CHAPTER 4

Probabilistic Strategies

1. It's Child's Play

By the end of this section, you will have considered new strategy concepts involving probabilities.

MARPS is an acronym for Monetary Asymmetric Rock-Paper-Scissors. It is based on the children's game **Rock-Paper-Scissors**. In both games, two players simultaneously shout "rock", "paper", or "scissors". The simultaneity of the shouts can be done by having both players rhythmically shout, "one, two, three" and then the chosen word. To help keep the rhythm and to add movement to the game, children often have one hand strike the open palm of their other hand with each number, and when they shout their chosen word, that hand is a fist to represent a rock, flat to represent paper, or partially closed with two fingers mimicking the two blades of scissors. Simultaneity can also be accomplished by having each player secretly write their chosen word on a piece of paper, and then the two players can show each other what they wrote. Of course, this latter approach is not as much fun as shouts and hands flying!

After the chosen words have been shouted, money may be exchanged. If both players shouted the same word, there is a tie and no money is exchanged. If the two players shout different words, one player wins and receives money from the other player. "Rock" crushes "scissors", and so the player shouting "rock" receives \$2 from the player shouting "scissors". "Scissors" cut "paper", and so the player shouting "scissors" receives \$2 from the player shouting "paper". "Paper" covers "rock", which is a somewhat wimpy victory, and so the player shouting "paper" receives only \$1 from the player shouting "rock". In contrast to the **Rock-Paper-Scissors** children's game, it is the exchange of money that adds the word "Monetary" and the differing amounts of money exchanged that adds the word "Asymmetric" to the title of the game.

Before you begin to read the dialogue below, you may want to play the game yourself. Write down the payoff matrix since it is different from that of the classical **Rock-Paper-Scissors** game, and find someone to play with. Before you play, think about how you might maximize your own payoff. Think about how you would maximize your average payoff if you played the game several times.

ROSE: This sounds like a fun game. Want to play?

COLIN: Sure! But we're talking inside of a game theory book, so we should probably first obtain the game matrices.

ROSE: Well, as long as it doesn't take too long.

COLIN: Because of the simultaneity of the shouts, a strategy can be described simply by our word choice. So, this is what I obtain for the outcome matrix:

MARPS Outcomes		Colin ROCK	Colin PAPER	Colin SCISSORS
	ROCK	Nothing	Colin receives $1 from Rose	Rose receives $2 from Colin
Rose	PAPER	Rose receives $1 from Colin	Nothing	Colin receives $2 from Rose
	SCISSORS	Colin receives $2 from Rose	Rose receives $2 from Colin	Nothing

ROSE (after comparing the matrix to the second paragraph of this section): Yes, I agree. Now we need to obtain a payoff matrix.

COLIN: What do you think each outcome is worth?

ROSE: You're a good friend, Colin, but for a game like this, I'm in it for the money.

COLIN: Me too.

ROSE: So, our cardinal payoffs should correspond directly to dollars received:

MARPS Cardinal Payoffs		Colin ROCK	Colin PAPER	Colin SCISSORS
	ROCK	$(0, 0)$	$(-1, 1)$	$(2, -2)$
Rose	PAPER	$(1, -1)$	$(0, 0)$	$(-2, 2)$
	SCISSORS	$(-2, 2)$	$(2, -2)$	$(0, 0)$

COLIN: That's true as long as we're also risk neutral.

ROSE: Huh?

COLIN: If you were given the choice, would you be indifferent between (1) receiving $0 with certainty, and (2) having a 50% chance of receiving $2 and a 50% chance of paying $2?

ROSE: Well, if instead of receiving or paying \$2 it had been receiving or paying \$200, I'd chose the \$0 with certainty.

COLIN: So, you're risk adverse.

ROSE: Yes, for large quantities of money. But for \$2, it doesn't make much difference to me. So, it's reasonable to assume that I am risk neutral (especially if that will allow us to play the game sooner).

COLIN: I think I'd prefer the receiving or paying \$2 lottery to the \$0 with certainty.

ROSE: So, you are risk loving?

COLIN: Yes, but probably not too much. I'd be willing to accept the above payoff matrix as a good approximation.

ROSE: Let's play!

COLIN: Should we first analyze the game?

ROSE: What's there to analyze?

COLIN: There are prudential strategies...

ROSE: If I shout "paper" or "scissors", I might lose \$2.00. But if I shout "rock", the most that I can lose is \$1, so ROCK is my prudential strategy.

COLIN: Successive elimination of dominated strategies...

ROSE: No strategy dominates another strategy. So, there are no new insights here.

COLIN: ... and Nash equilibrium.

ROSE: Here's the best response diagram:

MARPS Cardinal Payoffs		Colin ROCK	PAPER	SCISSORS
	ROCK	$(0,0)$	$(-1,\boxed{1})$	$(\boxed{2},-2)$
Rose	PAPER	$(\boxed{1},-1)$	$(0,0)$	$(-2,\boxed{2})$
	SCISSORS	$(-2,\boxed{2})$	$(\boxed{2},-2)$	$(0,0)$

Clearly, there are no Nash equilibria.

COLIN: Wow! So, I guess that what we learned in the last chapter is not too useful.

ROSE: I guess this game theory stuff doesn't give us much guidance for how to play the game.

COLIN: If I were to shout "rock"...

ROSE: ... and I knew that you were going to do that, I should shout "paper" and win \$1 from you.

