \Big(\sum_{i=1}^r \big(\frac{i-1}{n}\big)^2 \Big)^{1/2}. \end{aligned}$$

Enough said. □

Corollary 1.6.2. *Let X be an infinite dimensional Banach space, $k > 1$ and $r \in \mathbb{N}$. Then one can find $x_1, \dots, x_r \in X$ each of norm one such that $\|(x_1, \dots, x_r)\|_{\ell^2_{\mathrm{weak}}(X)} \leq k$.*

Proof. Let $n \in \mathbb{N}$ be chosen so that $n > r$ and

$$1 + \frac{r\sqrt{r}}{\sqrt{3}n} \leq k.$$

Let E be a linear subspace of X of dimension n. Apply Grothendieck's version of the Dvoretzky-Rogers Lemma. □

Theorem 1.6.3 (Grothendieck (1953/1956b), Theorem 2, p. 99). *Let X be an infinite dimensional Banach space, $(a_i)_i$ a sequence of non-negative reals with each $a_i < 1$ and $0 = \lim_i a_i$. Then there is a sequence $(x_i)_i$ in X such that $(x_i)_i \in \ell^2 \overset{\vee}{\otimes} X$ yet $\|x_i\| = a_i$ for each i.*

Proof. Set

$$\alpha = \frac{1 - \sup a_i}{2}.$$

$\alpha > 0$ since $\lim_i a_i = 0$ and each $a_i < 1$. For each $k \in \mathbb{N}$ let i_k be chosen ($> i_{k-1}$) such that

$$a_i < \alpha/2^k, \ if \ i \geq i_k. \tag{4}$$

Apply Corollary 1.6.2 to find for each $k \in \mathbb{N}$ vectors y_i, for $i_{k-1}+1 \leq i \leq i_k$, of norm one such that

$$\|(y_{i_{k-1}+1}, \ldots, y_{i_n})\|_{\ell^2_{\text{weak}}(X)} \leq \frac{1}{1-\alpha}. \tag{5}$$

Put $x_i = a_i y_i$ so that $\|x_i\| = a_i$. Assume $k \geq 2$. Denote by $\mathcal{X}_k$ the sequence in X given by

$$\mathcal{X}_{k,n} = \begin{cases} 0 & \text{if } n \leq i_{k-1}, \\ x_n & \text{if } i_{k-1}+1 \leq n \leq i_k, \\ 0 & \text{if } i_k < n. \end{cases}$$

Notice that, thanks to (4) and (5),

$$\|(\mathcal{X}_{k,n})_n\|_{\ell^2_{\text{weak}}(X)} \leq \frac{\alpha}{2^{k-1}} \frac{1}{1-\alpha},$$

$\mathcal{X}_1$? $\mathcal{X}_1 = (x_1, \ldots, x_{i_1}, 0, 0, \ldots)$. Naturally, since

$$\|(\mathcal{X}_{1,n})_n\|_{\ell^2_{\text{weak}}(X)} \leq \frac{\sup a_i}{1-\alpha} = \frac{1-2\alpha}{1-\alpha},$$

$(\mathcal{X}_k)$ is an absolutely summable sequence in the Banach space $\ell^2 \overset{\vee}{\otimes} X$ whose sum $\mathcal{X}$ satisfies

$$\|(\mathcal{X}_n)\|_{\ell^2_{\text{weak}}(X)} \leq \sum_k \|(\mathcal{X}_{k,n})_n\|_{\ell^2_{\text{weak}}(X)} \leq \frac{1}{1-\alpha}\big(1 - 2\alpha + \alpha \sum_k 2^{1-k}\big) = 1.$$

It is clear that $\mathcal{X} = (x_n)_n$ and so this proof is complete. □

Remark: Since $\ell^2 \overset{\vee}{\otimes} X \subseteq c_o \overset{\vee}{\otimes} X = c_o(X)$, the condition that $(a_i)_i \in c_o$ is essential. The point is that each such sequence serves as the term-by-term length of some sequence in $\ell^2 \overset{\vee}{\otimes} X$ — *regardless of how slowly $(a_i)_i$ tends to zero.*

1.6.2. Consequences.

Theorem 1.6.4 (Grothendieck (1953/1956b), Theorem 3, p. 100). *Let X be an infinite dimensional Banach space, $1 \leq p \leq 2$ and $q : \frac{1}{q} = \frac{1}{p} - \frac{1}{2}$ (thereby forcing q to be $2 \leq q \leq \infty$). Then for any sequence $(a_i)_i$ of non-negative reals with $(a_i)_i \in \ell^q$ (respectively c_o if $q = +\infty$), one can find $(x_i)_i \in \ell^p \overset{\vee}{\otimes} X$ such that $\|x_i\| = a_i$. What's more, given $\varepsilon > 0, (x_i)_i \in \ell^p \overset{\vee}{\otimes} X$ can be chosen so that $\|(x_i)_i\|_{\ell^p_{weak}} \leq \|(a_i)_i\|_q + \varepsilon$.*

