

Chapter 1

RECREATIONAL  
MATHEMATICS

*Recreational problems have survived,  
not because they were fostered by the  
textbook writers, but because of their  
inherent appeal to our love of mystery.*

**Vera Sanford**

Before taking up the noteworthy mathematical thinkers and their memorable problems, a brief overview of the history of mathematical recreations may benefit the reader. For more historical details see, *e.g.*, the books [6], [118], [133, Vol. 4], [153, Ch. VI], [167, Vol. II]. According to V. Sanford [153, Ch. VI], recreational mathematics comprises two principal divisions: those that depend on object manipulation and those that depend on computation.

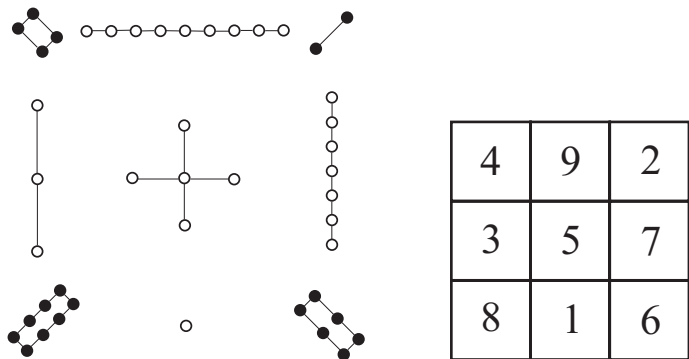


FIGURE 1.1. The oldest magic square—lo-shu

Perhaps the oldest known example of the first group is the magic square shown in the figure above. Known as *lo-shu* to Chinese mathematicians around 2200 B.C., the magic square was supposedly constructed during the reign of the Emperor Yü (see, *e.g.*, [61, Ch. II], or [167, Vol. I, p. 28]). Chinese myth [27] holds that Emperor Yü saw a tortoise of divine creation

swimming in the Yellow River with the *lo-shu*, or magic square figure, adorning its shell. The figure on the left shows the *lo-shu* configuration where the numerals from 1 to 9 are composed of knots in strings with black knots for even and white knots for odd numbers.

The Rhind (or Ahmes) papyrus,<sup>1</sup> dating to around 1650 B.C., suggests that the early Egyptians based their mathematics problems in puzzle form. As these problems had no application to daily life, perhaps their main purpose was to provide intellectual pleasure. One of the earliest instances, named “*As I was going to St. Ives*”, has the form of a nursery rhyme (see [153]):

“Seven houses; in each are 7 cats; each cat kills 7 mice; each mouse would have eaten 7 ears of spelt; each ear of spelt will produce 7 hekat. What is the total of all of them?”<sup>2</sup>

The ancient Greeks also delighted in the creation of problems strictly for amusement. One name familiar to us is that of Archimedes, whose the *cattle problem* appears on pages 41 to 43. It is one of the most famous problems in number theory, whose complete solution was not found until 1965 by a digital computer.

The classical Roman poet Virgil (70 B.C.–19 B.C.) described in the *Aeneid* the legend of the Phoenician princess Dido. After escaping tyranny in her home country, she arrived on the coast of North Africa and asked the local ruler for a small piece of land, only as much land as could be encompassed by a bull’s hide. The clever Dido then cut the bull’s hide into the thinnest possible strips, enclosed a large tract of land and built the city of Carthage that would become her new home. Today the problem of enclosing the maximum area within a fixed boundary is recognized as a classical *isoperimetric problem*. It is regarded as the first problem in a new mathematical discipline, established 17 centuries later, as calculus of variations. Jacob Steiner’s elegant solution of Dido’s problem is included in this book.

Another of the problems from antiquity is concerned with a group of men arranged in a circle so that if every  $k$ th man is removed going around the circle, the remainder shall be certain specified (favorable) individuals. This problem, appearing for the first time in Ambrose of Milan’s book *ca.* 370, is known as the Josephus problem, and it found its way not just into later European manuscripts, but also into Arabian and Japanese books. Depending on the time and location where the particular version of the Josephus problem was raised, the survivors and victims were sailors and

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<sup>1</sup>Named after Alexander Henry Rhind (1833–1863), a Scottish antiquarian, lawyer and Egyptologist who acquired the papyrus in 1858 in Luxor (Egypt).

<sup>2</sup>T. Eric Peet’s translation of *The Rhind Mathematical Papyrus*, 1923.

smugglers, Christians and Turks, sluggards and scholars, good guys and bad guys, and so on. This puzzle attracted attention of many outstanding scientists, including Euler, Tait, Wilf, Graham, and Knuth.

As Europe emerged from the Dark Ages, interest in the arts and sciences reawakened. In eighth-century England, the mathematician and theologian Alcuin of York wrote a book in which he included a problem that involved a man wishing to ferry a wolf, a goat and a cabbage across a river. The solution shown on pages 240–242 demonstrates how one can solve the problem accurately by using graph theory. River-crossing problems under specific conditions and constraints were very popular in medieval Europe. Alcuin, Tartaglia, Trenchant and Leurechon studied puzzles of this type. A variant involves how three couples should cross the river in a boat that cannot carry more than two people at a time. The problem is complicated by the jealousy of the husbands; each husband is too jealous to leave his wife in the company of either of the other men.

Four centuries later, mathematical puzzles appear in the third section of Leonardo Fibonacci's *Liber Abaci*, published in 1202. This medieval scholar's most famous problem, the *rabbit problem*, gave rise to the unending sequence that bears his name: the Fibonacci sequence, or Fibonacci numbers as they are also known, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... (see pages 12–13).

Yet another medieval mathematician, ibn Khallikan (1211–1282), formulated a brain teaser requiring the calculation of the total number of wheat grains placed on a standard  $8 \times 8$  chessboard. The placement of the grains must respect the following distribution: 1 grain is placed on the first square, 2 grains on the second, 4 on the third, 8 on the fourth, and so on, doubling the number for each successive square. The resulting number of grains is  $2^{64} - 1$ , or 18,446,744,073,709,551,615. Ibn Khallikan presented this problem in the form of the tale of the Indian king Shirham who wanted to reward the Grand Vizier Sissa ben Dahir for having invented chess. Sissa asked for the number of grains on the chessboard if each successive position is the next number in a geometric progression. However, the king could not fulfill Sissa's wish; indeed, the number of grains is so large that it is far greater than the world's annual production of wheat grains. Speaking in broad terms, ibn Khallikan's was one of the earliest chess problems.

Ibn Kallikan's problem of the number of grains is a standard illustration of geometric progressions, copied later by Fibonacci, Pacioli, Clavius and Tartaglia. Arithmetic progressions were also used in these entertaining problems. One of the most challenging problems appeared in Buteo's book *Logistica* (Lyons, 1559, 1560):<sup>3</sup>

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<sup>3</sup>The translation from Latin is given in [153], p. 64.

“A mouse is at the top of a poplar tree 60 braccia<sup>4</sup> high, and a cat is on the ground at its foot. The mouse descends  $\frac{1}{2}$  of a braccia a day and at night it turns back  $\frac{1}{6}$  of a braccia. The cat climbs one braccia a day and goes back  $\frac{1}{4}$  of a braccia each night. The tree grows  $\frac{1}{4}$  of a braccia between the cat and the mouse each day and it shrinks  $\frac{1}{8}$  of a braccia every night. In how many days will the cat reach the mouse and how much has the tree grown in the meantime, and how far does the cat climb?”

At about the same time Buteo showed enviable knowledge of the general laws of permutations and combinations; moreover, he constructed a combination lock with movable cylinders displayed in Figure 1.2.<sup>5</sup>

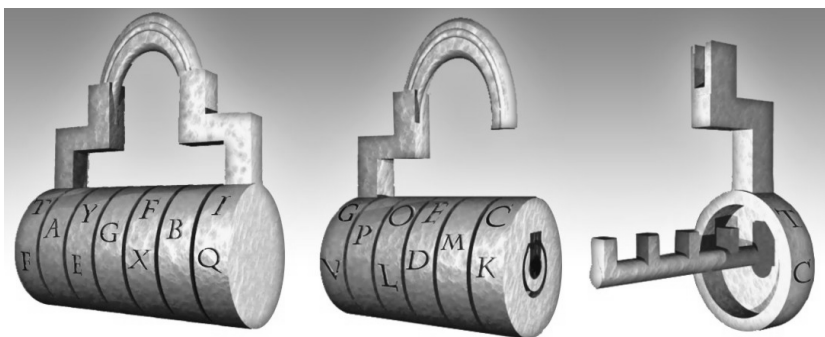
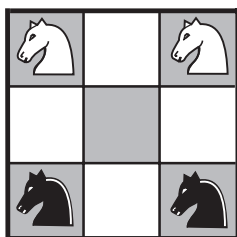


FIGURE 1.2. Buteo's combination lock (1559)

In 1512 Guarini devised a chessboard problem in which the goal is to effect the exchange of two black and two white knights, with each pair placed at the corners of a  $3 \times 3$  chessboard (see figure left), in the minimum number of moves. The solution of this problem by using graph theory is shown on pages 274–276. People's interest in chess problems and the challenge they provide has lasted from the Middle Ages, through the Renaissance and to the present day.



