Part 1

General Topology

Our goal in this part of the book is to teach the basics of the mathematical language. More specifically, one of its most important components: the language of set-theoretic topology, which treats the basic notions related to continuity. The term *general topology* means: this is the topology that is needed and used by most mathematicians. The permanent usage in the capacity of a common mathematical language has polished its system of definitions and theorems. Nowadays, studying general topology really more resembles studying a language rather than mathematics: one needs to learn a lot of new words, while proofs of most theorems are quite simple. On the other hand, the theorems are numerous because they play the role of rules regulating usage of words.

We have to warn students for whom this is one of their first mathematical subjects. Do not hurry to fall in love with it. Do not let an imprinting happen. This field may seem to be charming, but it is not very active nowadays. Other mathematical subjects are also nice and can give exciting opportunities for research. Check them out!

Chapter I

Structures and Spaces

1. Set-Theoretic Digression: Sets

We begin with a digression, which, however, we would like to consider unnecessary. Its subject is the first basic notions of the naive set theory. This is a part of the common mathematical language, too, but an even more profound part than general topology. We would not be able to say anything about topology without this part (look through the next section to see that this is not an exaggeration). Naturally, it may be expected that the naive set theory becomes familiar to a student when she or he studies Calculus or Algebra, two subjects of study that usually precede topology. If this is true in your case, then, please, just glance through this section and pass to the next one.

⌈1′1⌋ Sets and Elements

In an intellectual activity, one of the most profound actions is gathering objects in groups. The gathering is performed in mind and is not accompanied with any action in the physical world. As soon as the group has been created and assigned a name, it can be a subject of thoughts and arguments and, in particular, can be included into other groups. Mathematics has an elaborate system of notions, which organizes and regulates creating those groups and manipulating them. The system is called the *naive set theory*, which, however, is a slightly misleading name because this is rather a language than a theory.

The first words in this language are *set* and *element*. By a set we understand an arbitrary collection of various objects. An object included in the collection is an *element* of the set. A set *consists* of its elements. It is also *formed* by them. In order to diversify the wording, the word *set* is replaced by the word *collection*. Sometimes other words, such as *class*, *family*, and *group*, are used in the same sense, but this is not quite safe because each of these words is associated in modern mathematics with a more special meaning, and hence should be used instead of the word *set* with caution.

If x is an element of a set A, then we write $x \in A$ and say that x *belongs to* A and A *contains* x. The sign $\in$ is a variant of the Greek letter epsilon, which corresponds to the first letter of the Latin word *element*. To make the notation more flexible, the formula $x \in A$ is also allowed to be written in the form $A \ni x$. So, the origin of the notation is sort of ignored, but a more meaningful similarity to the inequality symbols $<$ and $>$ is emphasized. To state that x is not an element of A, we write $x \notin A$ or $A \not\ni x$.

⌈1′2⌋ Equality of Sets

A set is determined by its elements. The set is nothing but a collection of its elements. This manifests most sharply in the following principle: *two sets are considered equal if and only if they have the same elements*. In this sense, the word *set* has slightly disparaging meaning. When something is called a set, this shows, maybe unintentionally, a lack of interest to whatever organization of the elements of this set.

For example, when we say that a line is a set of points, we assume that two lines coincide if and only if they consist of the same points. On the other hand, we commit ourselves to consider all relations between points on a line (e.g., the distance between points, the order of points on the line, etc.) separately from the notion of a line.

We may think of sets as boxes that can be built effortlessly around elements, just to distinguish them from the rest of the world. The cost of this lightness is that such a box is not more than the collection of elements placed inside. It is a little more than just a name: it is a declaration of our wish to think about this collection of things as an entity and not to go into details about the nature of its member-elements. Elements, in turn, may also be sets, but as long as we consider them elements, they play the role of atoms, with their own original nature ignored.

In modern mathematics, the words *set* and *element* are very common and appear in most texts. They are even overused. There are instances when it is not appropriate to use them. For example, it is not good to use the word *element* as a replacement for other, more meaningful words. When you call something an *element*, then the *set* whose element is this one should be clear. The word *element* makes sense only in combination with the word *set*, unless we deal with a nonmathematical term (like *chemical element*), or a rare old-fashioned exception from the common mathematical terminology (sometimes the expression under the sign of integral is called an *infinitesimal element*; lines, planes, and other geometric images are also called *elements* in old texts). Euclid's famous book on geometry is called *Elements*, too.

⌈1′3⌋ The Empty Set

Thus, an element may not be without a set. However, a set may have no elements. Actually, there is such a set. This set is unique because a set is completely determined by its elements. It is the ***empty set*** denoted[1] by $\varnothing$.

⌈1′4⌋ Basic Sets of Numbers

In addition to $\varnothing$, there are some other sets so important that they have their own special names and designations. The set of all positive integers, i.e., 1, 2, 3, 4, 5, ..., etc., is denoted by $\mathbb{N}$. The set of all integers, both positive, and negative, and zero, is denoted by $\mathbb{Z}$. The set of all rational numbers (add to the integers the numbers that are presented by fractions, like 2/3 and $\frac{-7}{5}$) is denoted by $\mathbb{Q}$. The set of all real numbers (obtained by adjoining to rational numbers the numbers like $\sqrt{2}$ and $\pi = 3.14\dots$) is denoted by $\mathbb{R}$. The set of complex numbers is denoted by $\mathbb{C}$.

⌈1′5⌋ Describing a Set by Listing Its Elements

A set presented by a list a, b, ..., x of its elements is denoted by the symbol $\{a, b, \dots, x\}$. In other words, the list of objects enclosed in curly brackets denotes the set whose elements are listed. For example, $\{1, 2, 123\}$ denotes the set consisting of the numbers 1, 2, and 123. The symbol $\{a, x, A\}$ denotes the set consisting of three elements: a, x, and A, whatever objects these three letters denote.

1.1. What is $\{\varnothing\}$? How many elements does it contain?

1.2. Which of the following formulas are correct:

1) $\varnothing \in \{\varnothing, \{\varnothing\}\}$; 2) $\{\varnothing\} \in \{\{\varnothing\}\}$; 3) $\varnothing \in \{\{\varnothing\}\}$?

A set consisting of a single element is a ***singleton***. This is any set which is presented as $\{a\}$ for some a.

1.3. Is $\{\{\varnothing\}\}$ a singleton?

[1]Other designations, like Λ, are also in use, but $\varnothing$ has become a common one.

Notice that the sets $\{1, 2, 3\}$ and $\{3, 2, 1, 2\}$ are equal since they have the same elements. At first glance, lists with repetitions of elements are never needed. There even arises a temptation to prohibit usage of lists with repetitions in such notation. However this would not be wise. In fact, quite often one cannot say *a priori* whether there are repetitions or not. For example, the elements in the list may depend on a parameter, and under certain values of the parameter some entries of the list coincide, while for other values they don't.

1.4. How many elements do the following sets contain?

1) $\{1, 2, 1\}$;	2) $\{1, 2, \{1, 2\}\}$;	3) $\{\{2\}\}$;
4) $\{\{1\}, 1\}$;	5) $\{1, \varnothing\}$;	6) $\{\{\varnothing\}, \varnothing\}$;
7) $\{\{\varnothing\}, \{\varnothing\}\}$;	8) $\{x, 3x - 1\}$ for $x \in \mathbb{R}$.	

⌈1′6⌋ Subsets

If A and B are sets and every element of A also belongs to B, then we say that A is a *subset* of B, or B *includes* A, and write $A \subset B$ or $B \supset A$.

The inclusion signs $\subset$ and $\supset$ resemble the inequality signs $<$ and $>$ for a good reason: in the world of sets, the inclusion signs are obvious counterparts for the signs of inequalities.

1.A. Let a set A have a elements, and let a set B have b elements. Prove that if $A \subset B$, then $a \leq b$.

⌈1′7⌋ Properties of Inclusion

1.B Reflexivity of Inclusion. Any set includes itself: $A \subset A$ holds true for any A.

Thus, the inclusion signs are not completely true counterparts of the inequality signs $<$ and $>$. They are closer to $\leq$ and $\geq$. Notice that no number a satisfies the inequality $a < a$.

1.C The Empty Set Is Everywhere. The inclusion $\varnothing \subset A$ holds true for any set A. In other words, the empty set is present in each set as a subset.

Thus, each set A has two obvious subsets: the empty set $\varnothing$ and A itself. A subset of A different from $\varnothing$ and A is a *proper* subset of A. This word is used when we do not want to consider the obvious subsets (which are *improper*).

1.D Transitivity of Inclusion. If A, B, and C are sets, $A \subset B$, and $B \subset C$, then $A \subset C$.

⌈1′8⌋ To Prove Equality of Sets, Prove Two Inclusions

Working with sets, we need from time to time to prove that two sets, say A and B, which may have emerged in quite different ways, are equal. The most common way to do this is provided by the following theorem.

1.E Criterion of Equality for Sets.
$A = B$ if and only if $A \subset B$ and $B \subset A$.

⌈1′9⌋ Inclusion Versus Belonging

1.F. *$x \in A$ if and only if $\{x\} \subset A$.*

Despite this obvious relation between the notions of belonging $\in$ and inclusion $\subset$ and similarity of the symbols $\in$ and $\subset$, the concepts are quite different. Indeed, $A \in B$ means that A is an element in B (i.e., one of the indivisible pieces constituting B), while $A \subset B$ means that A is made of some of the elements of B.

In particular, we have $A \subset A$, while $A \notin A$ for any reasonable A. Thus, *belonging is not reflexive*. One more difference: *belonging is not transitive*, while inclusion is.

1.G Non-Reflexivity of Belonging. Construct a set A such that $A \notin A$. Cf. *1.B*.

1.H Non-Transitivity of Belonging. Construct three sets A, B, and C such that $A \in B$ and $B \in C$, but $A \notin C$. Cf. *1.D*.

⌈1′10⌋ Defining a Set by a Condition (Set-Builder Notation)

As we know (see Section 1′5), a set can be described by presenting a list of its elements. This simplest way may be not available or, at least, may not be the easiest one. For example, it is easy to say: "the set of all solutions of the following equation" and write down the equation. This is a reasonable description of the set. At least, it is unambiguous. Having accepted it, we may start speaking on the set, studying its properties, and eventually may be lucky to solve the equation and obtain the list of its solutions. (Though the latter task may be difficult, this should not prevent us from discussing the set.)

Thus, we see another way for a description of a set: to formulate properties that distinguish the elements of the set among elements of some wider and already known set. Here is the corresponding notation: the subset of a set A consisting of the elements x that satisfy a condition $P(x)$ is denoted by $\{x \in A \mid P(x)\}$.

1.5. Present the following sets by lists of their elements (i.e., in the form $\{a, b, \dots\}$)
(a) $\{x \in \mathbb{N} \mid x < 5\}$, (b) $\{x \in \mathbb{N} \mid x < 0\}$, (c) $\{x \in \mathbb{Z} \mid x < 0\}$.

⌈1′11⌋ Intersection and Union

The *intersection* of sets A and B is the set consisting of their common elements, i.e., elements belonging both to A and B. It is denoted by $A \cap B$ and is described by the formula

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

Two sets A and B are *disjoint* if their intersection is empty, i.e., $A \cap B = \varnothing$. In other words, they have no common elements.

The *union* of two sets A and B is the set consisting of all elements that belong to at least one of the two sets. The union of A and B is denoted by $A \cup B$. It is described by the formula

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Here the conjunction *or* should be understood in the inclusive way: the statement "$x \in A$ or $x \in B$" means that x belongs to *at least one* of the sets A and B, and, maybe, to both of them.[2]

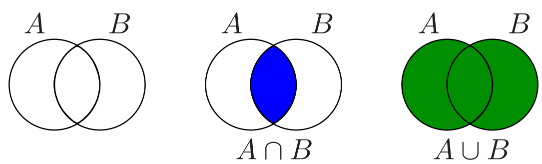


Figure 1. The sets A and B, their intersection $A \cap B$, and their union $A \cup B$.

1.I Commutativity of $\cap$ and $\cup$. For any two sets A and B, we have

$$A \cap B = B \cap A \quad \text{and} \quad A \cup B = B \cup A.$$

In the above figure, the first equality of Theorem *1.L* is illustrated by sketches. Such sketches are called *Venn diagrams* or *Euler circles*. They are quite useful, and we strongly recommend trying to draw them for each formula involving sets. (At least, for formulas with at most three sets, since in this case they can serve as proofs! (Guess why?)).

1.6. Prove that for any set A we have

$$A \cap A = A, \qquad A \cup A = A, \qquad A \cup \varnothing = A, \quad \text{and} \quad A \cap \varnothing = \varnothing.$$

1.7. Prove that for any sets A and B we have[3]

$$A \subset B, \quad \text{iff} \quad A \cap B = A, \quad \text{iff} \quad A \cup B = B.$$

[2]To make formulas clearer, sometimes we slightly abuse the notation and instead of, say, $A \cup \{x\}$, where x is an element outside A, we write just $A \cup x$. The same agreement holds true for other set-theoretic operations.

[3]Here, as usual, *iff* stands for "if and only if".

***1.J Associativity of* $\cap$ *and* $\cup$.** For any sets A, B, and C, we have

$$(A \cap B) \cap C = A \cap (B \cap C) \quad \text{and} \quad (A \cup B) \cup C = A \cup (B \cup C).$$

Associativity allows us to not care about brackets and sometimes even to omit them. We define $A \cap B \cap C = (A \cap B) \cap C = A \cap (B \cap C)$ and $A \cup B \cup C = (A \cup B) \cup C = A \cup (B \cup C)$. However, the intersection and union of an arbitrarily large (in particular, infinite) collection of sets can be defined directly, without reference to the intersection or union of two sets. Indeed, let Γ be a collection of sets. The *intersection* of the sets in Γ is the set formed by the elements that belong to *every* set in Γ. This set is denoted by $\bigcap_{A \in \Gamma} A$. Similarly, the *union* of the sets in Γ is the set formed by elements that belong to *at least one* of the sets in Γ. This set is denoted by $\bigcup_{A \in \Gamma} A$.