Proof. Suppose $2 \leq q < \infty, (a_i)_i \in \ell^q$ and $\varepsilon > 0$. There exists $(\lambda_i)_i \in c_o$ such that $0 \leq \lambda_i \leq 1$ for all i and

$$\left\|\left(\frac{a_i}{\lambda_i}\right)_i\right\|_q \leq \|(a_i)_i\|_q + \varepsilon.$$

Let $b_i = \frac{a_i}{\lambda_i}$ and let $(y_i)_i \in B_{\ell^2 \overset{\vee}{\otimes} X}$ be chosen so that $\|y_i\| = \lambda_i$ for each i; Theorem 1.6.3 lets us do this. Set

$$x_i = b_i y_i.$$

Then $\|x_i\| = a_i$. Hölder's inequality alerts us to $(x_i)_i$'s membership in $\ell^p \overset{\vee}{\otimes} X$ as well as

$$\|(x_i)_i\|_{\ell^p_{\text{weak}}} \le \|(b_i)_i\|_q \|(y_i)_i\|_{\ell^2_{\text{weak}}} \le \|(a_i)_i\|_q + \varepsilon.$$

Tra la, tra la! $\square$

Of course the case where $q = \infty$ is covered already by Theorem 1.6.3.

COROLLARY 1.6.5. *Let X be an infinite dimensional Banach space and $(a_i)_i$ be a square summable sequence of non-negative reals. Then there exists an unconditionally summable sequence $(x_i)_i$ in X so that $\|x_i\| = a_i$. What's more, given $\varepsilon > 0$ the unconditionally summable sequence $(x_i)_i$ can be chosen so that*

$$\|(x_i)_i\|_{\ell^1_{\text{weak}}} \le \|(a_i)_i\|_2 + \varepsilon.$$

COROLLARY 1.6.6. *Let X be an infinite dimensional Banach space and $1 \le p < \infty$. Then $\ell^p \overset{\vee}{\otimes} X$ and $\ell^p(X)$ are not the same. Hence, there is a weakly p-summable sequence of members of X which is not p-summable.*

PROOF. Suppose $1 \le p < 2$. Theorem 1.6.4 kicks in: if $\frac{1}{q} = \frac{1}{p} - \frac{1}{2}$, then $p < q$ and so we can choose $(a_i)_i \in \ell^q \backslash \ell^p$ and apply Theorem 1.6.4.

Suppose $2 \le p < \infty$. Let $(a_i)_i \in c_o \backslash \ell^p$. Use Theorem 1.6.3 to choose $(x_i)_i$ so that $(x_i)_i \in \ell^2 \overset{\vee}{\otimes} X$ and $\|x_i\| = |a_i|$. Then $(x_i)_i \in \ell^p \overset{\vee}{\otimes} X$, but $\sum \|x_i\|^p = +\infty$. $\square$

Remark: The case $p = \infty$ doesn't fall under the spell of the Dvoretzky-Rogers schtick.

Of course $\ell^\infty \overset{\vee}{\otimes} X \subseteq \ell^\infty(X)$, but they're never equal in the case $\dim X = \infty$ thanks to the F. Riesz Lemma. On the other hand, $c_o \overset{\vee}{\otimes} X = c_o(X)$.

A dual version of Theorem 1.6.4 holds.

THEOREM 1.6.7 (Grothendieck (1953/1956b), Theorem 4, p. 101). *Let X be an infinite dimensional Banach space, $2 \le p' \le \infty$ and $q' : \frac{1}{q'} = \frac{1}{p'} + \frac{1}{2}$ (forcing $q' : 1 \le q' \le 2$). Suppose $(a_i)_i$ is a sequence of non-negative reals that is not in $\ell^{q'}$. Then there is a sequence $(z_i)_i$ of members of X so that $\|z_i\| = a_i$ and (z_i) is not in $\ell^{p'} \overset{\wedge}{\otimes} X$.*