While the Italian mathematicians Niccolo Tartaglia (1500–1557) and Girolamo Cardano (1501–1576) labored jointly to discover the explicit formula for the solution of cubic algebraic equations, they also found time for recreational problems and games in their mathematical endeavors. Tartaglia's *General Trattato* (1556) described several interesting tasks; four of which,

<sup>4</sup>Braccia is an old Italian unit of length.

<sup>5</sup>Computer artwork, sketched according to the illustration from Buteo's *Logistica* (Lyons, 1559, 1560).

the weighing problem, the division of 17 horses, the wine and water problem, and the ferryboat problem, are described on pages 20, 24, 25 and 173.

Girolamo Cardano was one the most famous scientists of his time and an inventor in many fields. Can you believe that the joint connecting the gear box to the rear axle of a rear wheel drive car is known to the present day by a version of his name—the cardan shaft? In an earlier book, *De Subtilitate* (1550), Cardano presented a game, often called the *Chinese ring puzzle* (Figure 1.3), that made use of a bar with several rings on it that remains popular even now. The puzzle’s solution is closely related to Gray’s error-correcting binary codes introduced in the 1930s by the engineer Frank Gray. The Chinese ring puzzle also bears similarities to the *Tower of Hanoi*, invented in 1883 by Edouard Lucas (1842–1891), which is also discussed later in the book.

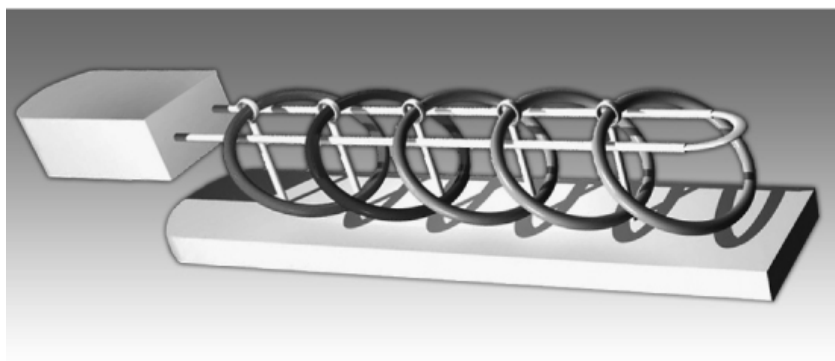


FIGURE 1.3. Chinese ring puzzle

Many scholars consider *Problèmes Plaisans et Délectables*, by Claude Gaspar Bachet (1581–1638), to be the first book on mathematical puzzles and tricks. Most of the famous puzzles and curious problems invented before the seventeenth century may be found in Bachet’s delightful book. In addition to Bachet’s original “delectable” problems, the book contains puzzles by Alcuin of York, Pacioli, Tartaglia and Cardano, and other puzzles of Asian origin. Bachet’s book, first published in 1612 and followed by the second edition published in 1624, probably served as the inspiration for subsequent works devoted to mathematical recreation.

Other important writers on the subject include the Jesuit scholar Jean Leurechon (1591–1670), who published under the name of Hendrik van Etten, and Jacques Ozanam (1640–1717). Etten’s work, *Mathematical Recreations, or a Collection of Sundry Excellent Problems Out of Ancient and Modern Philosophers Both Useful and Recreative*, first published in French

in 1624 with an English translation appearing in 1633, is a compilation of mathematical problems interspersed with mechanical puzzles and experiments in hydrostatics and optics that most likely borrowed heavily from Bachet's work.

Leonhard Euler (1707–1783), one of the world's greatest mathematicians whose deep and exacting investigations led to the foundation and development of new mathematical disciplines, often studied mathematical puzzles and games. Euler's results from the *seven bridges of Königsberg* problem (pages 230–232) presage the beginnings of graph theory. The *thirty-six officers problem* and orthogonal Latin squares (or Eulerian squares), discussed by Euler and later mathematicians, have led to important work in combinatorics. Euler's conjecture on the construction of mutually orthogonal squares found resolution nearly two hundred years after Euler himself initially posed the problem. These problems, and his examination of the chessboard *knight's re-entrant tour problem*, are described on pages 188 and 258. A knight's re-entrant path consists of moving a knight so that it moves successively to each square once and only once and finishes its tour on the starting square. This famous problem has a long history and dates back to the sixth century in India. P. M. Roget's half-board solution (1840), shown in Figure 1.4, offers a remarkably attractive design.

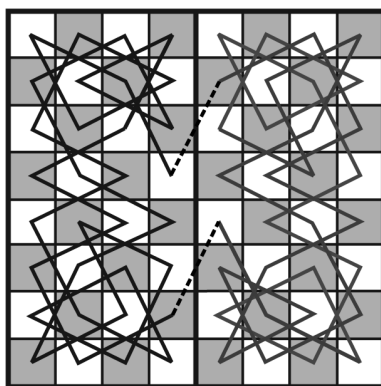


FIGURE 1.4. Knight's re-entrant path—Roget's solution

In 1850 Franz Nauck posed another classic chess problem, the *eight queens problem*, that calls for the number of ways to place eight queens on a chessboard so that no two queens attack each other. Gauss gave a solution of this problem, albeit incomplete in the first attempts. Further details about the *eight queens problem* appear on pages 269–273. In that same year, Thomas P. Kirkman (1806–1895) put forth the *schoolgirls problem* presented on pages

189 to 192. Several outstanding mathematicians, Steiner, Cayley and Sylvester among them, dealt with this combinatorial problem and other related problems. Although some of these problems remain unsolved even now, the subject continues to generate important papers on combinatorial design theory.

In 1857 the eminent Irish mathematician William Hamilton (1788–1856) invented the *icosian game* in which one must locate a path along the edges of a regular dodecahedron that passes through each vertex of the dodecahedron once and only once (see pages 234–237). As in Euler’s Königsberg bridges problem, the Hamiltonian game is related to graph theory. In modern terminology, this task requires a Hamiltonian cycle in a certain graph and it is one of the most important open problems not only in graph theory but in the whole mathematics. The Hamiltonian cycle problem is closely connected to the famous *traveling salesman problem* that asks for an optimal route between some places on a map with given distances.

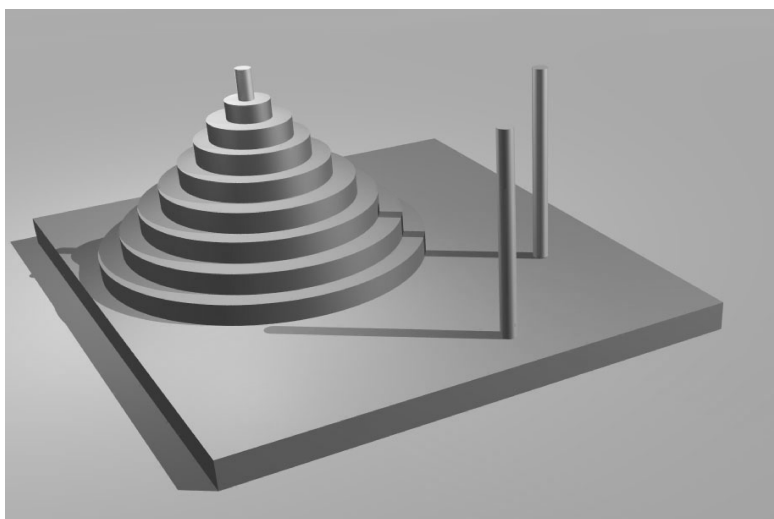
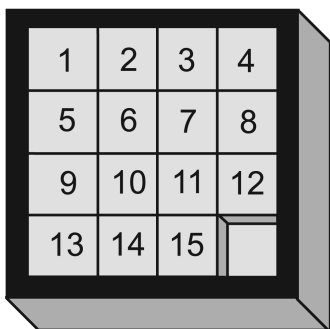


FIGURE 1.5. The Tower of Hanoi

The French mathematician François Edouard Lucas, best known for his results in number theory, also made notable contributions to recreational mathematics, among them, as already mentioned, the *Tower of Hanoi* (Figure 1.5), which is covered on pages 196–199, and many other amusing puzzles. Lucas’ four-volume book *Récréations Mathématiques* (1882–94), together with Rouse Ball’s, *Mathematical Recreations and Problems*, published in 1892, have become classic works on recreational mathematics.

No discussion of recreational mathematics would be complete without including Samuel Loyd (1841–1911) and Henry Ernest Dudeney (1857–1931), two of the most renowned creators of mathematical diversions. Loyd and Dudeney launched an impressive number of games and puzzles that remain as popular now as when they first appeared. Loyd’s ingenious toy-puzzle the “15 Puzzle” (known also as the “Boss Puzzle”, or “Jeu de Taquin”) is popular even today. The “15 Puzzle” (figure below) consists of a square divided into 16 small squares and holds 15 square blocks numbered from 1



to 15. The task is to start from a given initial arrangement and set these numbered blocks into the required positions (say, from 1 to 15), using the vacant square for moving blocks. For many years after its appearance in 1878, people all over the world were obsessed by this toy-puzzle. It was played in taverns, factories, in homes, in the streets, in the royal palaces, even in the Reichstag (see page 2430 in [133, Vol. 4].

Martin Gardner (b. 1914 Tulsa, OK), most certainly deserves mention as perhaps the greatest twentieth-century popularizer of mathematics and mathematical recreations. During the twenty-five years in which he wrote his *Mathematical Games* column for the *Scientific American*, he published quantities of amusing problems either posed or solved by notable mathematicians.