1.K. The notions of intersection and union of an arbitrary collection of sets generalize the notions of intersection and union of two sets: for $\Gamma = \{A, B\}$, we have

$$\bigcap_{C \in \Gamma} C = A \cap B \quad \text{and} \quad \bigcup_{C \in \Gamma} C = A \cup B.$$

1.8. Riddle. How are the notions of system of equations and intersection of sets related to each other?

1.L Two Distributivities. *For any sets A, B, and C, we have*

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C), \tag{1}$$

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C). \tag{2}$$

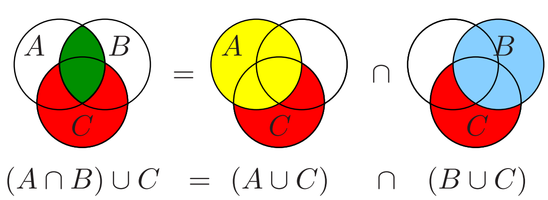


Figure 2. The left-hand side $(A \cap B) \cup C$ of equality (1) and the sets $A \cup C$ and $B \cup C$, whose intersection is the right-hand side of the equality (1).

1.M. Draw a Venn diagram illustrating (2). Prove (1) and (2) by tracing all details of the proofs in the Venn diagrams. Draw Venn diagrams illustrating all formulas below in this section.

1.9. Riddle. Generalize Theorem *1.L* to the case of arbitrary collections of sets.

1.N Yet Another Pair of Distributivities. *Let A be a set and let Γ be a set consisting of sets. Then we have*

$$A \cap \bigcup_{B \in \Gamma} B = \bigcup_{B \in \Gamma} (A \cap B) \quad \text{and} \quad A \cup \bigcap_{B \in \Gamma} B = \bigcap_{B \in \Gamma} (A \cup B).$$

⌈1′12⌋ Different Differences

The *difference* $A \smallsetminus B$ of two sets A and B is the set of those elements of A which do not belong to B. Here we do not assume that $A \supset B$.

If $A \supset B$, then the set $A \smallsetminus B$ is also called the *complement* of B in A.

1.10. Prove that for any sets A and B their union $A \cup B$ is the union of the following three sets: $A \smallsetminus B$, $B \smallsetminus A$, and $A \cap B$, which are pairwise disjoint.

1.11. Prove that $A \smallsetminus (A \smallsetminus B) = A \cap B$ for any sets A and B.

1.12. Prove that $A \subset B$ if and only if $A \smallsetminus B = \varnothing$.

1.13. Prove that $A \cap (B \smallsetminus C) = (A \cap B) \smallsetminus (A \cap C)$ for any sets A, B, and C.

The set $(A \smallsetminus B) \cup (B \smallsetminus A)$ is the *symmetric difference* of the sets A and B. It is denoted by $A \vartriangle B$.

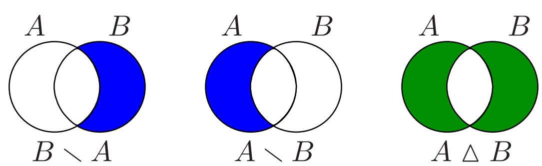


Figure 3. Differences of the sets A and B.

1.14. Prove that for any sets A and B we have

$$A \vartriangle B = (A \cup B) \smallsetminus (A \cap B).$$

1.15 Associativity of Symmetric Difference. Prove that for any sets A, B, and C we have

$$(A \vartriangle B) \vartriangle C = A \vartriangle (B \vartriangle C).$$

1.16. Riddle. Find a symmetric definition of the symmetric difference $(A \vartriangle B) \vartriangle C$ of three sets and generalize it to arbitrary finite collections of sets.

1.17 Distributivity. Prove that $(A \vartriangle B) \cap C = (A \cap C) \vartriangle (B \cap C)$ for any sets A, B, and C.

1.18. Does the following equality hold true for any sets A, B, and C:

$$(A \vartriangle B) \cup C = (A \cup C) \vartriangle (B \cup C)?$$

2. Topology on a Set

⌈2′1⌋ Definition of Topological Space

Let X be a set. Let Ω be a collection of its subsets such that:

(1) the union of any collection of sets that are elements of Ω belongs to Ω;

(2) the intersection of any finite collection of sets that are elements of Ω belongs to Ω;

(3) the empty set $\varnothing$ and the whole X belong to Ω.

Then

- Ω is a *topological structure* or just a *topology*[4] on X;
- the pair (X, Ω) is a *topological space*;
- elements of X are *points* of this topological space;
- elements of Ω are *open sets* of the topological space (X, Ω).

The conditions in the definition above are the *axioms of topological structure*.

⌈2′2⌋ Simplest Examples

A *discrete topological space* is a set with the topological structure consisting of all subsets.

2.A. Check that this is a topological space, i.e., all axioms of topological structure hold true.

An *indiscrete topological space* is the opposite example, in which the topological structure is the most meager. (It is also called *trivial topology*.) It consists only of X and $\varnothing$.

2.B. This is a topological structure, is it not?

Here are slightly less trivial examples.

2.1. Let X be the ray $[0, +\infty)$, and let Ω consist of $\varnothing$, X, and all rays $(a, +\infty)$ with $a \geq 0$. Prove that Ω is a topological structure.

2.2. Let X be a plane. Let Σ consist of $\varnothing$, X, and all open disks centered at the origin. Is Σ a topological structure?

2.3. Let X consist of four elements: $X = \{a, b, c, d\}$. Which of the following collections of its subsets are topological structures in X, i.e., satisfy the axioms of topological structure:

[4] Thus, Ω is important: it is called by the same word as the whole branch of mathematics. Certainly, this does not mean that Ω coincides with the subject of topology, but nearly everything in this subject is related to Ω.

(1) $\varnothing$, X, $\{a\}$, $\{b\}$, $\{a,c\}$, $\{a,b,c\}$, $\{a,b\}$;
(2) $\varnothing$, X, $\{a\}$, $\{b\}$, $\{a,b\}$, $\{b,d\}$;
(3) $\varnothing$, X, $\{a,c,d\}$, $\{b,c,d\}$?

The space of Problem *2.1* is the *arrow*. We denote the space of Problem *2.3* (1) by . It is a sort of toy space made of 4 points. (The meaning of the pictogram is explained below in Section 7′9.) Both spaces, as well as the space of Problem *2.2*, are not very important, but they provide nice simple examples.

⌈2′3⌋ The Most Important Example: Real Line

Let X be the set $\mathbb{R}$ of all real numbers, Ω the set of arbitrary unions of open intervals (a,b) with $a,b \in \mathbb{R}$.

2.C. Check whether Ω satisfies the axioms of topological structure.

This is the topological structure which is always meant when $\mathbb{R}$ is considered as a topological space (unless another topological structure is explicitly specified). This space is usually called the *real line*, and the structure is referred to as the *canonical* or *standard* topology on $\mathbb{R}$.

⌈2′4⌋ Additional Examples

2.4. Let X be $\mathbb{R}$, and let Ω consist of the empty set and all infinite subsets of $\mathbb{R}$. Is Ω a topological structure?

2.5. Let X be $\mathbb{R}$, and let Ω consists of the empty set and complements of all finite subsets of $\mathbb{R}$. Is Ω a topological structure?

The space of Problem *2.5* is denoted by $\mathbb{R}_{T_1}$ and called the *line with* T_1-*topology*.

2.6. Let (X,Ω) be a topological space, Y the set obtained from X by adding a single element a. Is

$$\{\{a\} \cup U \mid U \in \Omega\} \cup \{\varnothing\}$$

a topological structure in Y?

2.7. Is the set $\{\varnothing, \{0\}, \{0,1\}\}$ a topological structure in $\{0,1\}$?

If the topology Ω in Problem *2.6* is discrete, then the topology on Y is called a *particular point topology* or *topology of everywhere dense point*. The topology in Problem *2.7* is a particular point topology; it is also called the *topology of a connected pair of points* or *Sierpiński topology*.

2.8. List all topological structures in a two-element set, say, in $\{0,1\}$.

⌈2′5⌋ Using New Words: Points, Open Sets, Closed Sets

We recall that, for a topological space (X, Ω), elements of X are *points*, and elements of Ω are *open sets*.[5]

2.D. Reformulate the axioms of topological structure using the words *open set* wherever possible.

A set $F \subset X$ is *closed* in the space (X, Ω) if its complement $X \smallsetminus F$ is open (i.e., $X \smallsetminus F \in \Omega$).

⌈2′6⌋ Set-Theoretic Digression: De Morgan Formulas

2.E. Let Γ be an arbitrary collection of subsets of a set X. Then

$$X \smallsetminus \bigcup_{A \in \Gamma} A = \bigcap_{A \in \Gamma} (X \smallsetminus A), \tag{3}$$

$$X \smallsetminus \bigcap_{A \in \Gamma} A = \bigcup_{A \in \Gamma} (X \smallsetminus A). \tag{4}$$

Formula (4) is deduced from (3) in one step, is it not? These formulas are nonsymmetric cases of a single formulation, which contains, in a symmetric way, sets and their complements, unions, and intersections.

2.9. Riddle. Find such a formulation.

⌈2′7⌋ Properties of Closed Sets

2.F. Prove that:

(1) the intersection of any collection of closed sets is closed;

(2) the union of any finite number of closed sets is closed;

(3) the empty set and the whole space (i.e., the underlying set of the topological structure) are closed.

⌈2′8⌋ Being Open or Closed

Notice that the property of being closed is not the negation of the property of being open. (They are not exact antonyms in everyday usage, too.)

2.G. Find examples of sets that are

(1) both open and closed simultaneously (open-closed);

(2) neither open, nor closed.

[5]The letter Ω stands for the letter O which is the initial of the words with the same meaning: *Open* in English, *Otkrytyj* in Russian, *Offen* in German, *Ouvert* in French.

2.10. Give an explicit description of closed sets in
(1) a discrete space; (2) an indiscrete space;
(3) the arrow; (4) ;
(5) $\mathbb{R}_{T_1}$.

2.H. Is a closed segment $[a, b]$ closed in $\mathbb{R}$?

The concepts of closed and open sets are similar in a number of ways. The main difference is that the intersection of an infinite collection of open sets is not necessarily open, while the intersection of any collection of closed sets is closed. Along the same lines, the union of an infinite collection of closed sets is not necessarily closed, while the union of any collection of open sets is open.

2.11. Prove that the half-open interval $[0, 1)$ is neither open nor closed in $\mathbb{R}$, but is both a union of closed sets and an intersection of open sets.

2.12. Prove that the set $A = \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$ is closed in $\mathbb{R}$.

⌈2′9⌋ Characterization of Topology in Terms of Closed Sets

2.13. Suppose a collection $\mathcal{F}$ of subsets of X satisfies the following conditions:

(1) the intersection of any family of sets from $\mathcal{F}$ belongs to $\mathcal{F}$;
(2) the union of any finite number sets from $\mathcal{F}$ belongs to $\mathcal{F}$;
(3) $\varnothing$ and X belong to $\mathcal{F}$.

Prove that then $\mathcal{F}$ is the set of all closed sets of a topological structure (which one?).

2.14. List all collections of subsets of a three-element set such that there are topologies where these collections are complete sets of closed sets.

⌈2′10⌋ Neighborhoods

A *neighborhood* of a point in a topological space is any open set containing this point. Analysts and French mathematicians (following N. Bourbaki) prefer a wider notion of neighborhood: they use this word for any set containing a neighborhood in the above sense.

2.15. Give an explicit description of all neighborhoods of a point in
(1) a discrete space; (2) an indiscrete space;
(3) the arrow; (4) ;
(5) a connected pair of points; (6) particular point topology.

⌈2′11x⌋ Open Sets on Line

2.Ix. *Prove that every open subset of the real line is a union of disjoint open intervals.*

At first glance, Theorem *2.Ix* suggests that open sets on the line are simple. However, an open set may lie on the line in a quite complicated manner. Its complement may happen to be not that simple. The complement of an open set is a closed set. One can naively expect that a closed set on $\mathbb{R}$ is a union of closed intervals. The next important example shows that this is very far from being true.

⌈2′12x⌋ Cantor Set

Let K be the set of real numbers that are sums of series of the form $\sum_{k=1}^{\infty} a_k/3^k$ with $a_k \in \{0, 2\}$.

In other words, K consists of the real numbers that have the form $0.a_1a_2 \dots a_k \dots$ without the digit 1 in the number system with base 3.

2.Jx. Find a geometric description of K.

2.Jx.1. Prove that

(1) K is contained in $[0, 1]$,
(2) K does not meet $(1/3, 2/3)$,
(3) K does not meet $\left(\frac{3s+1}{3^k}, \frac{3s+2}{3^k}\right)$ for any integers k and s.

2.Jx.2. Present K as $[0, 1]$ with an infinite family of open intervals removed.

2.Jx.3. Try to sketch K.

The set K is the *Cantor set*. It has a lot of remarkable properties and is involved in numerous problems below.

2.Kx. Prove that K is a closed set in the real line.

⌈2′13x⌋ Topology and Arithmetic Progressions

2.Lx*. Consider the following property of a subset F of the set $\mathbb{N}$ of positive integers: there is $n \in \mathbb{N}$ such that F contains no arithmetic progressions of length n. Prove that subsets with this property together with the whole $\mathbb{N}$ form a collection of closed subsets in some topology on $\mathbb{N}$.

When solving this problem, you probably will need the following combinatorial theorem.

2.Mx Van der Waerden's Theorem*. For every $n \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that for any subset $A \subset \{1, 2, \dots, N\}$, either A or $\{1, 2, \dots, N\} \smallsetminus A$ contains an arithmetic progression of length n.

See [**3**].

3. Bases

⌈3′1⌋ Definition of Base

The topological structure is usually presented by describing its part, which is sufficient to recover the whole structure. A collection Σ of open sets is a ***base*** for a topology if each nonempty open set is a union of sets in Σ. For instance, all intervals form a base for the real line.

3.1. Can two distinct topological structures have the same base?

3.2. Find some bases for the topology of
(1) a discrete space; (2) ;
(3) an indiscrete space; (4) the arrow.
Try to choose the smallest possible bases.

