

## Preface

Decisions. We make them every day. They are made at the personal level when resolving family tensions, deciding where to take family vacations, choosing a college, planning for retirement, or partitioning an inheritance. We make decisions at a professional level as we make business decisions about products to market, where to locate a business, when to take on a partner, or where to invest scarce resources. We make decisions on a sociopolitical level as well, as we work to resolve environmental and social problems in international politics, and make decisions about how we should be represented in government.

Almost all of these many decisions are made in situations where the resulting outcome also depends on the decisions made by other people. For example, your decision about where to go to college is influenced by the decisions of your parents, of financial aid counselors, and of your friends.

Game theory, defined in its broadest sense, is the mathematical field that studies these types of decision making problems. Specifically, it asks the question: How does a player maximize his or her return in a situation where the outcome is at least partially determined by other, independently acting players? Game theory was developed during the 1930s and 1940s in the context of the study of economic behavior, but has since expanded in scope to include many social and political contexts. In recent years, it has even been used by biologists to better understand how behaviors evolve.

The process of representing a decision, behavior, or any other real-world phenomenon in mathematical terms is called mathematical modeling. The essential features of this process are making explicit assumptions to create the model, using mathematical techniques to arrive at results, interpreting those results, and verifying their reasonableness in the original context. A primary goal of this book is to help students to learn, practice, and use this process.

### **To the Student**

The study of mathematics is a participatory process, requiring full engagement with the material in order to learn it. **READ WITH ANTICIPATION.** As you read a sentence, ask yourself: What does this tell me? Where do I think that this idea leads me? How does it connect with previous material? Notice that we used the word “sentence” rather than “paragraph”, “page”, or “chapter” in the previous sentence. This is because reading mathematics, unlike reading literature, is a patient process. Individual sentences are meant to convey significant information

and frequently need to be read two or three times for complete comprehension. Furthermore, DO THE EXERCISES. Mathematical learning requires you to pick up pencil and paper to solve problems.

As a game theory text, this book offers additional, less traditional, opportunities to engage with the material. First among these is the opportunity to PLAY THE GAMES described in almost all of the sections. In several sections, there are activities that encourage you to do this. Find some friends with whom to play the games; you will find that the games are fun and thought provoking to play. As you play a game, reflect, both personally and with the other players, on the “best” ways to play them. Identify your goals and how you would attempt to achieve these goals.

Each chapter opens with a “dialogue”, an idealized conversation between two or more individuals about the topic of the chapter. Together with your friends, ACT OUT THE DIALOGUES. On the surface, these dialogues are merely a somewhat corny way to introduce the topic at hand, but at a deeper level they model the quantitatively literate conversations we hope that you will have in your life.

### **A Brief Description of the Chapters**

Each section includes a succinct statement of the learning objectives and exercises. Most exercises are directly linked to the learning objectives but some provide extra challenge. Following the dialogue section of each chapter, there are two or three sections that contain the core ideas of the chapter. These sections carefully lay out the critical concepts, methods and algebraic tools developing these ideas. The final section of each chapter is usually an extension of the topics that could be omitted in a first reading.

Chapters 1 and 2 are key to the entire book. Chapter 1 introduces the reader to deterministic games and uses them as a means to ease the reader into quantitative thinking. Readers are introduced to the distinct concepts of heuristics and strategies in this chapter and are encouraged to use them to find ways of winning several games, including the well-known game of Nim. They are also introduced to the idea of a game tree and its application to identifying winning strategies for a game. Chapter 2 develops the critical idea of player preferences. Before an individual can make informed decisions about their actions in situations involving conflict or cooperation, he or she must be able to determine preferences among the possible outcomes for all decision making. These preferences can be quantified in a variety of ways and provide the numerical basis for much of the work in the following chapters. This quantification of preferences is our first example of the mathematical modeling process, and the diagram depicting this process is referred to throughout the rest of the book.

Chapters 3, 4, and 5 are a basic study of problems of conflict. Chapter 3 introduces the reader to strategic games and provides many opportunities for the reader to develop models of simple real-world scenarios. This chapter also introduces various solution concepts for strategic games, including the Nash equilibrium. Chapter 4 extends this work by developing the algebraic background necessary to find Nash

equilibria in mixed strategies. Chapter 5 closes the theme of strategic games with a study of Prisoner's Dilemma, demonstrating the perpetual tension between rational self-interest and the collective good. This chapter investigates the repeated play solution of the dilemma, including a short study of infinite series.