## Chapter 10

## CHESS

*In many cases, mathematics as well as chess,  
is an escape from reality.*

**Stanislaw Ulam**

*Chess is the gymnasium of the mind.*

**Blaise Pascal**

*The chessboard is the world,  
the pieces are the phenomena of the Universe,  
the rules of the game are what we call the laws of Nature.*

**Thomas Huxley**

Puzzles concern the chessboards (of various dimensions and different shapes) and chess pieces have always lent themselves to mathematical recreations. Over the last five centuries so many problems of this kind have arisen. Find a re-entrant path on the chessboard that consists of moving a knight so that it moves successively to each square once and only once and finish its tour on the starting square. How to place 8 queens on the  $8 \times 8$  chessboard so that no queen can be attacked by another? For many years I have been interested in these types of chess-math problems and, in 1997, I wrote the book titled *Mathematics and Chess* (Dover Publications) [138] as a collection of such problems. Some of them are presented in this chapter.

Mathematics, the queen of the sciences, and chess, the queen of games, share an axiomatic approach and an abstract way of reasoning in solving problems. The logic of the rules of play, the chessboard's geometry, and the concepts "right" and "wrong" are reminiscent of mathematics. Some mathematical problems can be solved in an elegant manner using some elements of chess. Chess problems and chess-math puzzles can ultimately improve analytical reasoning and problem solving skills.

In its nature, as well as in the very structure of the game, chess resembles several branches of mathematics. Solutions of numerous problems and puzzles on a chessboard are connected and based on mathematical facts from graph theory, topology, number theory, arithmetic, combinatorial analysis, geometry, matrix theory, and other topics. In 1913, Ernst Zermelo used

these connections as a basis to develop his theory of game strategies, which is considered as one of the forerunners of game theory.

The most important mathematical challenge of chess has been how to develop algorithms that can play chess. Today, computer engineers, programmers and chess enthusiasts design chess-playing machines and computer programs that can defeat the world's best chess players. Recall that, in 1997, IBM's computer Deep Blue beat Garry Kasparov, the world champion of that time.

Many great mathematicians were interested in chess problems: Euler, Gauss, Vandermonde, de Moivre, Legendre. On the other hand, several world-class chess players have made contributions to mathematics, before all, Emanuel Lasker. One of the best English contemporary grandmasters and twice world champion in chess problem solving, John Nunn, received his Ph.D. in mathematics from Oxford University at the age of twenty-three.

The aim of this chapter is to present amusing puzzles and tasks that contain both mathematical and chess properties. We have mainly focused on those problems posed and/or solved by great mathematicians. The reader will see some examples of knight's re-entrant tours (or "knight's circles") found by Euler, de Moivre and Vandermond. We have presented a variant of knight's chessboard (uncrossed) tour, solved by the outstanding computer scientist Donald Knuth using a computer program. You will also find the famous eight queens problem, that caught Gauss' interest. An amusing chessboard problem on non-attacking rooks was solved by Euler.

None of the problems and puzzles exceed a high school level of difficulty; advanced mathematics is excluded. In addition, we presume that the reader is familiar with chess rules.

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***Abraham de Moivre*** (1667–1754) (→ p. 304)

***Pierre de Montmort*** (1678–1733) (→ p. 304)

***Alexandre Vandermonde*** (1735–1796) (→ p. 305)

***Leonhard Euler*** (1707–1783) (→ p. 305)

### **Knight's re-entrant route**

Among all re-entrant paths on a chessboard, the knight's tour is doubtless the most interesting and familiar to many readers.

**Problem 10.1.** Find a re-entrant route on a standard  $8 \times 8$  chessboard which consists of moving a knight so that it moves successively to each square once and only once and finishes its tour on the starting square.

Closed knight's tours are often called "knight's circles". This remarkable and very popular problem was formulated in the sixth century in India [181]. It was mentioned in the Arab, *Mansubas*<sup>1</sup>, of the ninth century A.D. There are well-known examples of the knight's circle in the Hamid I Mansubas (Istanbul Library) and the Al-Hakim Mansubas (Ryland Library, Manchester). This task has delighted people for centuries and continues to do so to this day. In his beautiful book *Across the Board: The Mathematics of Chessboard Problems* [181] J. J. Watkins describes his unforgettable experience at a Broadway theater when he was watching the famous sleight-of-hand artist Ricky Jay performing blindfolded a knights tour on stage.

The knight's circle also interested such great mathematicians as Euler, Vandermonde, Legendre, de Moivre, de Montmort, and others. De Montmort and de Moivre provided some of the earliest solutions at the beginning of the eighteenth century. Their method is applied to the standard  $8 \times 8$  chessboard divided into an inner square consisting of 16 cells surrounded by

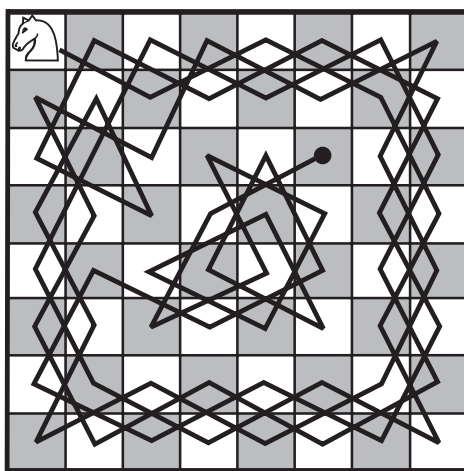


FIGURE 10.1. Knight's tour—de Moivre's solution

an outer ring of cells two deep. If the knight starts from a cell in the outer ring, it always moves along this ring filling it up and continuing into an inner ring cell only when absolutely necessary. The knight's tour, shown in

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<sup>1</sup>The Mansubas, a type of book, collected and recorded the games, as well as remarkably interesting positions, accomplished by well-known chess players.

Figure 10.1, was composed by de Moivre (which he sent to Brook Taylor). Surprisingly, the first 23 moves are in the outer two rows. Although it passes all 64 cells, the displayed route is not a re-entrant route.

Even though L. Bertrand of Geneva initiated the analysis, according to *Mémoires de Berlin* for 1759, Euler made the first serious mathematical analysis of this subject. In his letter to the mathematician Goldbach (April 26, 1757), Euler gave a solution to the knight's re-entrant path shown in Figure 10.2.

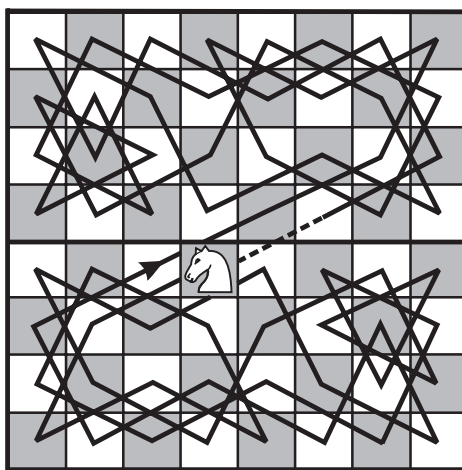
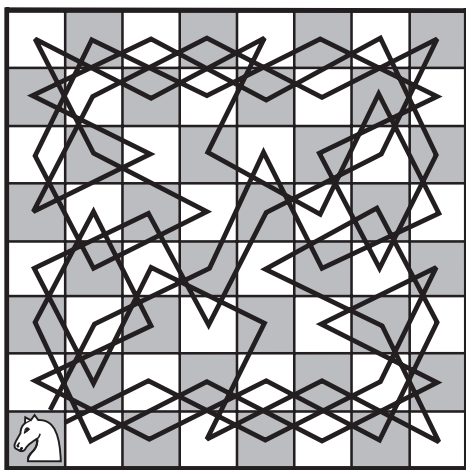


FIGURE 10.2. Euler's knight's circle solution

FIGURE 10.3. Euler's half-board solution

Euler's method consists of a knight's random movement over the board as long as it is possible, taking care that this route leaves the least possible number of untraversed cells. The next step is to interpolate these untraversed cells into various parts of the circuit to make the re-entrant route. Details on this method may be found in the books, *Mathematical Recreations and Essays* by Rouse Ball and Coxeter [150], *Across the Board* by J. J. Watkins [181] and *In the Czardom of Puzzles* (in Russian) [107] by E. I. Ignat'ev, the great Russian popularizer of mathematics. Figure 10.3 shows an example of Euler's modified method where the first 32 moves are restricted to the lower half of the board, then the same tour is repeated in a symmetric fashion for the upper half of the board.

Vandermonde's approach to solving the knight's re-entrant route uses fractions of the form  $x/y$ , where  $x$  and  $y$  are the coordinates of a traversed cell.<sup>2</sup>

<sup>2</sup> *L'Histoire de l'Académie des Sciences* for 1771, Paris 1774, pp. 566–574.