3.3. Prove that any base of the canonical topology on $\mathbb{R}$ can be decreased.

3.4. Riddle. What topological structures have exactly one base?

⌈3′2⌋ When a Collection of Sets is a Base

3.A. *A collection Σ of open sets is a base for the topology iff for every open set U and every point $x \in U$ there is a set $V \in \Sigma$ such that $x \in V \subset U$.*

3.B. *A collection Σ of subsets of a set X is a base for a certain topology on X iff X is the union of all sets in Σ and the intersection of any two sets in Σ is the union of some sets in Σ.*

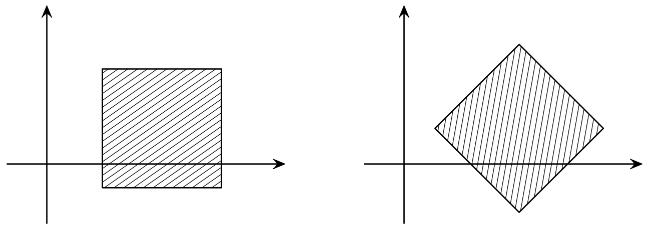
3.C. Show that the second condition in Theorem *3.B* (on the intersection) is equivalent to the following one: the intersection of any two sets in Σ contains, together with any of its points, a certain set in Σ containing this point (cf. Theorem *3.A*).

⌈3′3⌋ Bases for Plane

Consider the following three collections of subsets of $\mathbb{R}^2$:

- Σ^2, which consists of all possible open disks (i.e., disks without their boundary circles);
- Σ^∞, which consists of all possible open squares (i.e., squares without their sides and vertices) with sides parallel to the coordinate axes;
- Σ^1, which consists of all possible open squares with sides parallel to the bisectors of the coordinate angles.

(The squares in Σ^∞ and Σ^1 are determined by the inequalities $\max\{|x-a|, |y-b|\} < \rho$ and $|x-a| + |y-b| < \rho$, respectively.)



3.5. Prove that every element of Σ^2 is a union of elements of Σ^∞.

3.6. Prove that the intersection of any two elements of Σ^1 is a union of elements of Σ^1.

3.7. Prove that each of the collections Σ^2, Σ^∞, and Σ^1 is a base for some topological structure in $\mathbb{R}^2$, and that the structures determined by these collections coincide.

⌈3′4⌋ Subbases

Let (X, Ω) be a topological space. A collection Δ of its open subsets is a *subbase* for Ω provided that the collection

$$\Sigma = \{V \mid V = \bigcap_{i=1}^{k} W_i,\ k \in \mathbb{N},\ W_i \in \Delta\}$$

of all finite intersections of sets in Δ is a base for Ω.

3.8. Let X be a set, Δ a collection of subsets of X. Prove that Δ is a subbase for a topology on X iff $X = \bigcup_{W \in \Delta} W$.

⌈3′5⌋ Infiniteness of the Set of Prime Numbers

3.9. Prove that all (infinite) arithmetic progressions consisting of positive integers form a base for some topology on $\mathbb{N}$.

3.10. Using this topology, prove that the set of all prime numbers is infinite.

⌈3′6⌋ Hierarchy of Topologies

If Ω_1 and Ω_2 are topological structures in a set X such that $\Omega_1 \subset \Omega_2$, then Ω_2 is *finer* than Ω_1, and Ω_1 is *coarser* than Ω_2. For instance, the indiscrete topology is the coarsest topology among all topological structures in the same set, while the discrete topology is the finest one, is it not?

3.11. Show that the T_1-topology on the real line (see 2′4) is coarser than the canonical topology.

Two bases determining the same topological structure are *equivalent*.

3.D. Riddle. Formulate a necessary and sufficient condition for two bases to be equivalent without explicitly mentioning the topological structures determined by the bases. (Cf. *3.7*: the bases Σ^2, Σ^∞, and Σ^1 must satisfy the condition you are looking for.)

4. Metric Spaces

⌈4′1⌋ Definition and First Examples

A function[6] $\rho : X \times X \to \mathbb{R}_+ = \{ x \in \mathbb{R} \mid x \geq 0 \}$ is a *metric* (or *distance function*) on X if

(1) $\rho(x, y) = 0$ iff $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for any $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for any $x, y, z \in X$.

The pair (X, ρ), where ρ is a metric on X, is a *metric space*. Condition (3) is the *triangle inequality*.

4.A. Prove that the function

$$\rho : X \times X \to \mathbb{R}_+ : \ (x, y) \mapsto \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric for any set X.

4.B. Prove that $\mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ : (x, y) \mapsto |x - y|$ is a metric.

4.C. Prove that $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ is a metric.

The metrics of Problems *4.B* and *4.C* are always meant when $\mathbb{R}$ and $\mathbb{R}^n$ are considered as metric spaces, unless another metric is specified explicitly. The metric of Problem *4.B* is a special case of the metric of Problem *4.C*. All these metrics are called *Euclidean*.

⌈4′2⌋ Further Examples

4.1. Prove that $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \max_{i=1,\dots,n} |x_i - y_i|$ is a metric.

4.2. Prove that $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ : (x, y) \mapsto \sum_{i=1}^{n} |x_i - y_i|$ is a metric.

The metrics in $\mathbb{R}^n$ introduced in Problems *4.C*, *4.1*, *4.2* are members of an infinite sequence of metrics:

$$\rho^{(p)} : \ (x, y) \mapsto \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p}, \quad p \geq 1.$$

4.3. Prove that $\rho^{(p)}$ is a metric for any $p \geq 1$.

[6] The notions of function (mapping) and Cartesian square, as well as the corresponding notation, are discussed in detail below, in Sections 9 and 20. Nevertheless, we hope that the reader is acquainted with them, so we use them in this section without special explanations.

4.3.1 Hölder Inequality. Let $x_1, \dots, x_n, y_1, \dots, y_n \geq 0$, let $p, q > 0$, and let $1/p + 1/q = 1$. Prove that

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^p\right)^{1/p} \left(\sum_{i=1}^n y_i^q\right)^{1/q}.$$

The metric of *4.C* is $\rho^{(2)}$, that of *4.2* is $\rho^{(1)}$, and that of *4.1* can be denoted by $\rho^{(\infty)}$ and appended to the series since

$$\lim_{p \to +\infty} \left(\sum_{i=1}^n a_i^p\right)^{1/p} = \max a_i$$

for any positive $a_1, a_2, \dots, a_n$.

4.4. Riddle. How is this related to Σ^2, Σ^∞, and Σ^1 from Section 3?

For a real $p \geq 1$, denote by $l^{(p)}$ the set of sequences $x = \{x_i\}_{i=1,2,\dots}$ such that the series $\sum_{i=1}^\infty |x|^p$ converges.

4.5. Let $p \geq 1$. Prove that for any two sequences $x, y \in l^{(p)}$ the series $\sum_{i=1}^\infty |x_i - y_i|^p$ converges and that

$$(x, y) \mapsto \left(\sum_{i=1}^\infty |x_i - y_i|^p\right)^{1/p}$$

is a metric on $l^{(p)}$.

⌈4′3⌋ Balls and Spheres

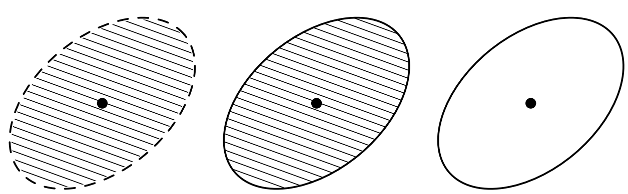
Let (X, ρ) be a metric space, $a \in X$ a point, r a positive real number. Then the sets

$$B_r(a) = \{ x \in X \mid \rho(a, x) < r \}, \quad (5)$$

$$D_r(a) = \{ x \in X \mid \rho(a, x) \leq r \}, \quad (6)$$

$$S_r(a) = \{ x \in X \mid \rho(a, x) = r \} \quad (7)$$

are, respectively, the ***open ball***, ***closed ball*** (or ***disk***), and ***sphere*** of the space (X, ρ) with center a and radius r.



⌈4′4⌋ Subspaces of a Metric Space

If (X, ρ) is a metric space and $A \subset X$, then the restriction of the metric ρ to $A \times A$ is a metric on A, and so $(A, \rho|_{A \times A})$ is a metric space. It is called a ***subspace*** of (X, ρ).

The disk $D_1(0)$ and the sphere $S_1(0)$ in $\mathbb{R}^n$ (with Euclidean metric, see *4.C*) are denoted by D^n and S^{n-1} and called the (***unit***) n-***disk*** and $(n-1)$-***sphere***. They are regarded as metric spaces (with the metric induced from $\mathbb{R}^n$).

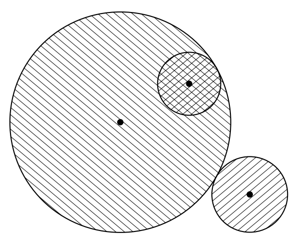
4.D. Check that D^1 is the segment $[-1, 1]$, D^2 is a plane disk, S^0 is the pair of points $\{-1, 1\}$, S^1 is a circle, S^2 is a sphere, and D^3 is a ball.

The last two assertions clarify the origin of the terms *sphere* and *ball* (in the context of metric spaces).

Some properties of balls and spheres in an arbitrary metric space resemble familiar properties of planar disks and circles and spatial balls and spheres.

4.E. Prove that for any points x and a of any metric space and any $r > \rho(a, x)$ we have

$$B_{r-\rho(a,x)}(x) \subset B_r(a) \text{ and } D_{r-\rho(a,x)}(x) \subset D_r(a).$$



4.6. Riddle. What if $r < \rho(x, a)$? What is an analog for the statement of Problem *4.E* in this case?

⌈4′5⌋ Surprising Balls

However, balls and spheres in other metric spaces may have rather surprising properties.

4.7. What are balls and spheres in $\mathbb{R}^2$ equipped with the metrics of *4.1* and *4.2*? (Cf. *4.4*.)

4.8. Find $D_1(a)$, $D_{1/2}(a)$, and $S_{1/2}(a)$ in the space of *4.A*.

4.9. Find a metric space and two balls in it such that the ball with the smaller radius contains the ball with the bigger one and does not coincide with it.

4.10. What is the minimal number of points in the space which is required to be constructed in *4.9*?

4.11. Prove that the largest radius in *4.9* is at most twice the smaller radius.

⌈4′6⌋ Segments (What Is Between)

4.12. Prove that the segment with endpoints $a, b \in \mathbb{R}^n$ can be described as

$$\{\, x \in \mathbb{R}^n \mid \rho(a, x) + \rho(x, b) = \rho(a, b) \,\},$$

where ρ is the Euclidean metric.

4.13. How does the set defined as in Problem *4.12* look if ρ is the metric defined in Problems *4.1* or *4.2*? (Consider the case where $n = 2$ if it seems to be easier.)

⌈4′7⌋ Bounded Sets and Balls

A subset A of a metric space (X, ρ) is ***bounded*** if there is a number $d > 0$ such that $\rho(x, y) < d$ for any $x, y \in A$. The greatest lower bound for such d is the ***diameter*** of A. It is denoted by $\operatorname{diam}(A)$.

4.F. Prove that a set A is bounded iff A is contained in a ball.

4.14. What is the relation between the minimal radius of such a ball and $\operatorname{diam}(A)$?

⌈4′8⌋ Norms and Normed Spaces

Let X be a vector space (over $\mathbb{R}$). A function $X \to \mathbb{R}_+ : x \mapsto \|x\|$ is a ***norm*** if
(1) $\|x\| = 0$ iff $x = 0$;
(2) $\|\lambda x\| = |\lambda| \|x\|$ for any $\lambda \in \mathbb{R}$ and $x \in X$;
(3) $\|x + y\| \le \|x\| + \|y\|$ for any $x, y \in X$.

4.15. Prove that if $x \mapsto \|x\|$ is a norm, then

$$\rho : X \times X \to \mathbb{R}_+ : (x, y) \mapsto \|x - y\|$$

is a metric.

A vector space equipped with a norm is a ***normed space***. The metric determined by the norm as in *4.15* transforms the normed space into a metric space in a canonical way.

4.16. Look through the problems of this section and figure out which of the metric spaces involved are, in fact, normed vector spaces.

4.17. Prove that every ball in a normed space is a convex[7] set symmetric with respect to the center of the ball.

4.18*. Prove that every convex closed bounded set in $\mathbb{R}^n$ that has a center of symmetry and is not contained in any affine space except $\mathbb{R}^n$ itself is a unit ball with respect to a certain norm, which is uniquely determined by this ball.

---

[7]Recall that a set A is ***convex*** if for any $x, y \in A$ the segment connecting x and y is contained in A. Certainly, this definition involves the notion of ***segment***, so it makes sense only for subsets of those spaces where the notion of segment connecting two points makes sense. This is the case in vector and affine spaces over $\mathbb{R}$.

⌈4′9⌋ Metric Topology

4.G. *The collection of all open balls in the metric space is a base for a certain topology.*

This topology is the *metric topology*. We also say that it is *generated* by the metric. This topological structure is always meant whenever the metric space is regarded as a topological space (for instance, when we speak about open and closed sets, neighborhoods, etc. in this space).

4.H. Prove that the standard topological structure in $\mathbb{R}$ introduced in Section 2 is generated by the metric $(x, y) \mapsto |x - y|$.

4.19. What topological structure is generated by the metric of *4.A*?

4.I. *A set U is open in a metric space iff, together with each of its points, the set U contains a ball centered at this point.*

⌈4′10⌋ Openness and Closedness of Balls and Spheres

4.20. Prove that a closed ball is closed (here and below, we mean the metric topology).

4.21. Find a closed ball that is open.

4.22. Find an open ball that is closed.

4.23. Prove that a sphere is closed.

4.24. Find a sphere that is open.

⌈4′11⌋ Metrizable Topological Spaces

A topological space is *metrizable* if its topological structure is generated by a certain metric.

4.J. An indiscrete space is not metrizable if it is not a singleton (otherwise, it has too few open sets).

4.K. A finite space X is metrizable iff it is discrete.

4.25. Which of the topological spaces described in Section 2 are metrizable?

⌈4′12⌋ Equivalent Metrics

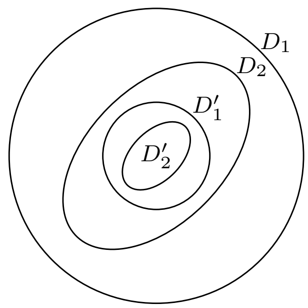
Two metrics in the same set are *equivalent* if they generate the same topology.