COLIN: But knowing you plan to shout "paper", I'll shout "scissors" instead and win \$2 from you.

ROSE: I'll be one step ahead of you, and shout "rock".

COLIN: I've got you covered with "paper".

ROSE: Very punny.

COLIN: I guess we are just seeing concretely what happens when there's no Nash equilibrium.

ROSE: Whatever our strategy choices, at least one of us will regret our choice.

COLIN: I'd really like to win \$2...

ROSE: Clever. You're trying to make me think that you'll shout "rock" or "scissors", which would mean I should shout "rock" so that I either tie or win \$2. Knowing this, you'll actually shout "paper". I won't be fooled.

COLIN: Darn! A guy can hope.

ROSE: So, are you ready to play?

COLIN: I might as well just choose my strategy randomly.

ROSE: Will you really choose your strategy randomly? Or will you make an *ad hoc* decision?

COLIN: I was just going to arbitrarily choose a word. I guess that is really *ad hoc*. But maybe I'll choose randomly. I have a normal 6-sided die here. I'll roll it (out of your sight). I'll shout "rock" if the roll is a 1 or 2, "paper" if the roll is a 3 or 4, and "scissors" if the roll is a 5 or 6.

ROSE: That's interesting. You'll choose each word with probability $\frac{1}{3}$.

COLIN: If I can't fool you, I might as well.

ROSE: So, if I plan to shout "rock", there'll be a $\frac{1}{3}$ chance that you'll also shout "rock" and I'll receive \$0, a $\frac{1}{3}$ chance that you'll shout "paper" and I'll lose \$1, and a $\frac{1}{3}$ chance that you'll shout "scissors" and I'll receive \$2.

COLIN: Yes, that's straight from the payoff matrix.

ROSE: By shouting "rock", my weighted average or expected payoff would be

$$(\frac{1}{3})(\$0) + (\frac{1}{3})(-\$1) + (\frac{1}{3})(\$2) = \$0.33.$$

COLIN: And my expected payoff must then be $-\$0.33$.

ROSE: Let me check my other strategies. If I shout "paper", my expected payoff would be

$$(\frac{1}{3})(\$1) + (\frac{1}{3})(\$0) + (\frac{1}{3})(-\$2) = -\$0.33.$$

If I shout "scissors", my expected payoff would be

$$(\frac{1}{3})(-\$2) + (\frac{1}{3})(\$2) + (\frac{1}{3})(\$0) = \$0.$$

So, my best approach would be to shout "rock" and win \$0.33 on average.

COLIN: This analysis is becoming a bit unpleasant.

ROSE: I'm actually starting to enjoy it.

COLIN: Well, if you plan to shout "rock", I could plan to shout "paper".

ROSE: But that just leads us back into more countermoves back and forth.

COLIN: Perhaps if I just assign more probability to "paper"... Yes, I'll shout "rock" with probability 0.3, shout "paper" with probability 0.4, and shout "scissors" with probability 0.3.

ROSE: And how do you plan to generate those probabilities?

COLIN: I have some colored beads here. I could mix three red, four purple, and three silver beads in a cup. If I draw a red bead, I shout "rock". If I draw a purple bead, I shout "paper". If I draw a silver bead, I shout "scissors".

ROSE: Okay, you've convinced me that you could carry out your plan.

COLIN: I'm sometimes blinded by my own brilliance.

ROSE: Very enlightening. Now if I shout "rock", my expected payoff would be

$$(0.3)(\$0) + (0.4)(-\$1) + (0.3)(\$2) = \$0.20.$$

If I shout "paper", my expected payoff would be

$$(0.3)(\$1) + (0.4)(\$0) + (0.3)(-\$2) = -\$0.30.$$

If I shout "scissors", my expected payoff would be

$$(0.3)(-\$2) + (0.4)(\$2) + (0.3)(\$0) = \$0.20.$$

It looks like I can still make \$0.20 on average by shouting "rock".

COLIN: That's a bit better on my wallet: \$0.20 is smaller than \$0.33. But now you have two options to earn that money.

ROSE: You're right! I could shout "scissors" instead of "rock".

COLIN: For that matter, you could even randomly choose between the two strategies.

ROSE: That's right! I could flip a coin to decide whether to shout "rock" or "scissors". Thanks for the idea, Colin!

COLIN: You're welcome. But why is it that I'm going into this game already planning to lose money?

ROSE: Perhaps because you're telling me what you plan to do, and I'm choosing the strategy that works the best for me against your plan.

COLIN: Well, I'm an "up front" and honest kind of guy.

ROSE: That's why you tried to fool me earlier.

COLIN: You caught me. Perhaps that deception idea has some merit. At this point, I have you thinking that randomly choosing between "rock" and "scissors" is a good idea.

ROSE: Only because you said that you planned to shout "rock" with probability 0.3, shout "paper" with probability 0.4, and shout "scissors" with probability 0.3.

COLIN: True, but to counter your "rock" or "scissors" choice, perhaps I should assign more probability to my choosing "rock".

ROSE: I'll just change my strategy choice to what is better.

COLIN: Fine. Suppose that I plan to shout "rock" with probability 0.4, shout "paper" with probability 0.4, and shout "scissors" with probability 0.2.