For example,  $1/1$  is the lower left corner square (a1) and  $8/8$  is the upper right corner square (h8). The values of  $x$  and  $y$  are limited by the dimensions of the chessboard and the rules of the knight's moves. Vandermonde's basic idea consists of covering the board with two or more independent paths taken at random. In the next step these paths are connected. Vandermonde has described a re-entrant route by the following fractions (coordinates):

$5/5, 4/3, 2/4, 4/5, 5/3, 7/4, 8/2, 6/1, 7/3, 8/1, 6/2, 8/3, 7/1, 5/2, 6/4, 8/5, 7/7, 5/8, 6/6, 5/4, 4/6, 2/5, 1/7, 3/8, 2/6, 1/8, 3/7, 1/6, 2/8, 4/7, 3/5, 1/4, 2/2, 4/1, 3/3, 1/2, 3/1, 2/3, 1/1, 3/2, 1/3, 2/1, 4/2, 3/4, 1/5, 2/7, 4/8, 3/6, 4/4, 5/6, 7/5, 8/7, 6/8, 7/6, 8/8, 6/7, 8/6, 7/8, 5/7, 6/5, 8/4, 7/2, 5/1, 6/3.$

The usual chess notation corresponding to the above fraction notation would be e5, d3, b4, d5, e3, and so on.

An extensive literature exists on the knight's re-entrant tour.<sup>3</sup> In 1823, H. C. Warnsdorff<sup>4</sup> provided one of its most elegant solutions. His method is very efficient, not only for the standard chessboard but also for a general  $n \times n$  board as well.

Recalling Problem 9.4 we immediately conclude that the knight's circles are in fact Hamiltonian cycles. There are 13,267,364,410,532 closed knight's tours, calculated in 2000 by Wegener [183]. The same number was previously claimed by Brendan McKay in 1997.<sup>5</sup> One of the ways to find a knight's

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<sup>3</sup>For instance, P. Volpicelli, *Atti della Reale Accademia dei Lincei* (Rome, 1872); C.F. de Jaenisch, *Applications de l'Analyse mathématique au Jeu des Echecks*, 3 vols. (Petrograd, 1862–63); A. van der Linde, *Geschichte und Literatur des Schachspiels*, vol. 2 (Berlin, 1874); M. Kraitchik, *La Mathématique des Jeux* (Brussels, 1930); W. W. Rouse Ball, *Mathematical Recreations and Essays*, rev. ed. (Macmillan, New York 1960); E. Gik, *Mathematics on the Chessboard* (in Russian, Nauka, Moscow 1976); E. I. Ignat'ev, *In the Czardom of Puzzles* (in Russian, Nauka, Moscow 1979); D'Hooge, *Les Secrets du cavalier* (Bruxelles-Paris, 1962); M. Petković, *Mathematics and Chess*, Dover Publications, Mineola (1997); I. Wegener, *Branching Programs and Binary Decision Diagrams*, SIAM, Philadelphia (2000); J. J. Watkins, *Across the Board: The Mathematics of Chessboard Problems*, Princeton University Press, Princeton and Oxford (2004); N. D. Elkies, R. P. Stanley, *The mathematical knight*, Mathematical Intelligencer 22 (2003), 22–34; A. Conrad, T. Hindrichs, H. Morsy, I. Wegener, *Solution of the knight's Hamiltonian path problem on chessboards*, Discrete Applied Mathematics 50 (1994), 125–134.

<sup>4</sup>*Des Rösselsprunges einfachste und allgemeinste Lösung*, Schalkalden 1823.

<sup>5</sup>A powerful computer, finding tours at a speed of 1 million tours per second, will have to run for more than 153 days and nights to reach the number of tours reported by McKay and Wegener.

tour is the application of *backtracking* algorithms<sup>6</sup>, but this kind of search is very slow so that even very powerful computers need considerable time. Another algorithm developed by A. Conrad *et al.* [40] is much faster and finds the knight's re-entrant tours on the  $n \times n$  board for  $n \geq 5$ .

An extensive study of the possibility of the knight's re-entrant routes on a general  $m \times n$  chessboard can be found in [181]. A definitive solution was given by Allen Schwenk [156] in 1991, summarized in the form of the following theorem.

**Theorem 10.1 (Schwenk).** *An  $m \times n$  chessboard ( $m \leq n$ ) has a knight's tour unless one or more of the following three conditions hold:*

- (i)  *$m$  and  $n$  are both odd;*
- (ii)  *$m = 1, 2$ , or  $4$ ; or*
- (iii)  *$m = 3$  and  $n = 4, 6$ , or  $8$ .*

One more remark. If a knight's closed tour exists, then it is obvious that any square on the considered chessboard can be taken as the starting point.

It is a high time for the reader to get busy and try to find the solution to the following problem.

**Problem 10.2.\*** *Prove the impossibility of knight's tours for  $4 \times n$  boards.*

The previous problem tell us that a knight's tour on a  $4 \times 4$  board is impossible. The question of existence of such a tour for the three-dimensional  $4 \times 4 \times 4$  board, consisting of four horizontal  $4 \times 4$  boards which lie one over the other, is left to the reader.

**Problem 10.3.\*** *Find a knight's re-entrant tour on a three-dimensional  $4 \times 4 \times 4$  board.*

Many composers of the knight's circles have constructed re-entrant paths of various unusual and intriguing shapes while also incorporating certain esthetic elements or other features. Among them, magic squares using a knight's tour (not necessarily closed) have attracted the most attention. J. J. Watkins calls the quest for such magic squares the *Holy Grail*. The Russian chess master and officer de Jaenisch (1813–1872) composed many notable problems concerning the knight's circles. Here is one of them [138, Problem 3.5], just connected with magic squares.

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<sup>6</sup>A backtracking algorithm searches for a partial candidate to the solution step by step and eliminates this candidate when the algorithm reaches an impasse, then backing up a number of steps to try again with another partial candidate—the knight's path in this particular case.

**Problem 10.4.** *Let the successive squares that form the knight's re-entrant path be denoted by the numbers from 1 to 64 consecutively, 1 being the starting square and 64 being the square from which the knight plays its last move, connecting the squares 64 and 1. Can you find a knight's re-entrant path such that the associated numbers in each row and each column add up to 260?*

The first question from the reader could be: *Must the sum be just 260?* The answer is very simple. The total sum of all traversed squares of the chessboard is

$$1 + 2 + \cdots + 64 = \frac{64 \cdot 65}{2} = 2,080,$$

and 2,080 divided by 8 gives 260. It is rather difficult to find magic or “semi-magic squares” (“semi-” because the sums over diagonals are not taken into account), so we recommend Problem 10.4 only to those readers who are well-versed in the subject. One more remark. De Jaenisch was not the first who constructed the semi-magic squares. The first semi-magic knight's tour, shown in Figure 10.4, was composed in 1848 by William Beverley, a British landscape painter and designer of theatrical effects.

There are 280 distinct arithmetical semi-magic tours (not necessarily closed). Taking into account that each of these semi-magic tours can be oriented by rotation and reflection in eight different ways, a total number of semi-magic squares is 2,240 ( $= 280 \times 8$ ).

Only a few knight's *re-entrant* paths possess the required “magic” properties. One of them, constructed by de Jaenisch, is given in Figure 10.5.

1	30	47	52	5	28	43	54
48	51	2	29	44	53	6	27
31	46	49	4	25	8	55	42
50	3	32	45	56	41	26	7
33	62	15	20	9	24	39	58
16	19	34	61	40	57	10	23
63	14	17	36	21	12	59	38
18	35	64	13	60	37	22	11

FIGURE 10.4. Beverley's tour

63	22	15	40	1	42	59	18
14	39	64	21	60	17	2	43
37	62	23	16	41	4	19	58
24	13	38	61	20	57	44	3
11	36	25	52	29	46	5	56
26	51	12	33	8	55	30	45
35	10	49	28	53	32	47	6
50	27	34	9	48	7	54	31

FIGURE 10.5. De Jaenisch's tour

The question of existence of a proper magic square (in which the sums over the two main diagonals are also equal to 260) on the standard  $8 \times 8$  chessboard has remained open for many years. However, in August 2003, Guenter Stertenbrink announced that an exhaustive search of all possibilities

using a computer program had led to the conclusion that no such knight's tour exists (see Watkins [181]).

Our discussion would be incomplete without addressing the natural question of whether a magic knight's tour exists on a board of any dimension  $n \times n$ . Where there is magic, there is hope. Indeed, it has been proved recently that such magic tours do exist on boards of size  $16 \times 16$ ,  $20 \times 20$ ,  $24 \times 24$ ,  $32 \times 32$ ,  $48 \times 48$ , and  $64 \times 64$  (see [181]).

The following problem concerns a knight's tour which is closed, but in another sense. Namely, we define a *closed knight's route* as a closed path consisting of knight's moves which do not intersect and do not necessarily traverse all squares. For example, such a closed route is shown in Figure 10.13.

**Problem 10.5.\*** *Prove that the area enclosed by a closed knight's route is an integral multiple of the area of a square of the  $n \times n$  ( $n \geq 4$ ) chessboard.*

*Hint:* Exploit the well-known Pick's theorem which reads: *Let  $A$  be the area of a non-self-intersecting polygon  $P$  whose vertices are points of a lattice. Assume that the lattice is composed of elementary parallelograms with the area  $S$ . Let  $B$  denote the number of points of the lattice on the polygon edges and  $I$  the number of points of the lattice inside  $P$ . Then*

$$A = \left(I + \frac{1}{2}B - 1\right)S. \quad (10.1)$$

Many generalizations of the knight's tour problem have been proposed which involve alteration of the size and shape of the board or even modifying the knight's standard move; see Kraitchik [118]. Instead of using the perpendicular components 2 and 1 of the knight's move, written as the pair (2,1), Kraitchik considers the  $(m, n)$ -move.