4.26. Are the metrics of *4.C*, *4.1*, and *4.2* equivalent?

4.27. Prove that two metrics ρ_1 and ρ_2 in X are equivalent if there are numbers $c, C > 0$ such that

$$c\rho_1(x, y) \leq \rho_2(x, y) \leq C\rho_1(x, y)$$

for any $x, y \in X$.



4.28. Generally speaking, the converse is not true.

4.29. Riddle. Hence, the condition of equivalence of metrics formulated in Problem *4.27* can be weakened. How?

4.30. The metrics $\rho^{(p)}$ in $\mathbb{R}^n$ defined right before Problem *4.3* are equivalent.

4.31*. Prove that the following two metrics ρ_1 and ρ_C in the set of all continuous functions $[0,1] \to \mathbb{R}$ are not equivalent:

$$\rho_1(f,g) = \int_0^1 \big|f(x) - g(x)\big| dx, \qquad \rho_C(f,g) = \max_{x\in[0,1]} \big|f(x) - g(x)\big|.$$

Is it true that one of the topological structures generated by them is finer than the other one?

⌈4′13⌋ Operations with Metrics

4.32. 1) Prove that if ρ_1 and ρ_2 are two metrics in X, then $\rho_1+\rho_2$ and $\max\{\rho_1,\rho_2\}$ also are metrics. 2) Are the functions $\min\{\rho_1,\rho_2\}$, $\rho_1\rho_2$, and ρ_1/ρ_2 metrics? (By definition, for $\rho = \rho_1/\rho_2$ we put $\rho(x,x) = 0$.)

4.33. Prove that if $\rho : X \times X \to \mathbb{R}_+$ is a metric, then

(1) the function $(x,y) \mapsto \dfrac{\rho(x,y)}{1+\rho(x,y)}$ is a metric;
(2) the function $(x,y) \mapsto \min\{\rho(x,y), 1\}$ is a metric;
(3) the function $(x,y) \mapsto f\big(\rho(x,y)\big)$ is a metric if f satisfies the following conditions:
   (a) $f(0) = 0$,
   (b) f is a monotone increasing function, and
   (c) $f(x+y) \le f(x) + f(y)$ for any $x, y \in \mathbb{R}$.

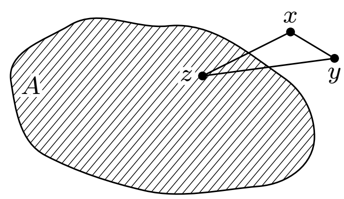
4.34. Prove that the metrics ρ and $\dfrac{\rho}{1+\rho}$ are equivalent.

⌈4′14⌋ Distances between Points and Sets

Let (X,ρ) be a metric space, $A \subset X$, and $b \in X$. The number $\rho(b,A) = \inf\{\, \rho(b,a) \mid a \in A \,\}$ is the *distance from the point b to the set A*.

4.L. Let A be a closed set. Prove that $\rho(b,A) = 0$ iff $b \in A$.

4.35. Prove that $|\rho(x,A) - \rho(y,A)| \le \rho(x,y)$ for any set A and any points x and y in a metric space.



$\rho(x,A) \leqslant \rho(x,z) \leqslant \rho(x,y)+\rho(y,z)$

⌈4′15x⌋ Distance between Sets

Let A and B be two bounded subsets in a metric space (X, ρ). We define

$$d_\rho(A,B) = \max\Big\{\sup_{a\in A} \rho(a,B),\ \sup_{b\in B} \rho(b,A)\Big\}.$$

This number is the *Hausdorff distance* between A and B.

***4.M*x.** Prove that the Hausdorff distance between bounded subsets of a metric space satisfies conditions (2) and (3) in the definition of a metric.

***4.N*x.** Prove that for every metric space the Hausdorff distance is a metric on the set of its closed bounded subsets.

Let A and B be two bounded polygons in the plane.[8] We define

$$d_\triangle(A,B) = S(A) + S(B) - 2S(A \cap B),$$

where $S(C)$ is the area of a polygon C.

***4.O*x.** Prove that $d_\triangle$ is a metric on the set of all bounded plane polygons.

We call $d_\triangle$ the *area metric*.

***4.P*x.** Prove that the area metric is *not* equivalent to the Hausdorff metric on the set of all bounded plane polygons.

***4.Q*x.** Prove that the area metric *is* equivalent to the Hausdorff metric on the set of *convex* bounded plane polygons.

⌈4′16x⌋ Ultrametrics and p-Adic Numbers

A metric ρ is an *ultrametric* if it satisfies the *ultrametric triangle inequality*:

$$\rho(x,y) \le \max\{\rho(x,z),\ \rho(z,y)\}$$

for any x, y, and z.

A metric space (X, ρ), where ρ is an ultrametric, is an *ultrametric space*.

[8] Although we assume that the notion of a bounded polygon is well known from elementary geometry, nevertheless, we recall the definition. A *bounded plane polygon* is the set of the points of a simple closed polygonal line γ and the points surrounded by γ. A *simple closed polygonal line* (or *polyline*) is a cyclic sequence of segments each of which starts at the point where the previous one ends and these are the only pairwise intersections of the segments.

4.Rx. Check that only one metric in *4.A–4.2* is an ultrametric. Which one?

4.Sx. Prove that all triangles in an ultrametric space are isosceles (i.e., for any three points a, b, and c, at least two of the three distances $\rho(a,b)$, $\rho(b,c)$, and $\rho(a,c)$ are equal).

4.Tx. Prove that spheres in an ultrametric space are not only closed (see Problem *4.23*), but also open.

The most important example of an ultrametric is the *p-adic metric* in the set $\mathbb{Q}$ of rational numbers. Let p be a prime number. For $x, y \in \mathbb{Q}$, present the difference $x - y$ as $\frac{r}{s}p^{\alpha}$, where r, s, and α are integers, and r and s are co-prime with p. We define $\rho(x,y) = p^{-\alpha}$.

4.Ux. Prove that ρ is an ultrametric.

⌈4′17x⌋ Asymmetrics

A function $\rho : X \times X \to \mathbb{R}_+$ is an *asymmetric* on a set X if

(1) $\rho(x,y) = 0$ and $\rho(y,x) = 0$, iff $x = y$;
(2) $\rho(x,y) \le \rho(x,z) + \rho(z,y)$ for any $x, y, z \in X$.

Thus, an asymmetric satisfies conditions 1 and 3 in the definition of a metric, but, maybe, does not satisfy condition 2.

Here is example of an asymmetric taken “from real life”: the length of the shortest path from one place to another by car in a city having one-way streets.

4.Vx. Prove that if $\rho : X \times X \to \mathbb{R}_+$ is an asymmetric, then the function

$$(x,y) \mapsto \rho(x,y) + \rho(y,x)$$

is a metric on X.

Let A and B be two bounded subsets of a metric space (X,ρ). The number $a_\rho(A,B) = \sup_{b\in B} \rho(b,A)$ is the *asymmetric distance from A to B*.

4.Wx. The function a_ρ on the set of bounded subsets of a metric space satisfies the triangle inequality in the definition of an asymmetric.

4.Xx. Let (X,ρ) be a metric space. A set $B \subset X$ is contained in all closed sets containing $A \subset X$ iff $a_\rho(A,B) = 0$.

4.Yx. Prove that a_ρ is an asymmetric on the set of all bounded closed subsets of a metric space (X,ρ).

Let A and B be two polygons on the plane. We define

$$a_\triangle(A,B) = S(B) - S(A \cap B) = S(B \smallsetminus A),$$

where $S(C)$ is the area of a polygon C.

4.36x. Prove that $a_\triangle$ is an asymmetric on the set of all planar polygons.

A pair (X, ρ), where ρ is an asymmetric on X, is an *asymmetric space*. Certainly, any metric space is an asymmetric space, too. Open and closed balls and spheres in an asymmetric space are defined as in a metric space, see Section 4′3.

4.Zx. *The set of all open balls of an asymmetric space is a base of a certain topology.*

We also say that this topology is *generated* by the asymmetric.

4.37x. Prove that the formula $a(x, y) = \max\{x - y, 0\}$ determines an asymmetric on $[0, \infty)$, and the topology generated by this asymmetric is the arrow topology, see Section 2′2.

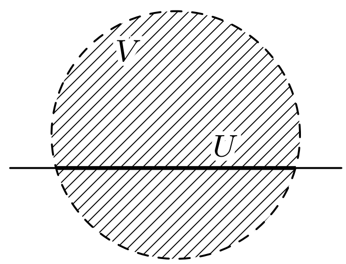
5. Subspaces

⌈5′1⌋ Topology for a Subset of a Space

Let (X, Ω) be a topological space, $A \subset X$. Denote by Ω_A the collection of sets $A \cap V$, where $V \in \Omega$: $\Omega_A = \{A \cap V \mid V \in \Omega\}$.

5.A. *The collection Ω_A is a topological structure in A.*

The pair (A, Ω_A) is a ***subspace*** of the space (X, Ω). The collection Ω_A is the ***subspace topology***, the ***relative topology***, or the topology ***induced*** on A by Ω, and its elements are said to be sets ***open*** in A.



5.B. The canonical topology on $\mathbb{R}^1$ coincides with the topology induced on $\mathbb{R}^1$ as on a subspace of $\mathbb{R}^2$.

5.1. Riddle. How to construct a base for the topology induced on A by using a base for the topology on X?

5.2. Describe the topological structures induced

(1) on the set $\mathbb{N}$ of positive integers by the topology of the real line;
(2) on $\mathbb{N}$ by the topology of the arrow;
(3) on the two-element set $\{1, 2\}$ by the topology of $\mathbb{R}_{T_1}$;
(4) on the same set by the topology of the arrow.

5.3. Is the half-open interval $[0, 1)$ open in the segment $[0, 2]$ regarded as a subspace of the real line?

5.C. *A set F is closed in a subspace $A \subset X$ iff F is the intersection of A and a closed subset of X.*

5.4. If a subset of a subspace is open (respectively, closed) in the ambient space, then it is also open (respectively, closed) in the subspace.

⌈5′2⌋ Relativity of Openness and Closedness

Sets that are open in a subspace are not necessarily open in the ambient space.

5.D. The unique open set in $\mathbb{R}^1$ which is also open in $\mathbb{R}^2$ is $\varnothing$.

However, the following is true.

5.E. An open set of an open subspace is open in the ambient space, i.e., if $A \in \Omega$, then $\Omega_A \subset \Omega$.

The same relation holds true for closed sets. Sets that are closed in the subspace are not necessarily closed in the ambient space. However, the following is true.

5.F. Closed sets of a closed subspace are closed in the ambient space.

5.5. Prove that a set U is open in X iff each point in U has a neighborhood V in X such that $U \cap V$ is open in V.

This allows us to say that the property of being open is *local*. Indeed, we can reformulate *5.5* as follows: a set is open iff it is open in a neighborhood of each of its points.

5.6. Show that the property of being closed is not local.

5.G Transitivity of Induced Topology. *Let (X, Ω) be a topological space, $X \supset A \supset B$. Then $(\Omega_A)_B = \Omega_B$, i.e., the topology induced on B by the relative topology of A coincides with the topology induced on B directly from X.*

5.7. Let (X, ρ) be a metric space, $A \subset X$. Then the topology on A generated by the induced metric $\rho|_{A \times A}$ coincides with the relative topology induced on A by the metric topology on X.

5.8. Riddle. The statement *5.7* is equivalent to a pair of inclusions. Which of them is less obvious?

⌈5′3⌋ Agreement on Notation for Topological Spaces

Different topological structures in the same set are considered simultaneously rather seldom. This is why a topological space is usually denoted by the same symbol as the set of its points, i.e., instead of (X, Ω) we write just X. The same applies to metric spaces: instead of (X, ρ) we write just X.

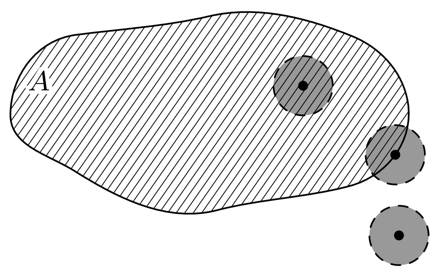
6. Position of a Point with Respect to a Set

This section is devoted to further expanding the vocabulary needed when we speak about phenomena in a topological space.

⌈6′1⌋ Interior, Exterior, and Boundary Points

Let X be a topological space, $A \subset X$ a subset, and $b \in X$ a point. The point b is

- an *interior* point of A if b has a neighborhood contained in A;
- an *exterior* point of A if b has a neighborhood disjoint with A;
- a *boundary* point of A if each neighborhood of b meets both A and the complement of A.



⌈6′2⌋ Interior and Exterior

The *interior* of a set A in a topological space X is the greatest (with respect to inclusion) open set in X contained in A, i.e., an open set that contains any other open subset of A. It is denoted by $\operatorname{Int} A$ or, in more detail, by $\operatorname{Int}_X A$.

6.A. *Every subset of a topological space has an interior. It is the union of all open sets contained in this set.*

6.B. *The interior of a set A is the set of interior points of A.*

6.C. *A set is open iff it coincides with its interior.*

6.D. Prove that in $\mathbb{R}$:

(1) $\operatorname{Int}[0,1) = (0,1)$,

(2) $\operatorname{Int} \mathbb{Q} = \varnothing$, and

(3) $\operatorname{Int}(\mathbb{R} \smallsetminus \mathbb{Q}) = \varnothing$.

6.1. Find the interior of $\{a, b, d\}$ in the space .

6.2. Find the interior of the interval $(0, 1)$ on the line with the Zariski topology.

The *exterior* of a set is the greatest open set disjoint with A. Obviously, the exterior of A is $\operatorname{Int}(X \smallsetminus A)$.

⌈6′3⌋ Closure

The *closure* of a set A is the smallest closed set containing A. It is denoted by $\operatorname{Cl} A$ or, more specifically, by $\operatorname{Cl}_X A$.

6.E. *Every subset of a topological space has a closure. It is the intersection of all closed sets containing this set.*

6.3. Prove that if A is a subspace of X and $B \subset A$, then $\operatorname{Cl}_A B = (\operatorname{Cl}_X B) \cap A$. Is it true that $\operatorname{Int}_A B = (\operatorname{Int}_X B) \cap A$?