ROSE: I'll just look at my expected payoffs again. If I shout "rock", my expected payoff would be

$$(0.4)(\$0) + (0.4)(-\$1) + (0.2)(\$2) = \$0.$$

If I shout "paper", my expected payoff would be

$$(0.4)(\$1) + (0.4)(\$0) + (0.2)(-\$2) = \$0.$$

If I shout "scissors", my expected payoff would be

$$(0.4)(-\$2) + (0.4)(\$2) + (0.2)(\$0) = \$0.$$

What happened to my positive payoffs?

COLIN: Looks like they're gone.

ROSE: And it looks like there's nothing I can do to counter your plan. No matter what I do, my expected payoff is $0.

COLIN: And if I discern what strategy you're using, I could vary my strategy to take advantage of my knowledge.

ROSE: In that case, perhaps I'll also shout "rock" with probability 0.4, shout "paper" with probability 0.4, and shout "scissors" with probability 0.2.

COLIN: I guess that would mean that whatever I tried to do, my expected payoff would be $0, too.

ROSE: So, if both of us adopt this plan, neither one of us could do any better by adopting a different plan.

COLIN: That makes the pair of plans sound like a Nash equilibrium.

ROSE: And if just one of us adopts the plan, that person already ensures him or herself an expected payoff of at least \$0.

COLIN: That makes the plan sound like a prudential strategy.

ROSE: Of course, that would be if the plan could be considered a strategy. But we already said that there were only three strategies: rock, paper, and scissors.

COLIN: But a strategy is a complete and unambiguous description of what to do in every possible situation. Our plan tells us to shout "rock" with probability 0.4, shout "paper" with probability 0.4, and shout "scissors" with probability 0.2. That seems complete and unambiguous.

ROSE: Especially if we add a description of the bead drawing mechanism for how to generate the specified probabilities.

COLIN: Cool! I'm now ready to play **MARPS**.

ROSE: So am I!

COLIN and ROSE together: One, two, three, ...

Exercises

(1) In Chapter 1, we stated that a strategy is a complete and unambiguous description of what to do in every possible situation. Explain why Rose and Colin's use of a probabilistic strategy, which involves random choices, does not violate this definition.
(2) If you played **MARPS** several times, determine the relative frequency for each strategy choice. How do your relative frequencies compare with the probabilities that Rose and Colin adopted?
(3) If "paper" against "rock" won \$2, instead of \$1, how might this have affected Rose and Colin's discussion?
(4) If "paper" against "rock" won \$0.50, instead of \$1, how might this have affected Rose and Colin's discussion?

2. Mixed Strategy Solutions

When we introduced strategic games, our examples seemed to have a small number of strategies. Each player in **MARPS** appears to have only three strategies: shout “rock”, shout “paper”, or shout “scissors”. But each player who is playing a strategic game has an infinite number of strategies because a player can always choose strategies in some probabilistic manner. Rose and Colin each decided to shout “rock” with probability 0.4, shout “paper” with probability 0.4, and shout “scissors” with probability 0.2. These probabilistic strategies are usually called mixed strategies, in order to distinguish them from the original strategies, which are then called pure strategies. Note that any pure strategy can be expressed as a mixed strategy: shout “rock” can be expressed as shout “rock” with probability 1.0. Formally, we have

> **Mixed Strategy:** A *mixed strategy* is an assignment of probabilities to strategies. It is usually expressed as a combination of the original, *pure*, strategies.

When we allow players to use mixed strategies, we will determine their payoffs using the Expected Utility Hypothesis. Therefore, the expected payoffs are only meaningful if cardinal, rather than ordinal, payoffs are used when we play the game.

> **Expected Payoff:** The payoff that a player receives when each is using a mixed strategy is calculated by computing the sum of the various pure strategy payoffs weighted by their probabilities. This payoff is called an *expected payoff.*

For **MARPS**, Colin finally suggested that he would shout “rock” with probability 0.4, shout “paper” with probability 0.4, and shout “scissors” with probability 0.2. We abbreviate this strategy with the notation

$$0.4 \text{ ROCK } + \ 0.4 \text{ PAPER } + \ 0.2 \text{ SCISSORS.}$$

Rose went on to calculate her expected payoff if she were to shout “rock” as

$$(0.4)(\$0) + (0.4)(-\$1) + (0.2)(\$2) = \$0.$$

There are at least three ways that we might interpret a player’s expected payoff for a given mixed strategy. First, the expected payoff is simply the cardinal utility that we assign to the lottery of outcomes associated with the mixed strategy; Rose should be indifferent between (1) Rose and Colin both shouting “scissors” giving her a payoff of \$0, and (2) Rose shouting “rock” and Colin shouting “rock”, “paper”, or “scissors” with probability 0.4, 0.4, and 0.2, respectively, giving her an expected payoff of \$0. Second, a player’s expected payoff may be interpreted as the average of his or her payoffs over many games when players consistently use the same mixed strategy; if Rose shouts “rock” and Colin shouts “rock”, “paper”, or “scissors” with probability 0.4, 0.4, and 0.2, respectively, Rose’s average payoff after several rounds of the game should be about \$0. Third, the expected payoff may be interpreted as the average payoff from many games played by a population of players who use pure strategies in proportion to the mixed strategy probabilities; if there are ten Rose clones each shouting “rock”, four Colin clones shouting “rock”, four Colin clones shouting “paper”, and two Colin clones shouting “scissors”, then the average payoff for the ten Rose clones would be \$0.