PROOF. Notationally speaking, let p, q be given by

$$\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}.$$

Then

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{p'} - \frac{1}{q'} = \frac{1}{p} - \frac{1}{2},$$

and so the spectre of Theorem 1.6.4 rises. Suppose (contrarily) that any time $(x_i)_i$ is a sequence of members of X for which $\|x_i\| = a_i$ for each i, then $(x_i)_i \in \ell^{p'} \overset{\wedge}{\otimes} X$. Then whenever (y_i) is a sequence of members of X and $\|y_i\| \le a_i$, there would be $(\lambda_i)_i \in B_{\ell^\infty}$ so that $y_i = \lambda_i x_i$ for some sequence $(x_i)_i \in \ell^{p'} \overset{\wedge}{\otimes} X$; it immediately follows that $(y_i)_i \in \ell^{p'} \overset{\wedge}{\otimes} X$, too. So given any $(z_i)_i \in c_o(X) = c_o \overset{\vee}{\otimes} X$ the sequence $(a_i z_i)_i$ is in $\ell^{p'} \overset{\wedge}{\otimes} X$: an operator is born; denote by u the operator from $c_o \overset{\vee}{\otimes} X$ to $\ell^{p'} \overset{\wedge}{\otimes} X$ given by

$$u\big((z_i)_i\big) = (a_i z_i)_i.$$

On $c_o \otimes X, u = v \otimes id_X$ where $v : c_o \to \ell^{p'}$ is given by $v\big((\lambda_i)_i\big) = (a_i\lambda_i)_i$, so that $u^* = \big(v \otimes id_X\big)^*$ takes $\big(\ell^{p'} \overset{\wedge}{\otimes} X\big)^*$ into $\big(c_o \overset{\vee}{\otimes} X\big)^*$. But $(c_o \overset{\vee}{\otimes} X)^* = c_o(X)^* = \ell^1(X^*) = \ell^1 \overset{\wedge}{\otimes} X^*$ and $\ell^p \overset{\vee}{\otimes} X^*$ lies isometrically inside $\big(\ell^{p'} \overset{\wedge}{\otimes} X\big)^*$; further, on $\ell^p \otimes X^*, (v \otimes id_X)^*$ is quickly seen to coincide with $v^* \otimes id_{X^*}$ and so u^* acts continuously from $\ell^p \overset{\vee}{\otimes} X^*$ to $\ell^1 \overset{\wedge}{\otimes} X^*$; in fact, if $(x_i^*)_i \in \ell^p \overset{\vee}{\otimes} X^*$, then $u^*\big((x_i^*)_i\big)$ is just the sequence $(a_i x_i^*)_i$. Hence we have that for each $(x_i^*)_i \in \ell^p \overset{\vee}{\otimes} X^*$,

$$\sum a_i \|x_i^*\| = \|u^*\big((x_i^*)_i\big)\|_{\ell^1(X^*)} < \infty. \tag{6}$$

Wow! Since $(a_i)_i \notin \ell^{q'}$ there is a $(b_i)_i \in \ell^q$ such that

$$\sum a_i |b_i| = +\infty.$$

Apply Theorem 1.6.4 to find a sequence $(w_i^*)_i \in \ell^p \overset{\vee}{\otimes} X^*$ such that $\|w_i^*\| = |b_i|$. Plainly,

$$\sum_i a_i \|w_i^*\| = +\infty.$$

But $(w_i^*)_i \in \ell^p \overset{\vee}{\otimes} X^*$ brings (6) to bear and so

$$\Big(\sum a_i|b_i| = \Big) \sum a_i \|w_i^*\| < \infty.$$

OOPS! Our contrariness leads to disaster. We must cease and desist in contrary behavior and conclude to the verity of the theorem. $\square$

Corollary 1.6.8. *Let X be an infinite dimensional Banach space.*

(1) *If $(a_i)_i$ is a sequence of non-negative reals that is not summable, then there is a sequence $(x_i)_i$ of members of X with $\|x_i\| = a_i$ yet $(x_i)_i \notin \ell^2 \overset{\wedge}{\otimes} X$.*
(2) *If $(a_i)_i$ is a sequence of non-negative reals that is not square summable, then there is a sequence $(x_i)_i$ of members of X with $\|x_i\| = a_i$, yet $(x_i)_i \notin \ell^\infty \overset{\wedge}{\otimes} X$.*

Proof. (1) Apply Theorem 1.6.7 to the case $q' = 1$ and $p' = 2$.
(2) Apply Theorem 1.6.7 to the case $q' = 2$ and $p' = +\infty$. $\square$

Corollary 1.6.9. *Let X be an infinite dimensional Banach space and $1 < p < \infty$. Then $\ell^p(X) \neq \ell^p \overset{\wedge}{\otimes} X$.*

Proof. Suppose $2 < p$. Take $(a_i)_i \in \ell^p \backslash \ell^q$ where $\frac{1}{q} = \frac{1}{p} + \frac{1}{2}$. Apply Theorem 1.6.7 to find a sequence of members of X so $\|x_i\| = |a_i|$, yet $(x_i)_i \notin \ell^p \overset{\wedge}{\otimes} X$.