A Persian manuscript, a translation of which can be found in Duncan Forbes' *History of Chess* (London, 1880), explains the complete rules of fourteenth-century Persian chess. A piece called the "camel", used in Persian chess and named the "cook" by Solomon Golomb, is actually a modified knight that moves three instead of two squares along a row or a file, then one square at right angles which may be written as (3,1). Obviously, this piece can move on the 32 black squares of the standard  $8 \times 8$  chessboard without leaving the black squares. Golomb posed the following task.

**Problem 10.6.\*** *Is there a camel's tour over all 32 black squares of the chessboard in such a way that each square is traversed once and only once?*



**Leonhard Euler** (1707–1783) ( $\rightarrow$  p. 305)

### Non-attacking rooks

Apart from the knight's re-entrant tours on the chessboard, shown on pages 258–264, another amusing chessboard problem caught Leonhard Euler's interest.

**Problem 10.7.** Let  $Q_n$  ( $n \geq 2$ ) be the number of arrangements of  $n$  rooks that can be placed on an  $n \times n$  chessboard so that no rook attacks any other and no rook lies on the squares of the main diagonal. One assumes that the main diagonal travels from the lower left corner square  $(1, 1)$  to the upper right corner square  $(n, n)$ . The task is to find  $Q_n$  for an arbitrary  $n$ .

The required positions of rooks for  $n = 2$  and  $n = 3$ , for example, are shown in Figure 10.6 giving  $Q_2 = 1$  and  $Q_3 = 2$ .

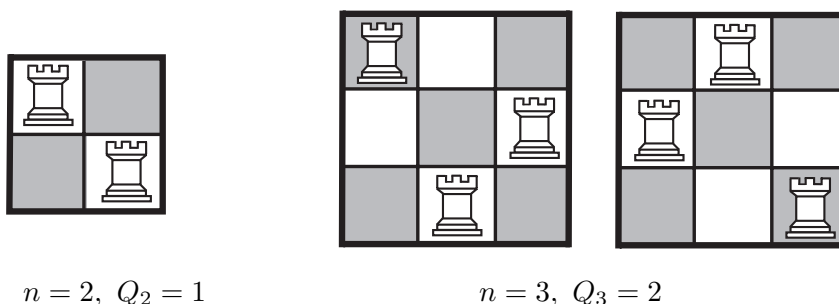


FIGURE 10.6. Non-attacking rooks outside the main diagonal

The above-mentioned problem is, in essence, the same one as that referred to as the Bernoulli–Euler *problem of misaddressed letters* appearing on page 184. Naturally, the same formula provides solutions to both problems. As our problem involves the placement of rooks on a chessboard, we will express the solution of the *problem of non-attacking rooks* in the context of the chessboard.

According to the task's conditions, every row and every column contain one and only one rook. For an arbitrary square  $(i, j)$ , belonging to the  $i$ th row and  $j$ th column, we set the square  $(j, i)$  symmetrical to the square  $(i, j)$  with respect to the main diagonal.

The rook can occupy  $n-1$  squares in the first column (all except the square belonging to the main diagonal). Assume that the rook in the first column is placed on the square  $(r, 1)$ ,  $r \in \{2, \dots, n\}$ . Depending on the arrangement of the rooks in the remaining  $n-1$  columns, we can distinguish

two groups of positions with non-attacking rooks: if the symmetrical square  $(1, r)$  (related to the rook on the square  $(r, 1)$ ) is not occupied by a rook, we will say that the considered position is of the first kind, otherwise, it is of the second kind. For example, the position on the left in Figure 10.7 (where  $n = 4$  and  $r = 2$ ) is of the first kind, while the position on the right is of the second kind.



FIGURE 10.7. Positions of the first kind (left) and second kind (right)

Let us now determine the number of the first kind positions. If we remove the  $r$ th row from the board and substitute it by the first row, and then remove the first column, a new  $(n - 1) \times (n - 1)$  chessboard is obtained. Each arrangement of rooks on the new chessboard satisfies the conditions of the problem. The opposite claim is also valid: for each arrangement of rooks on the new chessboard satisfying the conditions of the problem, the unique position of the first kind can be found. Hence, the number of the first kind positions is exactly  $Q_{n-1}$ .

To determine the number of second kind positions, let us remove the first column and the  $r$ th row, and also the  $r$ th column and the first row from the  $n \times n$  chessboard (regarding only positions of the second kind). If we join the remaining rows and columns without altering their order, a new  $(n - 2) \times (n - 2)$  chessboard is formed. It is easy to check that the arrangements of rooks on such  $(n - 2) \times (n - 2)$  chessboards satisfy the conditions of the posed problem. Therefore, it follows that there are  $Q_{n-2}$  positions of the second kind.

After consideration of the above, we conclude that there are  $Q_{n-1} + Q_{n-2}$  positions of non-attacking rooks on the  $n \times n$  chessboard, satisfying the problem's conditions and corresponding to the fixed position of the rook—the square  $(r, 1)$ —in the first column. Since  $r$  can take  $n - 1$  values ( $= 2, 3, \dots, n$ ), one obtains

$$Q_n = (n - 1)(Q_{n-1} + Q_{n-2}). \quad (10.2)$$

The above recurrence relation derived by Euler is a difference equation of the second order. It can be reduced to a difference equation of the first order in the following manner. Starting from (10.2) we find

$$\begin{aligned} Q_n - nQ_{n-1} &= (n-1)(Q_{n-1} + Q_{n-2}) - nQ_{n-1} \\ &= -(Q_{n-1} - (n-1)Q_{n-2}). \end{aligned}$$

Using successively the last relation we obtain

$$\begin{aligned} Q_n - nQ_{n-1} &= -(Q_{n-1} - (n-1)Q_{n-2}) \\ &= (-1)^2(Q_{n-2} - (n-2)Q_{n-3}) \\ &\vdots \\ &= (-1)^{n-3}(Q_3 - 3Q_2). \end{aligned}$$

Since  $Q_2 = 1$  and  $Q_3 = 2$  (see Figure 10.6), one obtains the difference equation of the first order

$$Q_n - nQ_{n-1} = (-1)^n. \quad (10.3)$$

To find the general formula for  $Q_n$ , we apply (10.3) backwards and obtain

$$\begin{aligned} Q_n &= nQ_{n-1} + (-1)^n = n((n-1)Q_{n-2} + (-1)^{n-1}) + (-1)^n \\ &= n(n-1)Q_{n-2} + n(-1)^{n-1} + (-1)^n \\ &= n(n-1)((n-2)Q_{n-3} + (-1)^{n-2}) + n(-1)^{n-1} + (-1)^n \\ &= n(n-1)(n-2)Q_{n-3} + n(n-1)(-1)^{n-2} \\ &\quad + n(-1)^{n-1} + (-1)^n \\ &\vdots \\ &= n(n-1)(n-2) \cdots 3 \cdot Q_2 + n(n-1) \cdots 4 \cdot (-1)^3 + \cdots \\ &\quad + n(-1)^{n-1} + (-1)^n \\ &= n! \left( \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!} \right), \end{aligned}$$

that is,

$$Q_n = n! \sum_{k=2}^n \frac{(-1)^k}{k!} \quad (n \geq 2).$$

The last formula gives

$$Q_2 = 1, \quad Q_3 = 2, \quad Q_4 = 9, \quad Q_5 = 44, \quad Q_6 = 265, \quad \text{etc.}$$

### ***Carl Friedrich Gauss*** (1777–1855) ( $\rightarrow$ p. 305)

Carl Friedrich Gauss indisputably merits a place among such illustrious mathematicians as Archimedes and Newton. Sometimes known as “the Prince of mathematicians,” Gauss is regarded as one of the most influential mathematicians in history. He made a remarkable contribution to many fields of mathematics and science (see short biography on page 305).

As a ten-year old schoolboy, Gauss was already exhibiting his formidable mathematical talents as the following story recounts. One day Gauss’ teacher Mr. Büttner, who had a reputation for setting difficult problems, set his pupils to the task of finding the sum of the arithmetic progression  $1 + 2 + \cdots + 100$ .<sup>7</sup> The lazy teacher assumed that this problem would occupy the class for the entire hour since the pupils knew nothing about arithmetical progression and the general sum formula. Almost immediately, however, gifted young Gauss placed his slate on the table.



Carl Friedrich Gauss  
1777–1855

When the astonished teacher finally looked at the results, he saw the correct answer, 5,050, with no further calculation. The ten-year-old boy had mentally computed the sum by arranging the addends in 50 groups  $(1+100)$ ,  $(2+99)$ ,  $\dots$ ,  $(50, 51)$ , each of them with the sum 101, and multiplying this sum by 50 in his head to obtain the required sum  $101 \cdot 50 = 5,050$ . Impressed by his young student, Büttner arranged for his assistant Martin Bartels (1769–1836), who later became a mathematics professor in Russia, to tutor Gauss.

Like Isaac Newton, Gauss was never a prolific writer. Being an ardent perfectionist, he refused to publish his works which he did not consider complete, presumably fearing criticism and controversy. His delayed publication of results, like the delays of Newton, led to many high profile controversies and disputes.

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<sup>7</sup>Some authors claim that the teacher gave the arithmetic progression  $81,297 + 81,495 + \cdots + 100,899$  with the difference 198. It does not matter!