The availability of mixed strategies increases the set of strategy choices that each player has. This means that there may be a mixed strategy that ensures a higher expected payoff than any of the pure strategies; that is, a mixed prudential strategy may be better than a pure prudential strategy. Also while some strategic games do not have a Nash equilibrium in pure strategies, Nash proved the following.

Nash Equilibrium Theorem: *Every strategic game has at least one Nash equilibrium in pure or mixed strategies.*

In the dialogue, it was argued that each player choosing the mixed strategy of shouting “rock”, “paper”, or “scissors” with probability 0.4, 0.4, and 0.2, respectively, would form a Nash equilibrium for **MARPS**.

By the end of this section, you will be able to calculate each player’s expected payoff when players use mixed strategies, to determine whether a pair of mixed strategies is a Nash equilibrium, and to determine whether a given mixed strategy is a prudential strategy.

Calculations

Mixed strategies, mixed prudential strategies, and mixed-strategy Nash equilibria were introduced using **MARPS** in the dialog between Rose and Colin. **MARPS** is a somewhat special game since the players’ interests are in complete opposition. We now develop these ideas further using a strategic game, **Impromptu**, in which the players’ interests sometimes coincide. Its cardinal payoff matrix is shown here.

Impromptu		Colin		
Cardinal Payoffs		*A*	*B*	*C*
	A	(10, 100)	(100, 50)	(60, 40)
Rose	*B*	(0, 60)	(80, 70)	(70, 0)
	C	(20, 0)	(40, 30)	(50, 90)

From the best response diagram

Impromptu		Colin		
Cardinal Payoffs		*A*	*B*	*C*
	A	(10, [100])	([100], 50)	(60, 40)
Rose	*B*	(0, 60)	(80, [70])	([70], 0)
	C	([20], 0)	(40, 30)	(50, [90])

it is clear that there is no pure-strategy Nash equilibrium. Any pair of pure strategies chosen by Rose and Colin will result in at least one of the players wishing he or she had chosen differently. The only hope for stability would be to employ a probabilistic approach to choosing pure strategies; that is, employ a mixed strategy. Thus, one mixed strategy for Colin would be

$$\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C,$$

in which he selects each of his pure strategies with probability $\frac{1}{3}$. Another mixed strategy for Colin is

$$0.7A + 0.1B + 0.2C,$$

in which he selects his first strategy more often than either of the other two.

When the context is clear, we will abbreviate mixed strategies by only listing the probabilities: the two mixed strategies for Colin described in the previous sentence could be abbreviated

$$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

and

$$(0.7, 0.1, 0.2),$$

respectively. A pure strategy can be thought of as a mixed strategy in which all of the probability is concentrated on a single strategy. For example, $(0, 1, 0)$, which represents $0A + 1B + 0C$, is the same as the pure strategy COLUMN B.

Suppose Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ consisting of a 10% chance of choosing strategy ROW A, a 30% chance of choosing strategy ROW B, and a 60% chance of choosing strategy ROW C. If Colin chooses COLUMN A, then the strategy pair (ROW A, COLUMN A) occurs with probability 0.1, the strategy pair (ROW B, COLUMN A) occurs with probability 0.3, and the strategy pair (ROW C, COLUMN A) occurs with probability 0.6. A good way to visualize this is

Impromptu Cardinal Payoffs		Colin A
	$0.1A$	$(10, 100)$
Rose	$0.3B$	$(0, 60)$
	$0.6C$	$(20, 0)$

So, Rose's expected payoff is

$$(0.1)(10) + (0.3)(0) + (0.6)(20) = 13$$

(found by multiplying the probabilities times the payoffs and summing), and Colin's expected payoff is

$$(0.1)(100) + (0.3)(60) + (0.6)(0) = 28.$$

Thus, using our second interpretation of expected payoffs, if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses COLUMN A in many repeated plays of **Impromptu**, then Rose's average payoff should be about 13 and Colin's average payoff should be about 28.

Similarly, if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses COLUMN B, then the strategy pair (ROW A, COLUMN B) occurs with probability 0.1, the strategy pair (ROW B, COLUMN B) occurs with probability 0.3, and the strategy pair (ROW C, COLUMN B) occurs with probability 0.6. So, Rose's expected payoff is

$$(0.1)(100) + (0.3)(80) + (0.6)(40) = 58,$$

and Colin's expected payoff is

$$(0.1)(50) + (0.3)(70) + (0.6)(30) = 44.$$

Again, if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses COLUMN B in many repeated plays of **Impromptu**, then Rose's average payoff should be about 58 and Colin's average payoff should be about 44.

What are the expected payoffs if both players choose mixed strategies? Suppose Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses the mixed strategy $(0.8, 0.2, 0.0)$. Since Rose and Colin are making independent decisions, the strategy pair (ROW A, COLUMN A) occurs with probability $(0.1)(0.8) = 0.08$. Similar calculations can be performed for every other strategy pair; they are summarized, along with the payoff pairs, in the following matrix.