Suppose $1 < p \leq 2$. Choose $(a_i)_i \in \ell^p \backslash \ell^1$. Corollary 1.6.8(1) provides us with a sequence $(x_i)_i$ of members of X so that $\|x_i\| = |a_i|$, yet $(x_i)_i \notin \ell^2 \overset{\wedge}{\otimes} X$. On the one hand, $\|x_i\| = |a_i|$ so $(x_i)_i \in \ell^p(X)$ while, on the other hand, $p < 2$ puts $\ell^p \overset{\wedge}{\otimes} X$ inside (set-wise) $\ell^2 \overset{\wedge}{\otimes} X$ and so $(x_i)_i \notin \ell^p \overset{\wedge}{\otimes} X$. $\square$

To summarize, we have the following:

Theorem 1.6.10. *If $1 < p < \infty$ and X is an infinite dimensional Banach space, then*

$$\ell^p \overset{\wedge}{\otimes} X \subsetneqq \ell^p(X) \subsetneqq \ell^p \overset{\vee}{\otimes} X.$$

Notes:

(1) As we mentioned before, the sequences in $\ell^p \overset{\vee}{\otimes} X$ are exactly the sequences in the subspace $\check{\ell}^p_{\text{weak}}(X)$ of $\ell^p_{\text{weak}}(X)$. An obvious question is: Which sequences in X correspond to the elements of $\ell^p \overset{\wedge}{\otimes} X$? To answer this question we consider the space $\ell^p\langle X\rangle$ of all sequences (x_n) in X such that $\big(x_n^*(x_n)\big) \in \ell^1$ for all $(x_n^*) \in \ell^{p'}_{\text{weak}}(X^*)$. Here, as always, p' is the index conjugate to p. It is not difficult to show that the space $\ell^p\langle X\rangle$ is a Banach space with respect to the norm defined by

$$\|(x_n)\|_{\ell^p\langle X\rangle} = \sup\{\sum_n |x_n^*(x_n)| : \|(x_n^*)\|_{\ell^{p'}_{\text{weak}}(X^*)} \leq 1\}.$$

The space $\ell^p\langle X\rangle$ was introduced by Cohen (1973).

(a) *For $1 < p < \infty$ the spaces $\ell^p\langle X\rangle$ and $\ell^p \overset{\wedge}{\otimes} X$ are isometrically isomorphic.*

(b) $\ell^1\langle X\rangle = \ell^1(X) = \ell^1 \overset{\wedge}{\otimes} X$ *(isometrically).*

We refer to [Fourie and Röntgen (2003)] and [Bu and Diestel (2001)] for details.

(2) There are large classes of tensor norms that are somewhat related to the norms in $\ell^p(X)$ and $\ell^p_{\text{weak}}(X)$. For $u \in X \otimes Y$ we define for any $1 \leq p \leq \infty$:

$$d_p(u) = \inf \|(x_i)\|_{\ell^p(X)} \|(y_i)\|_{\ell^{p'}_{\text{weak}}(Y)},$$
$$w_p(u) = \inf \|(x_i)\|_{\ell^p_{\text{weak}}(X)} \|(y_i)\|_{\ell^{p'}_{\text{weak}}(Y)}$$

where in each of the cases the infimum is taken over all representations of $u = \sum_{i=1}^n x_i \otimes y_i$ where $x_i \in X$ and $y_i \in Y$, $n \in \mathbb{N}$.

These tensor norms together with various generalizations (and the associated dual and transposed norms) were studied by various authors, in particular, Saphar (1970), Chevet (1969) and Lapresté (1976). In case $p \in \{1, 2, \infty\}$ these norms (together with their associated dual and transposed norms) are of particular interest, since they are equivalent to some of Grothendieck's "natural" tensor norms. Although most of this monograph is devoted to the study of these natural tensor norms (see Theorem 4.4.1, p. 163), we shall give no further attention to the norms g_p and w_p, and rather refer the interested reader to the mentioned references and to the book of Defant and Floret (1993), which contains a detailed exposition.

The α-integral and α-nuclear operators associated to g_p and w_p and their duals and transposes are obviously very interesting as well. The study of these operators led to factorization schemes similar to those of Grothendieck which occur so often later on. Again we refer the interested reader to the papers [Persson and Pietsch (1969)], [Gordon, Lewis, and Retherford (1973)] and [Fourie and Swart (1981)] and to the books of Defant and Floret (1993) and Diestel, Jarchow, and Tonge (1995).