Gauss' short dairy of only 19 pages, found after his death and published in 1901 by the renowned German mathematician Felix Klein, is regarded as one of the most valuable mathematical documents ever. In it he included 146 of his discoveries, written in a very concise form, without any traces of derivation or proofs. For example, he jotted down in his dairy "*Heureka! num =  $\triangle + \triangle + \triangle$ ,*" a coded form of his discovery that every positive integer is representable as a sum of at most three triangular numbers.

Many details about the work and life of Gauss can be found in G. W. Dunnington's book, *Carl Friedrich Gauss, Titan of Science* [58]. Here is a short list of monuments, objects and other things named in honour of Gauss:

- The CGS unit for magnetic induction was named *Gauss* in his honour,
- Asteroid 1001 *Gaussia*,
- The *Gauss* crater on the Moon,
- The ship *Gauss*, used in the Gauss expedition to the Antarctic,
- *Gaussberg*, an extinct volcano on the Antarctic,
- The *Gauss* Tower, an observation tower in Dransfeld, Germany,
- *Degaussing* is the process of decreasing or eliminating an unwanted magnetic field (say, from TV screens or computer monitors).

### The eight queens problem

One of the most famous problems connected with a chessboard and chess pieces is undoubtedly the *eight queens problem*. Although there are claims that the problem was known earlier, in 1848 Max Bezzel put forward this problem in the chess journal *Deutsche Schachzeitung* of Berlin:

**Problem 10.8.** *How does one place eight queens on an  $8 \times 8$  chessboard, or, for general purposes,  $n$  queens on an  $n \times n$  board, so that no queen is attacked by another. In addition, determine the number of such positions.*

Before we consider this problem, let us note that although puzzles involving non-attacking queens and similar chess-piece puzzles may be intriguing in their right, more importantly, they have applications in industrial mathematics; in maximum cliques from graph theory, and in integer programming (see, e.g., [65]).

The eight queens problem was posed again by Franz Nauck in the more widely read, *Illustrierte Zeitung*, of Leipzig in its issue of June 1, 1850. Four weeks later Nauck presented 60 different solutions. In the September issue he corrected himself and gave 92 solutions but he did not offer a proof that there are not more. In 1910 G. Bennett<sup>8</sup> concluded that there are only 12

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<sup>8</sup>G. Bennett, *The eight queens problem*, Messenger of Mathematics, 39 (1910), 19.

distinctly different solutions to the queens problem, that is, solutions that could not be obtained one from another by rotations for  $90^\circ$ ,  $180^\circ$  and  $270^\circ$ , and mirror images; T. Gosset later proved this in 1914.<sup>9</sup>

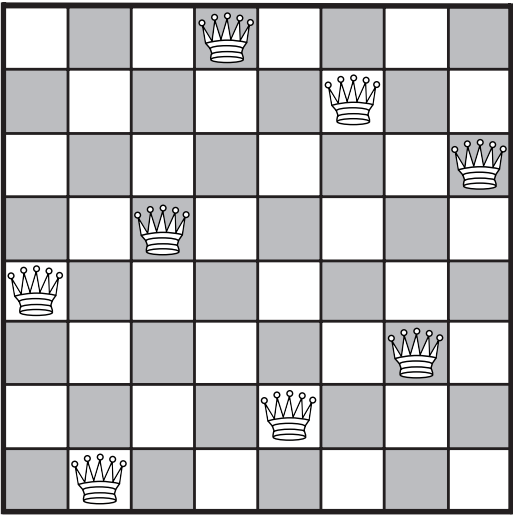


FIGURE 10.8. The 8-queens problem; one fundamental solution 41582736

Each position of the non-attacking queens on the  $8 \times 8$  board can be indicated by an array of 8 numbers  $k_1 k_2 \cdots k_8$ . The solution  $k_1 k_2 \cdots k_8$  means that one queen is on the  $k_1$ th square of the first column, one on the  $k_2$ th square of the second column, and so on. Therefore, twelve fundamental solutions can be represented as follows:

41582736	41586372	42586137
42736815	42736851	42751863
42857136	42861357	46152837
46827135	47526138	48157263

Each of the twelve basic solutions can be rotated and reflected to yield 7 other patterns (except the solution 10, which gives only 3 other patterns because of its symmetry). Therefore, counting reflections and rotations as different, there are 92 solutions altogether. One fundamental solution given by the first sequence 41582736 is shown in Figure 10.8.

Gauss himself also found great interest in the eight queens problem reading *Illustrierte Zeitung*. In September of 1850 he concluded that there were

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<sup>9</sup>T. Gosset, *The eight queens problem*, Messenger of Mathematics, 44 (1914), 48.

76 solutions. Only a week later, Gauss wrote to his astronomer friend H. C. Schumacher that four of his 76 solutions are false, leaving 72 as the number of true solutions. In addition, Gauss noted that there might be more, remembering that Franz Nauck did not prove his assertion that there are exactly 92 solutions. One can imagine that Gauss did not find all the solutions on the first attempt, presumably because at that time, he lacked the systematic and strongly supported methods necessary for solving problems of this kind. More details about Gauss and the eight queens problem can be found in [34] and [65].

Considering that the method of solving the eight queens problem via trail and error was inelegant, Gauss turned this problem into an arithmetical problem; see [34] and [86]. We have seen that each solution can be represented as a permutation of the numbers 1 through 8. Such a representation automatically confirms that there is exactly one queen in each row and each column. It was necessary to check in an easy way if any two queens occupy the same diagonal and Gauss devised such a method. We will illustrate his method with the permutation 41582736 that represents the eight-queens solution shown in Figure 10.8.

Let us form the following sums:

$$\begin{array}{cccccccc} & 4 & 1 & 5 & 8 & 2 & 7 & 3 & 6 \\ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ & - & - & - & - & - & - & - & - \\ \Sigma & 5 & 3 & 8 & 12 & 7 & 13 & 10 & 14 \end{array}$$

and

$$\begin{array}{cccccccc} & 4 & 1 & 5 & 8 & 2 & 7 & 3 & 6 \\ & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ & - & - & - & - & - & - & - & - \\ \Sigma & 12 & 8 & 11 & 13 & 6 & 10 & 5 & 7 \end{array}$$

In both cases the eight sums are distinct natural numbers, which means that no two queens lie on the same negative diagonal  $\backslash$  (the sums above) and no two queens lie on the same positive diagonal  $/$  (the sums below). According to these outcomes, Gauss concluded that the queens with positions represented by the permutation 41582736 are non-attacking.

In 1874 J. W. Glaisher<sup>10</sup> proposed expanding the eight queens problem to the *n*-queens problem, that is, solving the queens' puzzle for the general  $n \times n$  chessboard. He attempted to solve it using determinants. It was suspected

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<sup>10</sup>J. W. Glaisher, *On the problem of eight queens*, Philosophical Magazine, Sec. 4, 48 (1874), 457.

that exactly  $n$  non-attacking queens could be placed on an  $n \times n$  chessboard, but it was not until 1901 that Wilhelm Ahrens [2] could provide a positive answer. Other interesting proofs can be found in [104], [181] and [193]. In their paper [104] Hoffman, Loessi and Moore reduced the  $n$ -queens task to the problem of finding a maximum internally stable set of a symmetric graph, the vertices of which correspond to the  $n^2$  square elements of an  $n \times n$  matrix.

Considering the more general problem of the  $n \times n$  chessboard, first we verify that there is no solution if  $n < 4$  (except the trivial case of one queen on the  $1 \times 1$  square). Fundamental solutions for  $4 \leq n \leq 7$  are as follows:

$n = 4$  : 3142,

$n = 5$  : 14253, 25314,

$n = 6$  : 246135,

$n = 7$  : 1357246, 3572461, 5724613, 4613572, 3162574, 2574136.

The number of fundamental solutions  $F(n)$  and the number of all solutions  $S(n)$ , including those obtained by rotations and reflections, are listed below for  $n = 1, \dots, 12$ . A general formula for the number of solutions  $S(n)$  when  $n$  is arbitrary has not been found yet.

$n$	1	2	3	4	5	6	7	8	9	10	11	12
$F(n)$	1	–	–	1	2	1	6	12	46	92	341	1,784
$S(n)$	1	–	–	2	10	4	40	92	342	724	2,680	14,200

TABLE 10.1. The number of solutions to the  $n \times n$  queens problem

Some interesting relations between magic squares and the  $n$ -queens problem have been considered by Demirörs, Rafrat and Tanik in [48]. The authors have introduced a procedure for obtaining the arrangements of  $n$  non-attacking queens starting from magic squares of order  $n$  not divisible by 2 and 3.

The following two problems are more complicated modern variants of the eight queens problem and we leave them to the reader. In solving these problems, it is advisable to use a computer program.

In his *Mathematical Games* column, M. Gardner [79] presented a version of the  $n$ -queens problem with constraints. In this problem a queen may attack other queen directly (as in ordinary chess game) or by reflection from either the first or the  $(n + 1)$ -st horizontal virtual line. To put the reader at ease, we shall offer the special case  $n = 8$ .



**Problem 10.9.\*** *Place 8 chess queens on the  $8 \times 8$  board with a reflection strip in such a way that no two queens attack each other either directly or by reflection.*

A superqueen (known also as “Amazon”) is a chess piece that moves like a queen as well as a knight. This very powerful chess piece was known in some variants of chess in the Middle Ages. Obviously, the  $n$ -superqueen problem is an extension of the  $n$ -queen problem in which new restrictions should be taken into account. So it is not strange that the  $n$ -superqueen problem has no solution for  $n < 10$ .