Probability & Payoff Pair		Colin 0.8A	0.2B	0.0C
	0.1A	(0.08)(10, 100)	(0.02)(100, 50)	(0.00)(60, 40)
Rose	0.3B	(0.24)(0, 60)	(0.06)(80, 70)	(0.00)(70, 0)
	0.6C	(0.48)(20, 0)	(0.12)(40, 30)	(0.00)(50, 90)

So, Rose's expected payoff is

$$\begin{array}{rlclcl} & (0.08)(10) & + & (0.02)(100) & + & (0.00)(60) \\ + & (0.24)(\ 0) & + & (0.06)(\ 80) & + & (0.00)(70) \\ + & (0.48)(20) & + & (0.12)(\ 40) & + & (0.00)(50) \\ = & 22.0, & & & & \end{array}$$

and Colin's expected payoff is

$$\begin{array}{rlclcl} & (0.08)(100) & + & (0.02)(50) & + & (0.00)(40) \\ + & (0.24)(\ 60) & + & (0.06)(70) & + & (0.00)(\ 0) \\ + & (0.48)(\ \ 0) & + & (0.12)(30) & + & (0.00)(90) \\ = & 31.2. & & & & \end{array}$$

Summarizing, if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses the mixed strategy $(0.8, 0.2, 0.0)$ in many repeated plays of **Impromptu**, then Rose's average payoff should be about 22.0 and Colin's average payoff should be about 31.2.

We could have computed Rose's expected payoff in a different way. Before we do any arithmetic, Rose's expected payoff is given by the following calculation.

$$\begin{array}{rlclcl} & (0.1)(0.8)(10) & + & (0.1)(0.2)(100) & + & (0.1)(0.0)(60) \\ + & (0.3)(0.8)(\ 0) & + & (0.3)(0.2)(\ 80) & + & (0.3)(0.0)(70) \\ + & (0.6)(0.8)(20) & + & (0.6)(0.2)(\ 40) & + & (0.6)(0.0)(50) \end{array}$$

Regrouping these terms gives us the following.

$$\begin{array}{rlclcl} & (0.8)[(0.1)(\ 10) & + & (0.3)(\ 0) & + & (0.6)(20)] \\ + & (0.2)[(0.1)(100) & + & (0.3)(80) & + & (0.6)(40)] \\ + & (0.0)[(0.1)(\ 60) & + & (0.3)(70) & + & (0.6)(50)] \end{array}$$

Simplifying, we obtain

$$(0.8)(13) + (0.2)(58) + (0.0)(57) = 22.0.$$

Thus, Rose's expected payoff when Colin chooses mixed strategy $(0.8, 0.2, 0.0)$ is

(0.8)×[Rose's expected payoff when Colin chooses COLUMN A]
+
(0.2)×[Rose's expected payoff when Colin chooses COLUMN B]
+
(0.0)×[Rose's expected payoff when Colin chooses COLUMN C].

In general, the expected payoff associated with a mixed strategy is the weighted average of the expected payoffs associated with the pure strategies.

The conclusion of the previous paragraph implies that the expected payoff associated with a mixed strategy is no greater than the largest payoff associated with a pure strategy that is used with positive probability. In the example above, 22.0 is no greater than the larger of 13 and 58. If Rose were to choose a mixed strategy (p_A, p_B, p_C) to play against Colin's choice of $(0.8, 0.2, 0.0)$, then Rose's expected payoff would be $13p_A + 58p_B + 57p_C$, and this payoff must be no greater than 58, which is the payoff Rose would obtain by choosing the pure strategy ROW B. This shows us that there is always a pure strategy best response by a player to whatever strategy choices are made by the other players.

To illustrate these ideas from Colin's perspective, if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses the mixed strategy $(0.8, 0.2, 0.0)$, then Colin's expected payoff is

$$(0.8)(28) + (0.2)(44) + (0.0)(58) = 31.2,$$

obtained by multiplying the probability that he is in a given column times his expected payoff in that column (calculated above) and summing these products. More generally, if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$ and Colin chooses the mixed strategy (q_A, q_B, q_C), then Colin's expected payoff is

$$(q_A)(28) + (q_B)(44) + (q_C)(58),$$

which is never greater than 58. This implies that if Rose chooses the mixed strategy $(0.1, 0.3, 0.6)$, then Colin's best response is COLUMN C.

Solution Verifications

We have already seen that **Impromptu** has no Nash equilibrium using pure strategies, but the Nash Equilibrium Theorem tells us that it does have at least one Nash equilibrium in mixed strategies. Actually finding a Nash equilibrium can be a complex computational task that we will defer to the next two sections. For now we will show how we can determine whether a given pair of mixed strategies is a Nash equilibrium.

Suppose Bill claims that

$$\left(\frac{3}{8}A + 0B + \frac{5}{8}C, \frac{6}{7}A + \frac{1}{7}B + 0C\right) = \left(\frac{3}{8}A + \frac{5}{8}C, \frac{6}{7}A + \frac{1}{7}B\right)$$

is a Nash equilibrium for **Impromptu**. While not concerning ourselves with how Bill came up with this pair of strategies, we will check whether Bill's claim is correct. To do so, we need to determine (1) whether $\frac{3}{8}A + \frac{5}{8}C$ is Rose's best response to Colin choosing $\frac{6}{7}A+\frac{1}{7}B$, and (2) whether $\frac{6}{7}A+\frac{1}{7}B$ is Colin's best response to Rose choosing $\frac{3}{8}A + \frac{5}{8}C$.