A point b is an *adherent* point for a set A if all neighborhoods of b meet A.

6.F. *The closure of a set A is the set of the adherent points of A.*

6.G. *A set A is closed iff $A = \operatorname{Cl} A$.*

6.H. *The closure of a set A is the complement of the exterior of A. In formulas: $\operatorname{Cl} A = X \smallsetminus \operatorname{Int}(X \smallsetminus A)$, where X is the space and $A \subset X$.*

6.I. Prove that in $\mathbb{R}$ we have:

(1) $\operatorname{Cl}[0, 1) = [0, 1]$,
(2) $\operatorname{Cl} \mathbb{Q} = \mathbb{R}$, and
(3) $\operatorname{Cl}(\mathbb{R} \smallsetminus \mathbb{Q}) = \mathbb{R}$.

6.4. Find the closure of $\{a\}$ in .

⌈6′4⌋ Closure in Metric Space

Let A be a subset and b a point of a metric space (X, ρ). We recall that the distance $\rho(b, A)$ from b to A is $\inf\{\, \rho(b, a) \mid a \in A \,\}$ (see 4′14).

6.J. Prove that $b \in \operatorname{Cl} A$ iff $\rho(b, A) = 0$.

⌈6′5⌋ Boundary

The *boundary* of a set A is the set $\operatorname{Cl} A \smallsetminus \operatorname{Int} A$. It is denoted by $\operatorname{Fr} A$ or, in more detail, $\operatorname{Fr}_X A$.

6.5. Find the boundary of $\{a\}$ in .

6.K. The boundary of a set is the set of its boundary points.

6.L. Prove that a set A is closed iff $\operatorname{Fr} A \subset A$.

6.6. 1) Prove that $\operatorname{Fr} A = \operatorname{Fr}(X \smallsetminus A)$. 2) Find a formula for $\operatorname{Fr} A$ which is symmetric with respect to A and $X \smallsetminus A$.

6.7. The boundary of a set A equals the intersection of the closure of A and the closure of the complement of A: we have $\operatorname{Fr} A = \operatorname{Cl} A \cap \operatorname{Cl}(X \smallsetminus A)$.

⌈6′6⌋ Closure and Interior with Respect to a Finer Topology

6.8. Let Ω_1 and Ω_2 be two topological structures in X such that $\Omega_1 \subset \Omega_2$. Let Cl_i denote the closure with respect to Ω_i. Prove that $\operatorname{Cl}_1 A \supset \operatorname{Cl}_2 A$ for any $A \subset X$.

6.9. Formulate and prove a similar statement about the interior.

⌈6′7⌋ Properties of Interior and Closure

6.10. Prove that if $A \subset B$, then $\operatorname{Int} A \subset \operatorname{Int} B$.

6.11. Prove that $\operatorname{Int} \operatorname{Int} A = \operatorname{Int} A$.

6.12. Find out whether the following equalities hold true that for any sets A and B:

$$\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B, \tag{8}$$

$$\operatorname{Int}(A \cup B) = \operatorname{Int} A \cup \operatorname{Int} B. \tag{9}$$

6.13. Give an example in which one of equalities (8) and (9) is wrong.

6.14. In the example that you found when solving Problem *6.12*, an inclusion of one side into another one holds true. Does this inclusion hold true for arbitrary A and B?

6.15. Study the operator Cl in a way suggested by the investigation of Int undertaken in *6.10–6.14*.

6.16. Find $\operatorname{Cl}\{1\}$, $\operatorname{Int}[0,1]$, and $\operatorname{Fr}(2,+\infty)$ in the arrow.

6.17. Find $\operatorname{Int}\bigl((0,1] \cup \{2\}\bigr)$, $\operatorname{Cl}\{\, 1/n \mid n \in \mathbb{N} \,\}$, and $\operatorname{Fr} \mathbb{Q}$ in $\mathbb{R}$.

6.18. Find $\operatorname{Cl} \mathbb{N}$, $\operatorname{Int}(0,1)$, and $\operatorname{Fr}[0,1]$ in $\mathbb{R}_{T_1}$. How do you find the closure and interior of a set in this space?

6.19. Does a sphere contain the boundary of the open ball with the same center and radius?

6.20. Does a sphere contain the boundary of the closed ball with the same center and radius?

6.21. Find an example in which a sphere is disjoint with the closure of the open ball with the same center and radius.

⌈6′8⌋ Compositions of Closure and Interior

6.22 Kuratowski's Problem. How many pairwise distinct sets can one obtain from of a single set by using the operators Cl and Int?

The following problems will help you to solve Problem *6.22*.

6.22.1. Find a set $A \subset \mathbb{R}$ such that the sets A, $\operatorname{Cl} A$, and $\operatorname{Int} A$ are pairwise distinct.

6.22.2. Is there a set $A \subset \mathbb{R}$ such that

(1) A, $\operatorname{Cl} A$, $\operatorname{Int} A$, and $\operatorname{Cl}\operatorname{Int} A$ are pairwise distinct;
(2) A, $\operatorname{Cl} A$, $\operatorname{Int} A$, and $\operatorname{Int}\operatorname{Cl} A$ are pairwise distinct;
(3) A, $\operatorname{Cl} A$, $\operatorname{Int} A$, $\operatorname{Cl}\operatorname{Int} A$, and $\operatorname{Int}\operatorname{Cl} A$ are pairwise distinct?

If you find such sets, keep on going in the same way, and when you fail to proceed, try to formulate a theorem explaining the failure.

6.22.3. Prove that $\operatorname{Cl}\operatorname{Int}\operatorname{Cl}\operatorname{Int} A = \operatorname{Cl}\operatorname{Int} A$.

⌈6′9⌋ Sets with Common Boundary

6.23*. Find three open sets in the real line that have the same boundary. Is it possible to increase the number of such sets?

⌈6′10⌋ Convexity and Int, Cl, and Fr

Recall that a set $A \subset \mathbb{R}^n$ is *convex* if together with any two points it contains the entire segment connecting them (i.e., for any $x, y \in A$, every point z of the segment $[x, y]$ belongs to A).

Let A be a convex set in $\mathbb{R}^n$.

6.24. Prove that $\operatorname{Cl} A$ and $\operatorname{Int} A$ are convex.

6.25. Prove that A contains a ball if A is not contained in an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$.

6.26. When is $\operatorname{Fr} A$ convex?

⌈6′11⌋ Characterization of Topology by Operations of Taking Closure and Interior

6.27*. Suppose that Cl_* is an operator on the set of all subsets of a set X, which has the following properties:

(1) $\operatorname{Cl}_* \varnothing = \varnothing$,
(2) $\operatorname{Cl}_* A \supset A$,
(3) $\operatorname{Cl}_*(A \cup B) = \operatorname{Cl}_* A \cup \operatorname{Cl}_* B$,
(4) $\operatorname{Cl}_* \operatorname{Cl}_* A = \operatorname{Cl}_* A$.

Prove that $\Omega = \{ U \subset X \mid \operatorname{Cl}_*(X \smallsetminus U) = X \smallsetminus U \}$ is a topological structure and $\operatorname{Cl}_* A$ is the closure of a set A in the space (X, Ω).

6.28. Present a similar system of axioms for Int.

⌈6′12⌋ Dense Sets

Let A and B be two sets in a topological space X. A is *dense in* B if $\operatorname{Cl} A \supset B$, and A is *everywhere dense* if $\operatorname{Cl} A = X$.

6.M. *A set is everywhere dense iff it meets any nonempty open set.*

6.N. The set $\mathbb{Q}$ is everywhere dense in $\mathbb{R}$.

6.29. Give an explicit characterization of everywhere dense sets 1) in an indiscrete space, 2) in the arrow, and 3) in $\mathbb{R}_{T_1}$.

6.30. Prove that a topological space is discrete iff it contains a unique everywhere dense set. (By the way, which one?)

6.31. Formulate a necessary and sufficient condition on the topology of a space which has an everywhere dense point. Find spaces in Section 2 that satisfy this condition.

6.32. 1) Is it true that the union of everywhere dense sets is everywhere dense? 2) Is it true that the intersection of two everywhere dense sets is everywhere dense?

6.33. Prove that any two open everywhere dense sets have everywhere dense intersection.

6.34. Which condition in Problem *6.33* is redundant?

6.35*. 1) Prove that a countable intersection of open everywhere dense sets in $\mathbb{R}$ is everywhere dense. 2) Is it possible to replace $\mathbb{R}$ here by an arbitrary topological space?

6.36*. Prove that $\mathbb{Q}$ is not the intersection of countably many open sets in $\mathbb{R}$.

⌈6′13⌋ Nowhere Dense Sets

A set is *nowhere dense* if its exterior is everywhere dense.

6.37. Can a set be everywhere dense and nowhere dense simultaneously?

6.O. A set A is nowhere dense in X iff each neighborhood of each point $x \in X$ contains a point y such that the complement of A contains y together with a neighborhood of y.

6.38. Riddle. What can you say about the interior of a nowhere dense set?

6.39. Is $\mathbb{R}$ nowhere dense in $\mathbb{R}^2$?

6.40. Prove that if A is nowhere dense, then $\operatorname{Int}\operatorname{Cl} A = \varnothing$.

6.41. 1) Prove that the boundary of a closed set is nowhere dense. 2) Is this true for the boundary of an open set? 3) Is this true for the boundary of an arbitrary set?

6.42. Prove that a finite union of nowhere dense sets is nowhere dense.

6.43. Prove that for every set A there exists a greatest open set B in which A is dense. The extreme cases $B = X$ and $B = \varnothing$ mean that A is either everywhere dense or nowhere dense, respectively.

6.44*. Prove that $\mathbb{R}$ is not the union of countably many nowhere-dense subsets.

⌈6′14⌋ Limit Points and Isolated Points

A point b is a *limit point* of a set A if each neighborhood of b meets $A \smallsetminus b$.

6.P. Every limit point of a set is its adherent point.

6.45. Present an example in which an adherent point is not a limit one.

A point b is an *isolated point* of a set A if $b \in A$ and b has a neighborhood disjoint with $A \smallsetminus b$.

6.Q. *A set A is closed iff A contains all of its limit points.*

6.46. Find limit and isolated points of the sets $(0, 1] \cup \{2\}$ and $\{\, 1/n \mid n \in \mathbb{N} \,\}$ in $\mathbb{Q}$ and in $\mathbb{R}$.

6.47. Find limit and isolated points of the set $\mathbb{N}$ in $\mathbb{R}_{T_1}$.

⌈6′15⌋ Locally Closed Sets

A subset A of a topological space X is *locally closed* if each point of A has a neighborhood U such that $A \cap U$ is closed in U (cf. Problems *5.5*–*5.6*).

6.48. Prove that the following conditions are equivalent:

(1) A is locally closed in X;
(2) A is an open subset of its closure $\operatorname{Cl} A$;
(3) A is the intersection of open and closed subsets of X.

7. Ordered Sets

This section is devoted to orders. They are structures on sets and occupy a position in Mathematics almost as profound as topological structures. After a short general introduction, we focus on relations between structures of these two types. Similarly to metric spaces, partially ordered sets possess natural topological structures. This is a source of interesting and important examples of topological spaces. As we will see later (in Section 21), practically all finite topological spaces appear in this way.

⌈7′1⌋ Strict Orders

A *binary relation* on a set X is a set of ordered pairs of elements of X, i.e., a subset $R \subset X \times X$. Many relations are denoted by special symbols, like $\prec$, $\vdash$, $\equiv$, or $\sim$. When such notation is used, there is a tradition to write xRy instead of writing $(x, y) \in R$. So, we write $x \vdash y$, or $x \sim y$, or $x \prec y$, etc. This generalizes the usual notation for the classical binary relations $=$, $<$, $>$, $\leq$, $\subset$, etc.

A binary relation $\prec$ on a set X is a *strict partial order*, or just a *strict order* if it satisfies the following two conditions:

- *Irreflexivity*: There is no $a \in X$ such that $a \prec a$.
- *Transitivity*: $a \prec b$ and $b \prec c$ imply $a \prec c$ for any $a, b, c \in X$.

7.A Antisymmetry. Let $\prec$ be a strict partial order on a set X. There are no $x, y \in X$ such that $x \prec y$ and $y \prec x$ simultaneously.

7.B. Relation $<$ in the set $\mathbb{R}$ of real numbers is a strict order.

The formula $a \prec b$ is sometimes read as "a is less than b" or "b is greater than a", but it is often read as "b follows a" or "a precedes b". The advantage of the latter two ways of reading is that then the relation $\prec$ is not associated too closely with the inequality between real numbers.

⌈7′2⌋ Nonstrict Orders

A binary relation $\preceq$ on a set X is a *nonstrict partial order*, or just a *nonstrict order*, if it satisfies the following three conditions:

- *Transitivity*: If $a \preceq b$ and $b \preceq c$, then $a \preceq c$ for any $a, b, c \in X$.
- *Antisymmetry*: If $a \preceq b$ and $b \preceq a$, then $a = b$ for any $a, b \in X$.
- *Reflexivity*: $a \preceq a$ for any $a \in X$.

7.C. The relation $\leq$ on $\mathbb{R}$ is a nonstrict order.

7.D. In the set $\mathbb{N}$ of positive integers, the relation $a \mid b$ (a *divides* b) is a nonstrict partial order.

7.1. Is the relation $a \mid b$ a nonstrict partial order on the set $\mathbb{Z}$ of integers?

7.E. Inclusion determines a nonstrict partial order on the set of subsets of any set X.

⌈7′3⌋ Relation between Strict and Nonstrict Orders

7.F. For each strict order $\prec$, there is a relation $\preceq$ defined on the same set as follows: $a \preceq b$ if either $a \prec b$, or $a = b$. This relation is a nonstrict order.

The nonstrict order $\preceq$ of *7.F* is *associated* with the original strict order $\prec$.

7.G. For each nonstrict order $\preceq$, there is a relation $\prec$ defined on the same set as follows: $a \prec b$ if $a \preceq b$ and $a \neq b$. This relation is a strict order.

The strict order $\prec$ of *7.G* is *associated* with the original nonstrict order $\preceq$.