Chapters 6, 7, and 8 are a basic study of problems of cooperation. Chapter 6 develops solutions to the bargaining problem, in which two players negotiate among various alternatives. As the reader progresses through the sections in this chapter, various fairness properties are introduced, leading to three solution methods. Chapter 7 explores the situation in which there are three or more players and the value of cooperation is known for each subgroup of players. Two different solutions are presented and are characterized by differing concepts of fairness. Chapter 8 continues this investigation of fairness in problems where a group of players need to partition a set of objects.

As the book moves from its beginning to its end, the general tenor and organization of the material in the chapters changes. Early chapters tend to be more conversational in tone, while later chapters are a bit more formal. This transition mirrors the way the material is developed. In early chapters readers are initially provided with experiences and contexts and are asked to generalize from them, while in the later chapters readers are asked to think about the general properties and to draw conclusions from them. This transition works since readers become increasingly more comfortable in working in a mathematical mode. By the end of the book, readers are beginning to think like mathematicians!

### **To the Instructor**

For instructors using this book, the dialogues also provide built-in classroom activities for their students. In many sections of the book, there are activities that readers can do to develop a deeper understanding of the topic immediately at hand. Some take a longer time to complete, while others can be completed in just a few minutes. While they could easily be omitted by the reader, they represent significant opportunities to learn more by doing and can be used as homework exercises or as in-class activities by a course instructor. These are displayed in boxes for easy recognition.

Whenever they are presented, theorems are given names that reflect either their authorship or the content of the theorem. This enables the reader to readily identify the theorem and to contextualize it. We provide proofs of some of the theorems, and provide arguments based on generic examples for other theorems. Readers should practice, and learn, to work through technical material by carefully reading and, when possible, discussing these proofs and arguments with other people.

Chapter 1, in which *game* is defined, is a prerequisite to all other chapters. Chapter 2 is a prerequisite to Chapter 3 and a "soft" prerequisite for Chapters 7 and 8 (cardinal payoffs are being used but can be understood in terms of money rather than expected utilities for lotteries). Chapter 3 is a prerequisite for Chapters 4 and 5, which are independent of each other. Finally, Chapter 4 is a prerequisite for Chapter 6.

In the authors' experience, each section of each chapter requires two 50-minute periods of a course to cover satisfactorily. Thus, there is too much material in the book to be covered completely and effectively in a 3-credit course. In this setting, an instructor will need to make decisions about the material that can be omitted. The entire book can be covered in a 4-credit course, although instructors will need to identify how they want to use their time.

In either situation, there are plenty of topics omitted that individual course instructors might wish to include. There is no discussion of voting methods and Arrow's theorem, nor of the related topic of voting power. The use of linear programming methods to find Nash equilibria in strategic games and the nucleolus for coalition games is mentioned, but not covered; certainly the use of these tools to solve zero-sum games is within the reach of the general audience. Cake-cutting algorithms are not included nor are extensive discussions of multi-player games, evolutionary games, or simulation games. Any of these topics would be ideal for an instructor to use as an extension of the material, or to assign as a project for their students.

A complete set of annotated solutions can be obtained from the authors by writing either one using institutional letterhead. Please include the enrollment and a description of the course that you are teaching.

Also available from the authors is software for some of the game classes discussed in this book. The free packet includes software which can be distributed to students for installation on their personal machines. If you are interested in this software, please contact the authors for more information.

### **Acknowledgments**

We, the authors of this book, each began teaching game theory to a general audience (nonscience, nonmathematics, noneconomics majors) more than ten years ago. We made this decision, independently, because we saw great value in enabling our students to bring mathematical tools to bear on their fundamental decision-making processes. After conversing with each other, we realized that not only did we have similar visions of what should be taking place in such a course, but also agreed that none of the resources available at the time presented the material in a way that we believed was effective for this general population of students. In particular, we realized that the basic ideas of game theory can easily be communicated using elementary mathematical tools. That is, by systematically incorporating appropriate experiences into the course, we could help students become quantitatively literate. These experiences involve using the fundamental mathematical skills of quantitative literacy: logical reasoning, basic algebra and probability skills, geometric reasoning, and problem solving. Hence, this book is explicit in its intention to teach quantitative literacy, and the layout of each chapter is designed to promote quantitative literacy.

A book of this type does not happen without inspiration or without assistance. We thank Phil Straffin, Alan Taylor, and Steven Brams for their books on game theory which provided excellent models for our own work. We thank Mary Treanor and Dan Apple for their assistance in editing the text and in improving pedagogical

insights. We thank our home institutions, Valparaiso University and Goshen College, for supporting our work through internal grants, sabbatical leaves, and general encouragement. Finally, and most importantly, we thank the many students who, beginning very early in the process, patiently worked through our material as we developed ideas, examples, and text.