We begin by checking Rose's expected payoffs using each of her pure strategies when Colin chooses $\frac{6}{7}A + \frac{1}{7}B$:

Rose's Strategy	Rose's Expected Payoff When Colin Chooses $\frac{6}{7}A + \frac{1}{7}B$
A	$(\frac{6}{7})(10) + (\frac{1}{7})(100) = \frac{160}{7}$
B	$(\frac{6}{7})(\ 0) + (\frac{1}{7})(\ 80) = \frac{80}{7}$
C	$(\frac{6}{7})(20) + (\frac{1}{7})(\ 40) = \frac{160}{7}$

Since Rose's expected payoff is highest for pure strategies A and C, they are the best pure strategy responses for Rose. Any probabilistic mixture of A and C would also give Rose an expected payoff of $\frac{160}{7}$, and so any mixed strategy involving A and C is also a best response for Rose. In particular, $\frac{3}{8}A + \frac{5}{8}C$ is a best response by Rose to Colin choosing $\frac{6}{7}A + \frac{1}{7}B$.

Now we check Colin's expected payoffs using each of his pure strategies when Rose chooses $\frac{3}{8}A + \frac{5}{8}C$:

Colin's Strategy	Colin's Expected Payoff When Rose Chooses $\frac{3}{8}A + \frac{5}{8}C$
A	$(\frac{3}{8})(100) + (\frac{5}{8})(\ 0) = \frac{300}{8}$
B	$(\frac{3}{8})(\ 50) + (\frac{5}{8})(30) = \frac{300}{8}$
C	$(\frac{3}{8})(\ 40) + (\frac{5}{8})(90) = \frac{570}{8}$

Since Colin's expected payoff is highest for pure strategy C, it is the only best response for Colin. In particular, $\frac{6}{7}A + \frac{1}{7}B$ is not a best response of Colin to Rose choosing $\frac{3}{8}A + \frac{5}{8}C$. Hence, the answer to our second question is no. Therefore, $(\frac{3}{8}A + \frac{5}{8}C, \frac{6}{7}A + \frac{1}{7}B)$ is not a Nash equilibrium.

Suppose Andrea claims that $(0.5A+0.1B+0.4C, 0.5A+0.5C)$ is a Nash equilibrium. Again, we will not concern ourselves with how Andrea came up with this pair of strategies (nor will we be concerned with why Andrea uses decimals and Bill uses fractions). We only want to know whether Andrea's claim is correct. We need to ask (1) is $0.5A+0.1B+0.4C$ a best response by Rose to Colin choosing $0.5A+0.5C$, and (2) is $0.5A+0.5C$ a best response by Colin to Rose choosing $0.5A+0.1B+0.4C$?

In order to answer the first question, we check Rose's expected payoffs using each of her pure strategies when Colin chooses $0.5A + 0.5C$:

Rose's Strategy	Rose's Expected Payoff When Colin Chooses $0.5A + 0.5C$
A	$(.5)(10) + (.5)(60) = 35$
B	$(.5)(\ 0) + (.5)(70) = 35$
C	$(.5)(20) + (.5)(50) = 35$

Since Rose's expected payoff is the same for each pure strategy, each pure strategy and any probabilistic mixture of them is a best response for Rose. In particular, $0.5A+0.1B+0.4C$ is a best response by Rose to Colin choosing $0.5A+0.5C$. Hence, the answer to the first question is yes.

In order to answer the second question, we check Colin's expected payoffs using each of his pure strategies when Rose chooses $0.5A + 0.1B + 0.4C$:

Colin's Strategy	Colin's Expected Payoff When Rose Chooses $0.5A + 0.1B + 0.4C$
A	$(.5)(100) + (.1)(60) + (.4)(\ 0) = 56$
B	$(.5)(\ 50) + (.1)(70) + (.4)(30) = 44$
C	$(.5)(\ 40) + (.1)(\ 0) + (.4)(90) = 56$

Since Colin's expected payoff is highest for pure strategies A and C, they and any probabilistic mixture of them are best responses for Colin. In particular, $0.5A+0.5C$ is a best response by Colin to Rose choosing $0.5A+0.1B+0.4C$. Hence, the answer to the second question is yes. Since both answers are yes, that is, each strategy is a best response to the other, $(0.5A+0.1B+0.4C, 0.5A+0.5C)$ is a Nash equilibrium.

We can summarize our calculations in an extended payoff matrix:

	A	B	C	$\frac{6}{7}A + \frac{1}{7}B$	$.5A + .5C$
A	$(10, \boxed{100})$	$(\boxed{100}, 50)$	$(60, 40)$	$(\boxed{\frac{160}{7}}, \frac{650}{7})$	$(\boxed{35}, 70)$
B	$(0, 60)$	$(80, \boxed{70})$	$(\boxed{70}, 0)$	$(\frac{80}{7}, \frac{430}{7})$	$(\boxed{35}, 30)$
C	$(\boxed{20}, 0)$	$(40, 30)$	$(50, \boxed{90})$	$(\boxed{\frac{160}{7}}, \frac{30}{7})$	$(\boxed{35}, 45)$
$\frac{3}{8}A + \frac{5}{8}C$	$(16.25, 37.5)$	$(62.5, 37.5)$	$(53.75, \boxed{71.25})$		
$.5A + .1B + .4C$	$(13, \boxed{56})$	$(74, 44)$	$(57, \boxed{56})$		$(\boxed{35}, \boxed{56})$

The two new rows correspond to Rose using the mixed strategies $\frac{3}{8}A + \frac{5}{8}C$ and $0.5A+0.1B+0.4C$ against Colin using one of his pure strategies. We had previously computed Colin's expected payoffs; the table includes Rose's expected payoffs. The two new columns correspond to Colin using the mixed strategies $\frac{6}{7}A + \frac{1}{7}B$ and $0.5A + 0.5C$ against Rose using one of her pure strategies. We had previously computed Rose's expected payoffs; the table includes Colin's expected payoffs. One additional pair of expected payoffs is included to indicate the Nash equilibrium.