7.H. The constructions of Problems *7.F* and *7.G* are mutually inverse: applied one after another in any order, they give the initial relation.

Thus, strict and nonstrict orders determine each other. They are just different incarnations of the same structure of order. We have already met a similar phenomenon in topology: open and closed sets in a topological space determine each other and provide different ways for describing a topological structure.

A set equipped with a partial order (either strict or nonstrict) is a *partially ordered set* or, briefly, a *poset*. More formally speaking, a partially ordered set is a pair $(X, \prec)$ formed by a set X and a strict partial order $\prec$ on X. Certainly, instead of a strict partial order $\prec$ we can use the corresponding nonstrict order $\preceq$.

Which of the orders, strict or nonstrict, prevails in each specific case is a matter of convenience, taste, and tradition. Although it would be handy to keep both of them available, nonstrict orders conquer situation by situation. For instance, nobody introduces special notation for strict divisibility. Another example: the symbol $\subseteq$, which is used to denote nonstrict inclusion, is replaced by the symbol $\subset$, which is almost never understood as a designation solely for strict inclusion.

In abstract considerations, we use both kinds of orders: strict partial orders are denoted by the symbol $\prec$, nonstrict ones by the symbol $\preceq$.

⌈7′4⌋ Cones

Let $(X, \prec)$ be a poset and let $a \in X$. The set $\{x \in X \mid a \prec x\}$ is the *upper cone* of a, and the set $\{x \in X \mid x \prec a\}$ the *lower cone* of a. The element a does not belong to its cones. Adding a to them, we obtain *completed* cones: the *upper completed cone* or *star* $C_X^+(a) = \{x \in X \mid a \preceq x\}$ and the *lower completed cone* $C_X^-(a) = \{x \in X \mid x \preceq a\}$.

7.I Properties of Cones. *Let $(X, \prec)$ be a poset. Then we have:*

(1) $C_X^+(b) \subset C_X^+(a)$, *provided that* $b \in C_X^+(a)$;
(2) $a \in C_X^+(a)$ *for each* $a \in X$;
(3) $C_X^+(a) = C_X^+(b)$ *implies* $a = b$.

7.J Cones Determine an Order. *Let X be an arbitrary set. Suppose for each $a \in X$ we fix a subset $C_a \subset X$ so that*

(1) $b \in C_a$ *implies* $C_b \subset C_a$,
(2) $a \in C_a$ *for each* $a \in X$, *and*
(3) $C_a = C_b$ *implies* $a = b$.

We write $a \prec b$ if $b \in C_a$. Then the relation $\prec$ is a nonstrict order on X, and for this order we have $C_X^+(a) = C_a$.

7.2. Let $C \subset \mathbb{R}^3$ be a set. Consider the relation $\lhd_C$ on $\mathbb{R}^3$ defined as follows: $a \lhd_C b$ if $b - a \in C$. What properties of C imply that $\lhd_C$ is a partial order on $\mathbb{R}^3$? What are the upper and lower cones in the poset $(\mathbb{R}^3, \lhd_C)$?

7.3. Prove that each convex cone C in $\mathbb{R}^3$ with vertex $(0,0,0)$ and such that $P \cap C = \{(0,0,0)\}$ for some plane P satisfies the conditions found in the solution to Problem *7.2*.

7.4. Consider the space-time $\mathbb{R}^4$ of special relativity theory, where points represent moment-point events and the first three coordinates x_1, x_2 and x_3 are the spatial coordinates, while the fourth one, t, is the time. This space carries a relation, "the *event* (x_1, x_2, x_3, t) *precedes* (and may influence) the *event* $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{t})$". The relation is defined by the inequality

$$c(\tilde{t} - t) \geq \sqrt{(\widetilde{x}_1 - x_1)^2 + (\widetilde{x}_2 - x_2)^2 + (\widetilde{x}_3 - x_3)^2}.$$

Is this a partial order? If yes, then what are the upper and lower cones of an event?

7.5. Answer the versions of questions of the preceding problem in the case of two- and three-dimensional analogs of this space, where the number of spatial coordinates is 1 and 2, respectively.

⌈7′5⌋ Position of an Element with Respect to a Set

Let $(X, \prec)$ be a poset, $A \subset X$ a subset. Then b is the *greatest element* of A if $b \in A$ and $c \preceq b$ for every $c \in A$. Similarly, b is the *smallest element* of A if $b \in A$ and $b \preceq c$ for every $c \in A$.

7.K. An element $b \in A$ is the smallest element of A iff $A \subset C_X^+(b)$; an element $b \in A$ is the greatest element of A iff $A \subset C_X^-(b)$.

7.L. Each set has at most one greatest and at most one smallest element.

An element b of a set A is a *maximal* element of A if A contains no element c such that $b \prec c$. An element b is a *minimal* element of A if A contains no element c such that $c \prec b$.

7.M. An element b of A is maximal iff $A \cap C_X^-(b) = b$; an element b of A is minimal iff $A \cap C_X^+(b) = b$.

7.6. Riddle. 1) How are the notions of maximal and greatest elements related? 2) What can you say about a poset in which these notions coincide for each subset?

⌈7′6⌋ Linear Orders

Please, notice: the definition of a strict order does not require that for any $a, b \in X$ we have either $a \prec b$, or $b \prec a$, or $a = b$. The latter condition is called a *trichotomy*. In terms of the corresponding nonstrict order, it is reformulated as follows: any two elements $a, b \in X$ are *comparable*: either $a \preceq b$, or $b \preceq a$.

A strict order satisfying trichotomy is *linear* (or *total*). The corresponding poset is *linearly* ordered (or *totally* ordered). It is also called just an *ordered set*.[9] Some orders do satisfy trichotomy.

7.N. The order $<$ on the set $\mathbb{R}$ of real numbers is linear.

This is the most important example of a linearly ordered set. The words and images rooted in it are often extended to all linearly ordered sets. For example, cones are called *rays*, upper cones become *right rays*, while lower cones become *left rays*.

7.7. A poset $(X, \prec)$ is linearly ordered iff $X = C_X^+(a) \cup C_X^-(a)$ for each $a \in X$.

7.8. The order $a \mid b$ on the set $\mathbb{N}$ of positive integers is not linear.

7.9. For which X is the relation of inclusion on the set of all subsets of X a linear order?

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[9]Quite a bit of confusion was brought into the terminology by Bourbaki. At that time, linear orders were called orders, nonlinear orders were called partial orders, and, in occasions when it was not known if the order under consideration was linear, the fact that this was unknown was explicitly stated. Bourbaki suggested to drop the word *partial*. Their motivation for this was that a partial order is a phenomenon more general than a linear order, and hence deserves a shorter and simpler name. This suggestion was commonly accepted in the French literature, but in English literature it would imply abolishing a nice short word, *poset*, which seems to be an absolutely impossible thing to do.

⌈7′7⌋ Topologies Determined by Linear Order

7.O. *Let $(X, \prec)$ be a linearly ordered set. Then the set X itself and all right rays of X, i.e., sets of the form $\{x \in X \mid a \prec x\}$, where a runs through X, constitute a base for a topological structure in X.*

The topological structure determined by this base is the *right ray topology* of the linearly ordered set $(X, \prec)$. The *left ray topology* is defined similarly: it is generated by the base consisting of X and sets of the form $\{x \in X \mid x \prec a\}$ with $a \in X$.

7.10. The topology of the arrow (see Section 2) is the right ray topology of the half-line $[0, \infty)$ equipped with the order $<$.

7.11. Riddle. To what extent is the assumption that the order be linear necessary in Theorem *7.O*? Find a weaker condition that implies the conclusion of Theorem *7.O* and allows us to speak about the topological structure described in Problem *2.2* as the right ray topology of an appropriate partial order on the plane.

7.P. *Let $(X, \prec)$ be a linearly ordered set. Then the subsets of X having the forms*

- $\{x \in X \mid a \prec x\}$, *where a runs through X,*
- $\{x \in X \mid x \prec a\}$, *where a runs through X,*
- $\{x \in X \mid a \prec x \prec b\}$, *where a and b run through X*

constitute a base for a topological structure in X.

The topological structure determined by this base is the *interval topology* of the linearly ordered set $(X, \prec)$.

7.12. Prove that the interval topology is the smallest topological structure containing the right ray and left ray topological structures.

7.Q. *The canonical topology of the line is the interval topology of $(\mathbb{R}, <)$.*

⌈7′8⌋ Poset Topology

7.R. *Let $(X, \preceq)$ be a poset. Then the subsets of X having the form $\{x \in X \mid a \preceq x\}$, where a runs through the entire X, constitute a base for a topological structure in X.*

The topological structure generated by this base is the *poset topology*.

7.S. *In the poset topology, each point $a \in X$ has the smallest (with respect to inclusion) neighborhood. This is $\{x \in X \mid a \preceq x\}$.*

7.T. *The following properties of a topological space are equivalent:*

(1) *each point has a smallest neighborhood,*

(2) *the intersection of any collection of open sets is open,*

(3) *the union of any collection of closed sets is closed.*

A space satisfying the conditions of Theorem *7.T* is a ***smallest neighborhood space***.[10] In such a space, open and closed sets satisfy the same conditions. In particular, the set of all closed sets of a smallest neighborhood space is also a topological structure, which is ***dual*** to the original one. It corresponds to the opposite partial order.

7.13. How to characterize points open in the poset topology in terms of the partial order? Answer the same question about closed points. (Slightly abusing the terminology, here by *points* we mean the corresponding singletons.)

7.14. Directly describe open sets in the poset topology of $\mathbb{R}$ with order $<$.

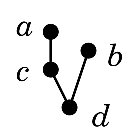
7.15. Consider a partial order on the set $\{a, b, c, d\}$ where the strict inequalities are: $c \prec a$, $d \prec c$, $d \prec a$, and $d \prec b$. Check that this is a partial order and the corresponding poset topology is the topology of described in Problem *2.3* (1).

7.16. Describe the closure of a point in a poset topology.

7.17. Which singletons are dense in a poset topology?

⌈7′9⌋ How to Draw a Poset

Now we can explain the pictogram , by which we denote the space introduced in Problem *2.3*(1). It describes the partial order on $\{a, b, c, d\}$ that determines the topology of this space by *7.15*. Indeed, if we place a, b, c, and d, i.e., the elements of the set under consideration, at vertices of the graph of the pictogram, as shown in the picture, then the vertices marked by comparable elements are connected by a segment or ascending broken line, and the greater element corresponds to the higher vertex.

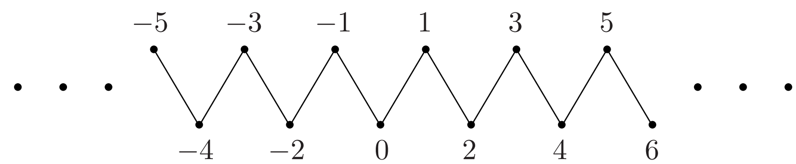


In this way, we can represent any finite poset by a diagram. Elements of the poset are represented by points. We have $a \prec b$ if and only if the following two conditions are fulfilled: 1) the point representing b lies above the point representing a, and 2) the two points are connected either by a segment or by a polyline consisting of segments that connect points representing intermediate elements of a chain $a \prec c_1 \prec c_2 \prec \cdots \prec c_n \prec b$. We could have connected by a segment any two points corresponding to comparable elements, but this would make the diagram excessively cumbersome. This is why the segments that are determined by the others via transitivity are not drawn. Such a diagram representing a poset is its ***Hasse diagram***.

7.U. Prove that any finite poset is determined by a Hasse diagram.

[10] This class of topological spaces was introduced and studied by P. S. Alexandrov in 1935. Alexandrov called them ***discrete***. Nowadays, the term discrete space is used for a much narrower class of topological spaces (see Section 2). The term ***smallest neighborhood space*** was introduced by Christer Kiselman.

7.V. Describe the poset topology on the set $\mathbb{Z}$ of integers defined by the following Hasse diagram:



The space of Problem *7.V* is the ***digital line***, or ***Khalimsky line***. In this space, each even number is closed and each odd one is open.

7.18. Associate with each even integer $2k$ the interval $(2k-1, 2k+1)$ of length 2 centered at this point, and with each odd integer $2k-1$, the singleton $\{2k-1\}$. Prove that a set of integers is open in the Khalimsky topology iff the union of sets associated to its elements is open in $\mathbb{R}$ with the standard topology.

7.19. Among the topological spaces described in Section 2, find all those obtained as posets with the poset topology. In the cases of finite sets, draw Hasse diagrams describing the corresponding partial orders.

8. Cyclic Orders

⌈8′1⌋ Cyclic Orders in Finite Sets

Recall that a *cyclic order* on a finite set X is a linear order considered up to cyclic permutation. The linear order allows us to enumerate elements of the set X by positive integers, so that $X = \{x_1, x_2, \dots, x_n\}$. A cyclic permutation transposes the first k elements with the last $n-k$ elements without changing the order inside each of the two parts of the set:

$$(x_1, x_2, \dots, x_k, x_{k+1}, x_{k+2}, \dots, x_n) \mapsto (x_{k+1}, x_{k+2}, \dots, x_n, x_1, x_2, \dots, x_k).$$

When we consider a cyclic order, it makes no sense to say that one of its elements is greater than another one, since an appropriate cyclic permutation puts the two elements in the opposite order. However, it makes sense to say that an element *immediately* precedes another one. Certainly, the very last element immediately precedes the very first one: indeed, a nontrivial cyclic permutation puts the first element immediately after the last one.

In a cyclically ordered finite set, each element a has a unique element b next to a, i.e., which follows a immediately. This determines a map of the set onto itself, namely, the simplest cyclic permutation

$$x_i \mapsto \begin{cases} x_{i+1} & \text{if } i < n, \\ x_1 & \text{if } i = n. \end{cases}$$

This permutation acts transitively (i.e., any element is mapped to any other one by an appropriate iteration of the permutation).

8.A. *Any map $T : X \to X$ that transitively acts on X determines a cyclic order on X such that each $a \in X$ precedes $T(a)$.*

8.B. *An n-element set possesses exactly $(n-1)!$ pairwise distinct cyclic orders.*

In particular, a two-element set has only one cyclic order (which is so uninteresting that sometimes it is said to make no sense), while any three-element set possesses two cyclic orders.