**Problem 10.10.\*** *Place 10 superqueens on the  $10 \times 10$  chessboard so that no superqueen can attack any other.*

There is just one fundamental solution for the case  $n = 10$ . Can you find this solution?

A variation of the chess that would be worth mentioning is one in which the game is played on a cylindrical board. The pieces in so-called cylindrical chess are arranged as on an ordinary chessboard, and they move following the same rules. But the board is in a cylindrical form because its vertical edges are joined (“vertical cylindrical chess”) so that the verticals  $a$  and  $h$  are juxtaposed. Also, it is possible to join the horizontal edges of the board (“horizontal cylindrical chess”) so that the first and the eighth horizontal are connected.

We have already seen that the eight queens problem on the standard  $8 \times 8$  chessboard has 92 solutions. The following problem on a cylindrical chessboard was considered by the outstanding chess journalist and chessmaster Edvard Gik [84].

**Problem 10.11.\*** *Solve the problem of non-attacking queens on a cylindrical chessboard that is formed of an  $8 \times 8$  chessboard.*

**Donald Knuth** (1938– ) ( $\rightarrow$  p. 310)

### The longest uncrossed knight's tour

On pages 258–262 we previously considered a knight's tour over a chessboard such that all 64 squares are traversed once and only once. The difficult problem presented below imposes certain restrictions on the knight's tour:

**Problem 10.12.** *Find the largest uncrossed knight's tour on a chessboard.*

Apparently T. R. Dawson once posed this problem, but L. D. Yarbrough launched the same problem again in the July 1968 *Journal of Recreational Mathematics*.

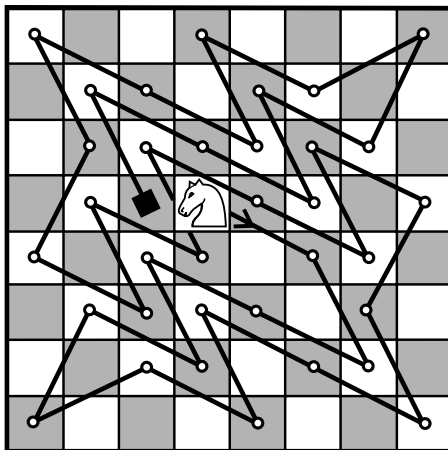


FIGURE 10.9. Knuth's solution for the longest uncrossed knight's tour

Donald E. Knuth wrote a “backtrack” computer program to find four fundamental solutions for the knight's tour. To find these tours, the computer examined 3,137,317,289 cases. One of these solutions is shown in Figure 10.9 (see, *e.g.*, the book [138, p. 61]).

### Guarini's knight-switching problem

We end this chapter with Guarini's classic knight-switching problem from 1512, mentioned in Chapter 1. A number of mathematicians have considered problems of this type, in modern times most frequently in connection with planar graphs. No matter how unexpected it sounds, a kind of “graph approach” was known to al-Adli (*ca.* 840 A.D.) who considered in his work on chess a simple circuit that corresponds to the knight-move network on a  $3 \times 3$  board.

**Problem 10.13.** *The task is to interchange two white knights on the top corner squares of a  $3 \times 3$  chessboard and two black knights on the bottom corner squares. The white knights should move into the places occupied initially by the black knights—and vice versa—in the minimum number of moves. The knights may be moved in any order, regardless of their color. Naturally, no two of them can occupy the same square.*

**Solution.** This puzzle belongs to the class of problems that can be solved in an elegant manner using the theory of planar graphs. Possibly this problem could find its place in Chapter 9 on graphs, but we regard that it is an unimportant dilemma.

The squares of the chessboard represent *nodes* of a graph, and the possible moves of the pieces between the corresponding squares (the nodes of the graph) are interpreted as the *connecting lines* of the graph. The corresponding graph for the board and the initial positions of the knights are shown in Figure 10.10(a).

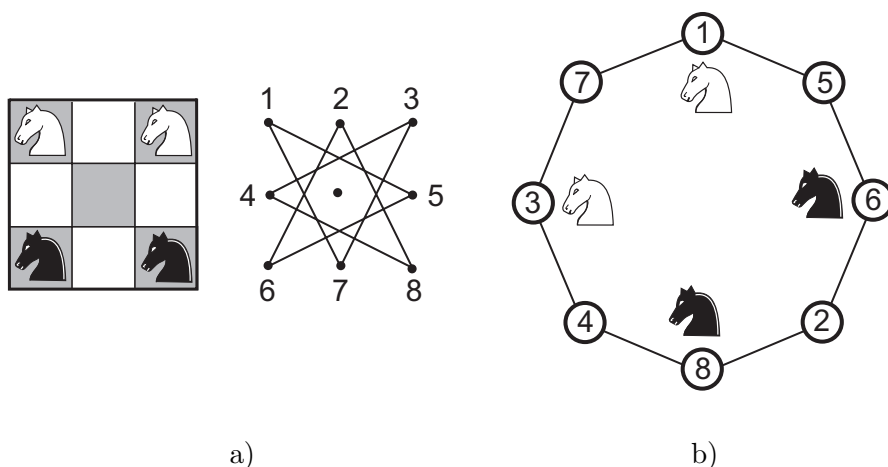


FIGURE 10.10. a) Graph to Guarini's problem b) Equivalent simplified graph

The initial positions of the knights are indicated and all possible moves of the knights between the squares (the nodes of the graph) are marked by lines. Using Dudeney's famous "method of unraveling a graph,"<sup>11</sup> starting from any node, the graph 10.10(a) can be "unfolded" to the equivalent graph 10.10(b), which is much clearer and more convenient for the analysis. Obviously, the topological structure and the connectedness are preserved. To find the solution it is necessary to write down the moves (and reproduce them on the  $3 \times 3$  board according to some correspondence), moving the knights along the circumference of the graph until they exchange places. The minimum number of moves is 16 although the solution is not unique (because the movement of the knights along the graph is not unique). Here is one solution:

<sup>11</sup>This "method" was described in detail by E. Gik in the journal *Nauka i Zhizn* 12 (1976); see also M. Gardner, *Mathematical Puzzles and Diversions* (New York: Penguin, 1965).

1-5 6-2 3-7 8-4 5-6 2-8 7-1 4-3  
1-5 8-4 6-2 3-7 5-6 7-1 2-8 4-3.

A similar problem also involves two white and two black chess knights and requires their interchange in the fewest possible moves.

**Problem 10.14.\*** *Two white and two black knights are placed on a board of an unusual form, as shown in Figure 10.11. The goal is to exchange the white and black knights in the minimum number of moves.*

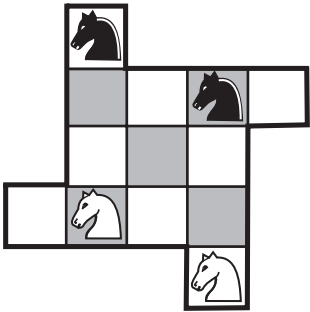


FIGURE 10.11. Knight-switching problem

### Answers to Problems

**10.2.** Suppose that the required knight’s re-entrant route exists. We assume that this board is colored alternately white and black (in a chess order). The upper and lower row will be called the *outer* lines (*O*), and the two remaining rows the *middle* lines (*M*). Since a knight, starting from any *outer* square, can land only on a *middle* square, it follows that among  $4n$  moves that should make the route,  $2n$  moves must be played from the *outer* to the *middle* squares. Therefore, there remain exactly  $2n$  moves that have to be realized from the *middle* to the *outer* squares.

Since any square of the closed knight’s tour can be the starting square, without loss of generality, we can assume that we start from a “middle” square. The described tour gives an alternate sequence

$$M(\text{start}) - O - M - O - \dots - M - O - M(\text{finish}), \quad (10.4)$$

ending at the starting square. We emphasize that a knight can’t dare visit two middle squares in a row anywhere along the tour because of the following. Assume that we start with this double move  $M - M$  (which is always possible because these moves belong to the circuit), then we will have the sequence

$M - M - (2n - 1) \times (O - M)$ . In this case we have  $2n + 1$   $M$  moves and  $2n - 1$   $O$  moves, thus each different from  $2n$ . Note further that the double move  $O - O$  in the parenthesis in the last sequence is impossible because a knight cannot jump from the outside line to the outside line.

On the other hand, the same knight's tour alternates between white and black squares, say,

$$\text{black} - \text{white} - \text{black} - \text{white} - \cdots - \text{black} - \text{white} - \text{black} \quad (10.5)$$

(or opposite). Comparing the sequences (10.4) and (10.5) we note that all squares of the outer lines are in one color and all squares of the middle lines are in the other color. But this is a *contradiction* since the board is colored alternately. Thus, the required path is impossible.

**10.3.** One solution is displayed in Figure 10.12. The three-dimensional  $4 \times 4 \times 4$  board is represented by the four horizontal  $4 \times 4$  boards, which lie one over the other; the lowest board is indicated by I, the highest by IV. The knight's moves are denoted by the numbers from 1 (starting square) to 64 (ending square). The knight can make a re-entrant tour because the squares 64 and 1 are connected by the knight's moves.