Since there are an infinite number of mixed strategies available to each player, completing the payoff table would require an infinite number of rows and columns. While we can imagine such a payoff table, we would not want to try writing it down. Furthermore, the whole table is not needed to determine whether a strategy pair is a Nash equilibrium. As we have observed, best response payoffs always occur at pure strategy payoffs. Given a mixed strategy for Rose, all of Colin's best responses are probabilistic mixtures of his pure strategy best responses. Similarly, given a mixed strategy for Colin, all of Rose's best responses are probabilistic mixtures of her pure strategy best responses. If the two strategies are best responses to each other, we have a Nash equilibrium.

It is a bit more difficult to determine whether a given mixed strategy is a prudential strategy, but it can be done. In **Impromptu**, we can show ROW C is a prudential strategy for Rose. If Rose chooses ROW C, she could receive anywhere from 20 to 50 depending on which pure or mixed strategy Colin chooses. Thus, by choosing ROW C, Rose ensures herself a payoff of at least 20. However, if Colin chooses COLUMN A, Rose will receive at most 20, depending on what pure or mixed strategy she uses. We see that Rose can guarantee herself a payoff of at least 20 by selecting ROW C, but cannot guarantee anything better since Colin may select COLUMN A. Thus, ROW C is prudential for Rose.

We claim that Colin's prudential strategy is $\frac{1}{45}A + \frac{30}{45}B + \frac{14}{45}C$. To verify this claim, we first determine that, by using $\frac{1}{45}A + \frac{30}{45}B + \frac{14}{45}C$, Colin receives a payoff of 48 regardless of what Rose chooses to do (see exercise 4). Further, if Rose chooses the mixed strategy $0.3A + 0.3B + 0.4C$, Colin will receive a payoff of 48 no matter what strategy he selects (see exercise 4). Thus, Rose's choice of strategy can reduce Colin's payoff to no more than 48. Since Colin can guarantee at least 48 by choosing $\frac{1}{45}A + \frac{30}{45}B + \frac{14}{45}C$, but he cannot ensure more than this, 48 is his security level and the mixed strategy $\frac{1}{45}A + \frac{30}{45}B + \frac{14}{45}C$ is prudential.

In the previous several paragraphs, we gave you strategies that turned out to form a Nash equilibrium and other strategies that turned out to be prudential strategies. Given a set of mixed strategies for a game, it is easy to determine whether they are a Nash equilibrium, and somewhat more difficult to determine whether either is a prudential strategy. However, it is generally a difficult task to actually find strategies that either are part of a Nash equilibrium or are prudential. In the next two sections, we will demonstrate how to do this in the two simplest cases: 2×2 and $m \times 2$ strategic games. The solution methods described in those sections illustrate the general solution method, but for larger games the algebraic manipulations become significantly more complicated.

Implementation and Interpretation

In **Impromptu**, the security levels for Rose and Colin are 20 and 48, respectively, while the payoffs from the Nash equilibrium described earlier are 35 and 56. Hence, there is no real incentive for either player to choose their prudential strategy instead of the Nash equilibrium strategies.

In the Nash equilibrium, Rose uses the strategy $0.5A + 0.1B + 0.4C$ and Colin uses the strategy $0.5A+0.5C$. How do Rose and Colin implement these mixed strategies? Colin could toss a fair coin, choose A if the coin lands heads, and choose C if the coin lands tails. Rose could place five azure, one blue, and four cyan beads in a can, shake the can, choose a bead from the can without looking, choose A if the chosen bead is azure, choose B if the chosen bead is blue, and choose C if the chosen bead is cyan. Rose and Colin could also implement their mixed strategies using dice, cards, or a computer random number generator.

But what will happen? It all depends on the random coin flip and bead choice. If the bead is blue and the coin lands heads, Rose will choose B and Colin will choose A, resulting in a payoff of 0 for Rose and 60 for Colin. Wouldn't Rose now regret her choice? Yes, she would: given that Colin chose A, Rose would have preferred to have chosen C. But her regret is *ex post* (after the bead has been chosen). When Rose was choosing her strategy, she may have thought that Colin would use the mixed strategy $0.5A + 0.5C$. Given that thought, Rose receives an expected payoff of 35 no matter what strategy she chooses (as shown earlier). So, *ex ante* (before the bead has been chosen), Rose does not regret choosing the mixed strategy $0.5A+0.1B + 0.4C$.

Since choosing a strategy randomly feels like a loss of control for most people and it is the *ex post* payoff that a player will actually receive at the end of the game, it is tempting to abandon using a mixed strategy. It is the desire to avoid the overt appearance of randomness that is the reason why humans throughout history have consulted astrologers, shamans, and counselors. But as soon as Colin decides on the pure strategy A in **Impromptu** and Rose realizes that Colin would make that decision, then Rose would choose C, whereupon Colin making this realization would switch to C, and once again there is a cycling of changing choices. If she knows that Colin will choose the mixed strategy $0.5A + 0.5C$, Rose will be willing to stick with the mixed strategy $0.5A + 0.1B + 0.4C$.