⌈8′2x⌋ Cyclic Orders in Infinite Sets

One can consider cyclic orders on an infinite set. However, most of what was said above does not apply to cyclic orders on infinite sets without an adjustment. In particular, most of them cannot be described by showing pairs of elements that are next to each other. For example, points of a circle can be cyclically ordered clockwise (or counter-clockwise), but no point immediately follows another point with respect to this cyclic order.

Such "continuous" cyclic orders are defined almost in the same way as cyclic orders on finite sets were defined above. The difference is that sometimes it is impossible to define *cyclic permutations of a set* in the necessary quantity, and we have to replace them by *cyclic transformations of linear orders*. Namely, a cyclic order is defined as a linear order considered up to cyclic transformations, where by a *cyclic transformation* of a linear order $\prec$ on a set X we mean a passage from $\prec$ to a linear order $\prec'$ such that X splits into subsets A and B such that the restrictions of $\prec$ and $\prec'$ to each of them coincide, while $a \prec b$ and $b \prec' a$ for any $a \in A$ and $b \in B$.

8.Cx. *Existence of a cyclic transformation transforming linear orders to each other determines an equivalence relation on the set of all linear orders on a set.*

A *cyclic order* on a set is an equivalence class of linear orders with respect to the above equivalence relation.

8.Dx. Prove that for a finite set this definition is equivalent to the definition in the preceding section.

8.Ex. Prove that the cyclic "counter-clockwise" order on a circle can be defined along the definition of this section, but cannot be defined as a linear order modulo cyclic transformations of the set for whatever definition of cyclic transformations of circle. Describe the linear orders on the circle that determine this cyclic order up to cyclic transformations of orders.

8.Fx. Let A be a subset of a set X. If two linear orders $\prec'$ and $\prec$ on X are obtained from each other by a cyclic transformation, then their restrictions to A are also obtained from each other by a cyclic transformation.

8.Gx Corollary. A cyclic order on a set induces a well-defined cyclic order on every subset of this set.

8.Hx. *A cyclic order on a set X can be recovered from the cyclic orders induced by it on all three-element subsets of X.*

> ***8.Hx.1.*** A cyclic order on a set X can be recovered from the cyclic orders induced by it on all three-element subsets of X containing a fixed element $a \in X$.

Theorem *8.Hx* allows us to describe a cyclic order as a ternary relation. Namely, let a, b, and c be elements of a cyclically ordered set. Then we write $[a \prec b \prec c]$ if the induced cyclic order on $\{a, b, c\}$ is determined by the linear order in which the inequalities in the brackets hold true (i.e., b follows a and c follows b).

8.Ix. *Let X be a cyclically ordered set. Then the ternary relation $[a \prec b \prec c]$ on X has the following properties:*

(1) *for any pairwise distinct* $a, b, c \in X$*, we have either* $[a \prec b \prec c]$*, or* $[b \prec a \prec c]$*, but not both;*

(2) $[a \prec b \prec c]$*, iff* $[b \prec c \prec a]$*, iff* $[c \prec a \prec b]$*, for any* $a, b, c \in X$*;*

(3) *if* $[a \prec b \prec c]$ *and* $[a \prec c \prec d]$*, then* $[a \prec b \prec d]$.

Vice versa, a ternary relation on X *having these four properties determines a cyclic order on the set* X.

⌈8′3x⌋ Topology of Cyclic Order

8.Jx. *Let* X *be a cyclically ordered set. Then the sets that belong to the interval topology of every linear order determining the cyclic order on* X *constitute a topological structure in* X.

The topology defined in *8.Jx* is the *cyclic order topology*.

8.Kx. The cyclic order topology determined by the cyclic counterclockwise order of S^1 is the topology generated by the metric $\rho(x, y) = |x - y|$ on $S^1 \subset \mathbb{C}$.

Proofs and Comments

1.A The question is so elementary that it is difficult to find more elementary facts which we could use in the proof. What does it mean that A consists of a elements? This means, say, that we can count elements of A one by one, assigning to them numbers 1, 2, 3, ... and the last element will receive number a. It is known that the result does not depend on the order in which we count. (In fact, one can develop a set theory which would include a theory of counting, and in which this is a theorem. However, since we have no doubts about this fact, let us use it without proof.) Therefore, we can start counting elements of B by counting those in A. Counting the elements in A is done first, and then, if there are some elements of B that are not in A, counting is continued. Thus, the number of elements in A is less than or equal to the number of elements in B.

1.B Recall that, by the definition of an inclusion, $A \subset B$ means that each element of A is an element of B. Therefore, the statement that we must prove can be rephrased as follows: each element of A is an element of A. This is a tautology.

1.C Recall that, by the definition of an inclusion, $A \subset B$ means that each element of A is an element of B. Thus, we need to prove that any element of $\varnothing$ belongs to A. This is true because $\varnothing$ does not contain any elements. If you are not satisfied with this argument (since it may seem a little bit strange), then let us resort to the question whether this can be wrong. How can it happen that $\varnothing$ is not a subset of A? This is possible only if $\varnothing$ contains an element which is not an element of A. However, $\varnothing$ does not contain such elements because $\varnothing$ has no elements at all.

1.D We must prove that each element of A is an element of C. Let $x \in A$. Since $A \subset B$, it follows that $x \in B$. Since $B \subset C$, the latter belonging (i.e., $x \in B$) implies $x \in C$. This is what we had to prove.

1.E We have already seen that $A \subset A$. Hence, if $A = B$, then, indeed, $A \subset B$ and $B \subset A$. On the other hand, $A \subset B$ means that each element of A belongs to B, while $B \subset A$ means that each element of B belongs to A. Hence, A and B have the same elements, i.e., they are equal.

1.G It is easy to construct a set A with $A \notin A$. Take $A = \varnothing$, or $A = \mathbb{N}$, or $A = \{1\}$, ...

1.H Take $A = \{1\}$, $B = \{\{1\}\}$, and $C = \{\{\{1\}\}\}$. It is more difficult to construct sets A, B, and C such that $A \in B$, $B \in C$, and $A \in C$. Take $A = \{1\}$, $B = \{\{1\}\}$, and $C = \{\{1\}, \{\{1\}\}\}$.

2.A What should we check? The first axiom reads here that the union of any collection of subsets of X is a subset of X. Well, this is true. If $A \subset X$ for each $A \in \Gamma$, then, obviously, $\bigcup_{A\in\Gamma} A \subset X$. We check the second axiom exactly in the same way. Finally, we obviously have $\varnothing \subset X$ and $X \subset X$.

2.B Yes, it is. If one of the united sets is X, then the union is X, otherwise the union in empty. If one of the sets to intersect is $\varnothing$, then the intersection is $\varnothing$. Otherwise, the intersection equals X.

2.C First, show that $\bigcup_{A\in\Gamma} A \cap \bigcup_{B\in\Sigma} B = \bigcup_{A\in\Gamma, B\in\Sigma}(A\cap B)$. Therefore, if A and B are intervals, then the right-hand side is a union of intervals. This proves that Ω satisfies the second axiom of topological structure. The first and third axioms are obvious here.

If you think that a set which is a union of intervals is too simple, then, please try to answer the following question (which has nothing to do with the problem under consideration, though). Let $\{r_n\}_{n=1}^{\infty} = \mathbb{Q}$ (i.e., we numbered all rational numbers). Prove that $\bigcup_{n=1}^{\infty}(r_n - 2^{-n}, r_n + 2^{-n})$ does not contain all real numbers, although this is a union of intervals that contains all (!) rational numbers.

2.D The union of any collection of open sets is open. The intersection of any finite collection of open sets is open. The empty set and the whole space are open.

2.E

(3)

$$x \in \bigcap_{A\in\Gamma}(X \smallsetminus A) \iff \forall\, A \in \Gamma: \ x \in X \smallsetminus A$$
$$\iff \ \forall\, A \in \Gamma: \ x \notin A \ \iff x \notin \bigcup_{A\in\Gamma} A \iff x \in X \smallsetminus \bigcup_{A\in\Gamma} A.$$

(4) Replace both sides of the formula by their complements in X and put $B = X \smallsetminus A$.

2.F (1) Let $\Gamma = \{F_\alpha\}$ be a collection of closed sets. We must verify that $\bigcap_\alpha F_\alpha$ is closed, i.e., $X \smallsetminus \bigcap_\alpha F_\alpha$ is open. Indeed, by the second De Morgan formula we have

$$X \smallsetminus \bigcap_\alpha F_\alpha = \bigcup_\alpha (X \smallsetminus F_\alpha),$$

which is open by the first axiom of topological structure.

(2) Similar to (1); use the first De Morgan formula and the second axiom of topological structure.

(3) Obvious.

2.G In any topological space, the empty set and the whole space are both open and closed. Any set in a discrete space is both open and closed. Half-open intervals on the line are neither open nor closed. Cf. the next problem.

2.H Yes, it is, because its complement $\mathbb{R} \smallsetminus [a,b] = (-\infty, a) \cup (b, +\infty)$ is open.

2.Ix Let $U \subset \mathbb{R}$ be an open set. For each $x \in U$, let $(m_x, M_x) \subset U$ be the largest open interval containing x (take the union of all open intervals in U that contain x). Since U is open, such intervals exist. Any two such intervals either coincide or are disjoint.

2.Lx Conditions (a) and (c) from Problem *2.13* are obviously fulfilled. To prove (b), we use Theorem *2.Mx* and argue by contradiction. Suppose that two sets A and B contain no arithmetic progressions of length n. If $A \cup B$ contains a sufficiently long progression, then A or B contains a progression of length n, a contradiction.

3.A To prove an equivalence of two statements, prove two implications.
$\Rightarrow$ Present U as a union of elements of Σ. Each point $x \in U$ is contained in at least one of these sets. Such a set can be taken for V. It is contained in U since it participates in a union equal to U.
$\Leftarrow$ We must prove that each $U \in \Omega$ is a union of elements of Σ. For each point $x \in U$, choose according to the assumption a set $V_x \in \Sigma$ such that $x \in V_x \subset U$ and consider $\bigcup_{x \in U} V_x$. Notice that $\bigcup_{x \in U} V_x \subset U$ because $V_x \subset U$ for each $x \in U$. On the other hand, each point $x \in U$ is contained in its own V_x and hence in $\bigcup_{x \in U} V_x$. Therefore, $U \subset \bigcup_{x \in U} V_x$. Thus, $U = \bigcup_{x \in U} V_x$.

3.B $\Rightarrow$ X, being an open set in any topology, is the union of some sets in Σ. The intersection of any two sets in Σ is open, and, therefore, it also is a union of base sets.
$\Leftarrow$ Let us prove that the set of unions of all collections of elements of Σ satisfies the axioms of topological structure. The first axiom is obviously fulfilled since the union of unions is a union. Let us prove the second axiom (the intersection of two open sets is open). Let $U = \bigcup_\alpha A_\alpha$ and $V = \bigcup_\beta B_\beta$, where $A_\alpha, B_\beta \in \Sigma$. Then

$$U \cap V = \Big(\bigcup_\alpha A_\alpha\Big) \cap \Big(\bigcup_\beta B_\beta\Big) = \bigcup_{\alpha,\beta} (A_\alpha \cap B_\beta),$$

and since, by assumption, $A_\alpha \cap B_\beta$ is a union of elements of Σ, so is the intersection $U \cap V$. In the third axiom, we need to check only the part concerning the entire X. By assumption, X is a union of sets in Σ.

3.D Let Σ_1 and Σ_2 be bases of topological structures Ω_1 and Ω_2 in a set X. Obviously, $\Omega_1 \subset \Omega_2$ iff $\forall U \in \Sigma_1\ \forall x \in U\ \exists V \in \Sigma_2 :\ x \in V \subset U$. Now recall that $\Omega_1 = \Omega_2$ iff $\Omega_1 \subset \Omega_2$ and $\Omega_2 \subset \Omega_1$.

4.A Indeed, it makes sense to check that all conditions in the definition of a metric are fulfilled for *each* triple of points x, y, and z.

4.B The triangle inequality in this case takes the form $|x-y| \le |x-z| + |z-y|$. Putting $a = x - z$ and $b = z - y$, we transform the triangle inequality into the well-known inequality $|a + b| \le |a| + |b|$.

4.C As in the solution of Problem *4.B*, the triangle inequality takes the form: $\sqrt{\sum_{i=1}^n (a_i + b_i)^2} \le \sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2}$. Two squarings and an obvious simplification reduce this inequality to the well-known Cauchy inequality $(\sum a_i b_i)^2 \le \sum a_i^2 \sum b_i^2$.

4.E We must prove that every point $y \in B_{r-\rho(a,x)}(x)$ belongs to $B_r(a)$. In terms of distances, this means that $\rho(y,a) < r$ if $\rho(y,x) < r - \rho(a,x)$ and $\rho(a,x) < r$. By the triangle inequality, $\rho(y,a) \le \rho(y,x) + \rho(x,a)$. Replacing the first summand on the right-hand side of the latter inequality by a greater number $r - \rho(a,x)$, we obtain the required inequality. The second inclusion is proved in a similar way.

4.F $\boxed{\Longrightarrow}$ Show that if $d = \operatorname{diam} A$ and $a \in A$, then $A \subset D_d(a)$.
$\boxed{\Longleftarrow}$ Use the fact that $\operatorname{diam} D_d(a) \le 2d$. (Cf. *4.11*.)

4.G This follows from Problem *4.E*, Theorem *3.B* and Assertion *3.C*.

4.H For this metric, the balls are open intervals. Each open interval in $\mathbb{R}$ is a ball. The standard topology on $\mathbb{R}$ is determined by the base consisting of all open intervals.

4.I $\boxed{\Longrightarrow}$ If $a \in U$, then $a \in B_r(x) \subset U$ and $B_{r-\rho(a,x)}(a) \in B_r(x) \subset U$, see *4.E*.
$\boxed{\Longleftarrow}$ U is a union of balls, and, therefore, U is open in the metric topology.

4.J An indiscrete space does not have sufficiently many open sets. For $x, y \in X$ and $r = \rho(x,y) > 0$, the ball $D_r(x)$ is nonempty and does not coincide with the whole space (it does not contain y).