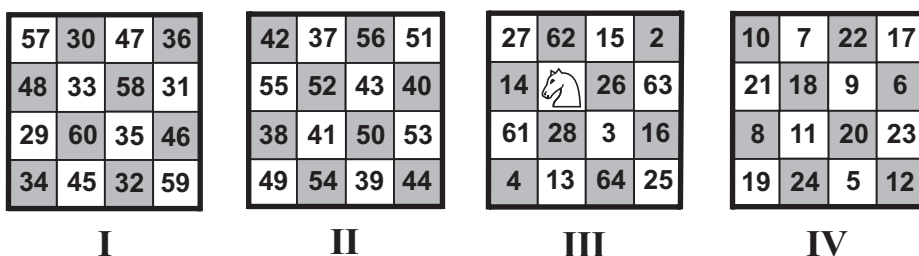


FIGURE 10.12. Knight's re-entrant path on the  $4 \times 4 \times 4$  board

**10.5.** Let  $S$  be the area of a square of the  $n \times n$  chessboard. Considering formula (10.1) in Pick's theorem, it is sufficient to prove that the number of boundary points  $B$  is even. Since the knight's tour alternates between white ( $w$ ) and black squares ( $b$ ), in the case of any closed tour (the starting square coincides with the ending square) it is easy to observe that the number of traversed squares must be even. Indeed, the sequence  $b(\text{start}) - w - b - w - \cdots - b - w - b(\text{finish})$ , associated to the closed knight's path, always has an even number of moves (= traversed squares); see Figure 10.13. Since the number of squares belonging to the required closed knight's path is equal to the number of boundary points  $B$ , the proof is completed.



**10.9.** If you have not succeeded in solving the given problem, see the following solutions found by Y. Kusaka [120]. Using a computer program and backtracking algorithm he established that there are only 10 solutions in this eight queens problem with constraints (we recall that this number is 92 for the ordinary case; see Table 10.1 for  $n = 8$ ):

25741863 27581463 36418572 36814752 36824175  
37286415 42736815 51468273 51863724 57142863

**10.10.** The author of this book provided in his book *Mathematics and Chess* [138] a computer program in the computer language C that can find all possible solutions: the fundamental one and similar ones obtained by the rotations of the board and by the reflections in the mirror. The program runs for arbitrary  $n$  and solves the standard  $n$ -queens problem as well as the  $n$ -superqueens problem. We emphasize that the running time increases very quickly if  $n$  increases.

The fundamental solution is shown in Figure 10.15, which can be denoted as the permutation (3,6,9,1,4,7,10,2,5,8). As before, such denotation means that one superqueen is on the third square of the first column, one on the sixth square of the second column, and so on.

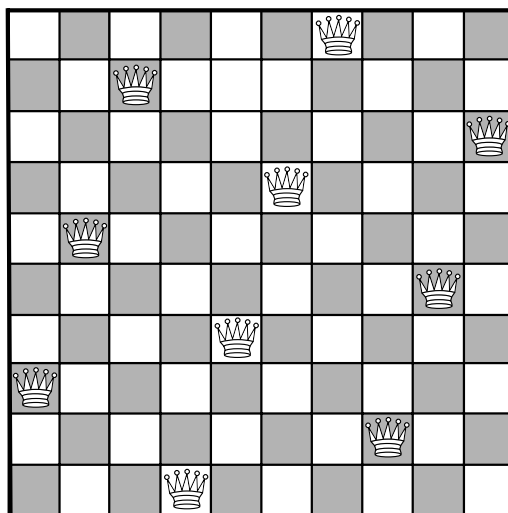


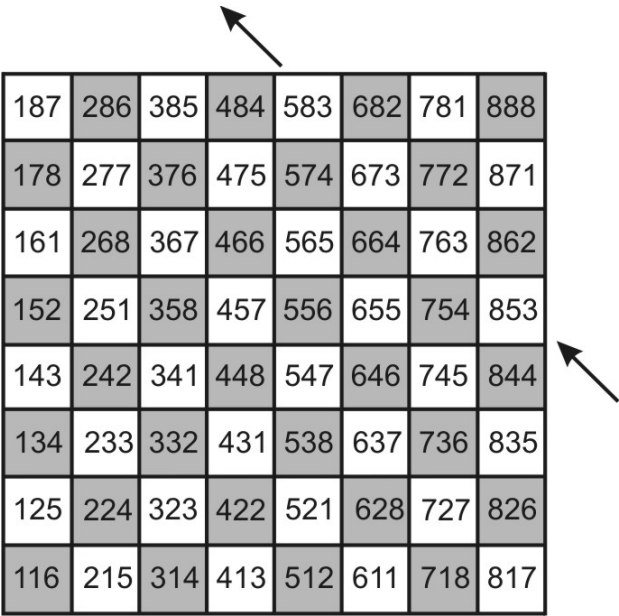
FIGURE 10.15. The fundamental solution of the superqueens problem for  $n = 10$

The three remaining solutions (found by the computer) arise from the fundamental solution, and they can be expressed as follows:

(7,3,10,6,2,9,5,1,8,4) (4,8,1,5,9,2,6,10,3,7) (8,5,2,10,7,4,1,9,6,3)

**10.11.** There is no solution of the eight queens problem on the cylindrical chessboard of the order 8. We follow the ingenious proof given by E. Gik [84].

Let us consider an ordinary chessboard, imagining that its vertical edges are joined (“vertical cylindrical chess”). Let us write in each of the squares three digits  $(i, j, k)$ , where  $i, j, k \in \{1, \dots, 8\}$  present column, row, and diagonal (respectively) of the traversing square (Figure 10.16). Assume that there is a replacement of 8 non-attacking queens and let  $(i_1, j_1, k_1), \dots, (i_8, j_8, k_8)$  be the ordered triples that represent 8 occupied squares. Then the numbers  $i_1, \dots, i_8$  are distinct and belong to the set  $\{1, \dots, 8\}$ ; therefore,  $\sum i_m = 1 + \dots + 8 = 36$ . The same holds for the numbers from the sets  $\{j_1, \dots, j_8\}$  and  $\{k_1, \dots, k_8\}$ .



187	286	385	484	583	682	781	888
178	277	376	475	574	673	772	871
161	268	367	466	565	664	763	862
152	251	358	457	556	655	754	853
143	242	341	448	547	646	745	844
134	233	332	431	538	637	736	835
125	224	323	422	521	628	727	826
116	215	314	413	512	611	718	817

FIGURE 10.16. Gik’s solution

We see that the sum  $(i_1 + \dots + i_8) + (j_1 + \dots + j_8) + (k_1 + \dots + k_8)$  of all 24 digits written in the squares occupied by the queens is equal to  $(1 + \dots + 8) \times 3 = 108$ . Since the sum  $i_\nu + j_\nu + k_\nu$  of the digits on each of the squares is divided by 8 (see Figure 10.16), it follows that the sum of the mentioned 24 digits must be divisible by 8. But 108 is not divisible by 8—a contradiction, and the proof is completed, we are home free.

**10.14.** Although the chessboard has an unusual form, the knight-switching problem is effectively solved using graphs, as in the case of Guar-



ini's problem 10.13. The corresponding graph for the board and the knight's moves is shown in Figure 10.17(a), and may be reduced to the equivalent (but much simpler) graph 10.17(b).

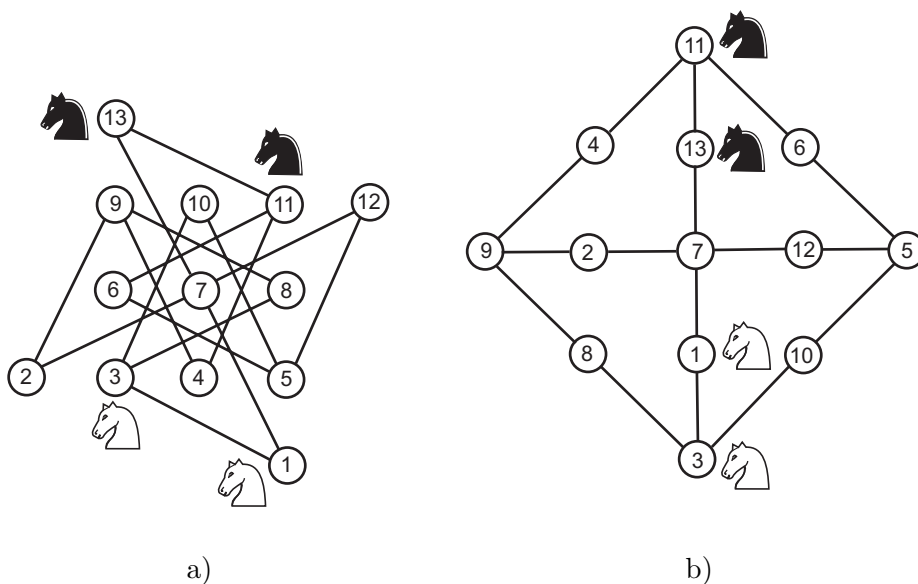


FIGURE 10.17. A graph of possible moves and a simplified graph

The symmetry of the simplified graph and the alternative paths of the knights along the graph permit a number of different solutions, but the minimum number of 14 moves cannot be decreased. Here is one solution:

**13-7-2 11-4-9-8 1-7-13-11 3-1-7-13 8-3 2-7-1**