4.K $\boxed{\Longrightarrow}$ For $x \in X$, put $r = \min\{\rho(x,y) \mid y \in X \smallsetminus x\}$. Which points are in $B_r(x)$? $\boxed{\Longleftarrow}$ Obvious. (Cf. *4.19*.)

4.L $\boxed{\Longrightarrow}$ The condition $\rho(b,A) = 0$ means that each ball centered at b meets A, i.e., b does not belong to the complement of A (since A is closed, the complement of A is open). $\boxed{\Longleftarrow}$ Obvious.

4.Mx Condition (2) is obviously fulfilled. Put $r(A,B) = \sup\limits_{a\in A} \rho(a,B)$, so that $d_\rho(A,B) = \max\{r(A,B), r(B,A)\}$. To prove that (3) is also fulfilled, it suffices to prove that $r(A,C) \le r(A,B) + r(B,C)$ for any $A, B, C \subset X$. We easily see that $\rho(a, C) \le \rho(a, b) + \rho(b, C)$ for all $a \in A$ and $b \in B$. Hence, we have $\rho(a, C) \le \rho(a, b) + r(B, C)$, whence

$$\rho(a, C) \le \inf_{b \in B} \rho(a, b) + r(B, C) = \rho(a, B) + r(B, C) \le r(A, B) + r(B, C),$$

which implies the required inequality.

***4.N*x** By *4.Mx*, d_ρ satisfies conditions (2) and (3) from the definition of a metric. From *4.L* it follows that if the Hausdorff distance between two closed sets A and B equals zero, then $A \subset B$ and $B \subset A$, i.e., $A = B$. Thus, d_ρ satisfies the condition (1).

***4.O*x** $d_\triangle(A, B)$ is the area of the symmetric difference $A \triangle B = (A \smallsetminus B) \cup (B \smallsetminus A)$ of A and B. The first two axioms of metric are obviously fulfilled. Prove the triangle inequality by using the inclusion $A \smallsetminus B \subset (C \smallsetminus B) \cup (A \smallsetminus C)$.

***4.R*x** Clearly, the metric in *4.A* is an ultrametric. The other metrics are not: for each of them you can find three points x, y, and z such that $\rho(x, y) = \rho(x, z) + \rho(z, y)$.

***4.S*x** The definition of an ultrametric implies that none of the pairwise distances between the points a, b, and c is greater than each of the other two.

***4.T*x** By *4.Sx*, if $y \in S_r(x)$ and $r > s > 0$, then $B_s(y) \subset S_r(x)$.

***4.U*x** Let $x - z = \frac{r_1}{s_1} p^{\alpha_1}$ and $z - y = \frac{r_2}{s_2} p^{\alpha_2}$, where $\alpha_1 \le \alpha_2$. Then we have

$$x - y = p^{\alpha_1} \left(\frac{r_1}{s_1} + \frac{r_2}{s_2} p^{\alpha_2 - \alpha_1} \right) = p^{\alpha_1} \frac{r_1 s_2 + r_2 s_1 p^{\alpha_2 - \alpha_1}}{s_1 s_2},$$

whence $p(x, y) \le p^{-\alpha_1} = \max\{\rho(x, z), \rho(z, y)\}$.

5.A We must check that Ω_A satisfies the axioms of topological structure. Consider the first axiom. Let $\Gamma \subset \Omega_A$ be a collection of sets in Ω_A. We must prove that $\bigcup_{U \in \Gamma} U \in \Omega_A$. For each $U \in \Gamma$, find $U_X \in \Omega$ such that $U = A \cap U_X$. This is possible due to the definition of Ω_A. Transform the union under consideration: $\bigcup_{U \in \Gamma} U = \bigcup_{U \in \Gamma} (A \cap U_X) = A \cap \bigcup_{U \in \Gamma} U_X$. The union $\bigcup_{U \in \Gamma} U_X$ belongs to Ω (i.e., is open in X) as the union of sets open in X. (Here we use the fact that Ω, being a topology on X, satisfies the first axiom of topological structure.) Therefore, $A \cap \bigcup_{U \in \Gamma} U_X$ belongs to Ω_A. Similarly we can check the second axiom. The third axiom: $A = A \cap X$, and $\varnothing = A \cap \varnothing$.

5.B Let us prove that a subset of $\mathbb{R}^1$ is open in the relative topology iff it is open in the canonical topology.

$\Longrightarrow$ The intersection of an open disk with $\mathbb{R}^1$ is either an open interval or the empty set. Any open set in the plane is a union of open disks. Therefore, the intersection of any open set of the plane with $\mathbb{R}^1$ is a union

of open intervals. Thus, it is open in $\mathbb{R}^1$.
$\Leftarrow$ Prove this part on your own.

5.C $\Rightarrow$ The complement $A \smallsetminus F$ is open in A, i.e., $A \smallsetminus F = A \cap U$, where U is open in X. What closed set cuts F on A? It is cut by $X \smallsetminus U$. Indeed, we have $A \cap (X \smallsetminus U) = A \smallsetminus (A \cap U) = A \smallsetminus (A \smallsetminus F) = F$.
$\Leftarrow$ This is proved in a similar way.

5.D No disk of $\mathbb{R}^2$ is contained in $\mathbb{R}$.

5.E If $A \in \Omega$ and $B \in \Omega_A$, then $B = A \cap U$, where $U \in \Omega$. Therefore, $B \in \Omega$ is the intersection of two sets, A and U, belonging to Ω.

5.F Follow the solution to the preceding Problem, *5.E*, but use *5.C* instead of the definition of the relative topology.

5.G The core of the proof is the equality $(U \cap A) \cap B = U \cap B$. It holds true because $B \subset A$, and we apply it to $U \in \Omega$. When U runs through Ω, the right-hand side of the equality $(U \cap A) \cap B = U \cap B$ runs through Ω_B, while the left-hand side runs through $(\Omega_A)_B$. Indeed, elements of Ω_B are intersections $U \cap B$ with $U \in \Omega$, and elements of $(\Omega_A)_B$ are intersections $V \cap B$ with $V \in \Omega_A$, but V, in turn, being an element of Ω_A, is the intersection $U \cap A$ with $U \in \Omega$.

6.A The union of all open sets contained in A, firstly, is open (as a union of open sets), and, secondly, contains every open set that is contained in A (i.e., it is the greatest one among those sets).

6.B Let x be an interior point of A (i.e., there exists an open set U_x such that $x \in U_x \subset A$). Then $U_x \subset \operatorname{Int} A$ (because $\operatorname{Int} A$ is *the greatest* open set contained in A), whence $x \in \operatorname{Int} A$. Vice versa, if $x \in \operatorname{Int} A$, then the set $\operatorname{Int} A$ itself is a neighborhood of x contained in A.

6.C $\Rightarrow$ If U is open, then U is the greatest open subset of U, and hence coincides with the interior of U.
$\Leftarrow$ A set coinciding with its interior is open since the interior is open.

6.D

(1) $[0, 1)$ is not open in the line, while $(0, 1)$ is. Therefore, $\operatorname{Int}[0, 1) = (0, 1)$.
(2) Since any interval contains an irrational point, $\mathbb{Q}$ contains no nonempty set open in the classical topology of $\mathbb{R}$. Therefore, $\operatorname{Int} \mathbb{Q} = \varnothing$.
(3) Since any interval contains rational points, $\mathbb{R} \smallsetminus \mathbb{Q}$ does not contain a nonempty set open in the classical topology of $\mathbb{R}$. Therefore, $\operatorname{Int}(\mathbb{R} \smallsetminus \mathbb{Q}) = \varnothing$.

6.E The intersection of all closed sets containing A, firstly, is closed (as an intersection of closed sets), and, secondly, is contained in every closed set that contains A (i.e., it is the smallest one among those sets). Cf. the proof of Theorem *6.A*. In general, properties of closure can be obtained from properties of interior by replacing unions with intersections and vice versa.

6.F If $x \notin \operatorname{Cl} A$, then there exists a closed set F such that $F \supset A$ and $x \notin F$, whence $x \in U = X \smallsetminus F$. Thus, x is not an adherent point for A. Prove the converse implication on your own, cf. *6.H*.

6.G Cf. the proof of Theorem *6.C*.

6.H The intersection of all closed sets containing A is the complement of the union of all open sets contained in $X \smallsetminus A$.

6.I (1) The half-open interval $[0, 1)$ is not closed, and $[0, 1]$ is closed; (2)–(3) The exterior of each of the sets $\mathbb{Q}$ and $\mathbb{R} \smallsetminus \mathbb{Q}$ is empty since each interval contains both rational and irrational numbers.

6.J $\boxed{\Rightarrow}$ If b is an adherent point for A, then $\forall\, \varepsilon > 0\ \exists\, a \in A \cap D_\varepsilon(b)$, whence $\forall\, \varepsilon > 0\ \exists\, a \in A :\ \rho(a, b) < \varepsilon$. Thus, $\rho(b, A) = 0$.
$\boxed{\Leftarrow}$ This is an easy exercise.

6.K If $x \in \operatorname{Fr} A$, then $x \in \operatorname{Cl} A$ and $x \notin \operatorname{Int} A$. Hence, firstly, each neighborhood of x meets A, secondly, no neighborhood of x is contained in A, and therefore each neighborhood of x meets $X \smallsetminus A$. Thus, x is a boundary point of A. Prove the converse on your own.

6.L Since $\operatorname{Int} A \subset A$, it follows that $\operatorname{Cl} A = A$ iff $\operatorname{Fr} A \subset A$.

6.M $\boxed{\Rightarrow}$ Argue by contradiction. A set A disjoint with an open set U is contained in the closed set $X \smallsetminus U$. Therefore, if U is nonempty, then A is not everywhere dense.
$\boxed{\Leftarrow}$ A set meeting each nonempty open set is contained in only one closed set: the entire space. Hence, its closure is the whole space, and this set is everywhere dense.

6.N This is *6.I*(2).

6.O The condition means that each neighborhood of each point contains an exterior point of A. This, in turn, means that the exterior of A is everywhere dense.

6.Q $\boxed{\Rightarrow}$ This is Theorem *6.P*.
$\boxed{\Leftarrow}$ Hint: any point of $\operatorname{Cl} A \smallsetminus A$ is a limit point of A.

7.F We need to check that the relation "$a \prec b$ or $a = b$" satisfies the three conditions from the definition of a nonstrict order. Doing this, we can use only the fact that $\prec$ satisfies the conditions from the definition of a strict order. Let us check the transitivity. Suppose that $a \preceq b$ and $b \preceq c$. This means that either 1) $a \prec b \prec c$, or 2) $a = b \prec c$, or 3) $a \prec b = c$, or 4) $a = b = c$.
1) In this case, $a \prec c$ by transitivity of $\prec$, and so $a \preceq c$. 2) We have $a \prec c$,

whence $a \preceq c$. 3) We have $a \prec c$, whence $a \preceq c$. 4) Finally, $a = c$, whence $a \preceq c$. Other conditions are checked similarly.

7.I Assertion (1) follows from transitivity of the order. Indeed, consider an arbitrary element $c \in C_X^+(b)$. By the definition of a cone, $b \preceq c$, while the condition $b \in C_X^+(a)$ means that $a \preceq b$. By transitivity, this implies that $a \preceq c$, i.e., $c \in C_X^+(a)$. We have thus proved that each element of $C_X^+(b)$ belongs to $C_X^+(a)$. Hence, $C_X^+(b) \subset C_X^+(a)$, as required.
Assertion (2) follows from the definition of a cone and the reflexivity of order. Indeed, by definition, $C_X^+(a)$ consists of all b such that $a \preceq b$, and, by reflexivity of order, $a \preceq a$.
Assertion (3) follows similarly from the antisymmetry: the assumption $C_X^+(a) = C_X^+(b)$ together with assertion (2) implies that $a \preceq b$ and $b \preceq a$, which together with antisymmetry implies that $a = b$.

7.J By Theorem *7.I*, cones in a poset have the properties that form the hypothesis of the theorem to be proved. When proving Theorem *7.I*, we showed that these properties follow from the corresponding conditions in the definition of a partial nonstrict order. In fact, they are equivalent to these conditions. Permuting words in the proof of Theorem *7.I*, we obtain a proof of Theorem *7.J*.

7.O By Theorem *3.B*, it suffices to prove that the intersection of any two right rays is a union of right rays. Let $a, b \in X$. Since the order is linear, either $a \prec b$, or $b \prec a$. Let $a \prec b$. Then

$$\{x \in X \mid a \prec x\} \cap \{x \in X \mid b \prec x\} = \{x \in X \mid b \prec x\}.$$

7.R By Theorem *3.C*, it suffices to prove that each element of the intersection of two cones, say, $C_X^+(a)$ and $C_X^+(b)$, is contained in the intersection together with a whole cone of the same kind. Assume that $c \in C_X^+(a) \cap C_X^+(b)$ and $d \in C_X^+(c)$. Then $a \preceq c \preceq d$ and $b \preceq c \preceq d$, whence $a \preceq d$ and $b \preceq d$. Therefore, $d \in C_X^+(a) \cap C_X^+(b)$. Hence, $C_X^+(c) \subset C_X^+(a) \cap C_X^+(b)$.

7.T Equivalence of the second and third properties follows from the De Morgan formulas, as in *2.F*. Let us prove that the first property implies the second one. Consider the intersection of an arbitrary collection of open sets. For each of its points, every set in this collection is a neighborhood. Therefore, its smallest neighborhood is contained in each of the sets to be intersected. Hence, the smallest neighborhood of the point is contained in the intersection, which we denote by U. Thus, each point of U lies in U together with its neighborhood. Since U is the union of these neighborhoods, it is open.

Now let us prove that if the intersection of any collection of open sets is open, then any point has a smallest neighborhood. Where can one get such a neighborhood from? How to construct it? Take all neighborhoods of a point x and consider their intersection U. By assumption, U is open. It contains x. Therefore, U is a neighborhood of x. This neighborhood, being the intersection of all neighborhoods, is contained in each of the neighborhoods. Thus, U is the smallest neighborhood.

7.V The minimal base of this topology consists of singletons of the form $\{2k-1\}$ with $k \in \mathbb{Z}$ and three-element sets of the form $\{2k-1, 2k, 2k+1\}$, where again $k \in \mathbb{Z}$.