Of course, if Colin is using $0.5A + 0.5C$, Rose would receive the same expected payoff regardless of which pure or mixed strategy she uses because all of her pure strategies are best responses. So, Rose should be willing to choose any strategy, not just $0.5A+0.1B+0.4C$. The reason she is choosing the specific mixed strategy $0.5A + 0.1B + 0.4C$ is to make sure that Colin would not regret choosing $0.5A + 0.5C$.

How would Rose know that Colin is using a mixed strategy? After all, in the end, she only knows what pure strategy Colin announces. Even if Rose sees Colin flipping a coin before announcing his choice, she does not know whether Colin actually makes use of the coin flip information (maybe Colin just likes to flip coins). However, if Rose and Colin publicly announce that they will be using $0.5A + 0.1B + 0.4C$ and $0.5A + 0.5C$, then neither has an incentive do anything other than what they publicly announced. This is the inherent meaning in $(0.5A + 0.1B + 0.4C, 0.5A + 0.5C)$ being a Nash equilibrium. Of course, if Rose has an ability to predict Colin's choice (perhaps because his eyes twitch faster when he is about to choose B), she should make use of that information in order to make her choice. But if Rose cannot predict Colin's choice, Nash tells her to randomize her choices with the mixed strategy $0.5A + 0.1B + 0.4C$.

If **Impromptu** is played repeatedly, then it seems even more reasonable for Rose and Colin to choose the mixed strategies $0.5A + 0.1B + 0.4C$ and $0.5A + 0.5C$, because even though their payoffs will vary in each round, on average Rose will obtain 35 and Colin will receive 56, and neither player can obtain a higher payoff by unilaterally changing her or his strategy. The difference between the *ex ante* and *ex post* payoffs has been eliminated by the averaging that occurs in repeated play.

Instead of playing once or repeatedly involving sentient players, suppose we replace Rose and Colin with populations of mindless Rose and Colin clones that play each other. Suppose 50%, 10%, and 40% of the Rose clones are genetically programmed to use pure strategies A, B, and C, respectively. We call these Rose A clones, Rose B clones, and Rose C clones, respectively. Similarly, suppose that 50% of Colin's clones are genetically programmed to use pure strategy A, and the other 50% of Colin's clones are genetically programmed to use pure strategy C. When a Rose clone interacts with a Colin clone, progeny are produced. For example, when a Rose C clone interacts with a Colin B clone, this corresponds to the strategy choices (C, B), and the payoff pair $(40, 30)$ results in 40 new Rose C clones and 30 new Colin B clones.

Since the initial distributions of clones correspond to the Nash equilibrium, the population distributions will remain the same. That is, there will continue to be 50%, 10%, and 40% of the Rose clones genetically programmed to use pure strategies A, B, and C, and equal numbers of Colin clones genetically programmed to use pure strategies A and C. Any other non-Nash equilibrium distribution of clones would change over time. Biologists call our Nash equilibrium distributions evolutionarily stable genetic population distributions.

> To experience the population interpretation for mixed strategies, play **Impromptu** in the following manner. Gather a group of at least five friends. Have each person play the role of Rose by secretly writing down on one piece of paper their name and ROW A, ROW B, or ROW C; have each person play the role of Colin by secretly writing down on one piece of paper their name and COLUMN A, COLUMN B, or COLUMN C. Collect the two sets of papers and publicly display the relative frequency distribution of Rose's strategy choices and the relative frequency distribution of Colin's strategy choices. Each player's Rose payoff is computed using their written Rose strategy versus Colin's relative frequency distribution, and similarly, each player's Colin payoff is computed using their written Colin strategy versus Rose's relative frequency distribution.

Therefore, we have at least three reasonable interpretations of the Nash equilibrium: *ex ante* regret free in a single play of the game, long term regret free in repeated play of the game, or stable population distributions. Which one is applicable depends on the original scenario that was modeled as a strategic game.

Unfortunately, difficulties arise in all three of these interpretations. When there is more than one Nash equilibrium, none of the interpretations help us select among them. Second, if the players are playing repeatedly, there could be important interactions across rounds. These two ideas are explored more in Chapter 5. Third, if populations of Rose and Colin clones do not start in a Nash equilibrium distribution, the distribution may not converge to the Nash equilibrium distribution. This final difficulty is beyond the scope of this book.

Exercises

(1) Consider the following game.

Cardinal Payoffs (Rose, Colin)		Colin A	B
Rose	A	$(2,4)$	$(1,0)$
	B	$(3,1)$	$(0,4)$

(a) What are the players' expected payoffs if Rose chooses A and Colin chooses the mixed strategy $\frac{4}{7}A + \frac{3}{7}B$?
(b) What are the players' expected payoffs if Rose chooses B and Colin chooses the mixed strategy $\frac{4}{7}A + \frac{3}{7}B$?
(c) What are the players' expected payoffs if Rose chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$ and Colin chooses the mixed strategy $\frac{4}{7}A + \frac{3}{7}B$?
(d) What is Rose's best response(s) to Colin choosing the mixed strategy $\frac{4}{7}A+\frac{3}{7}B$?
(e) What is the lowest payoff Colin could receive if he chooses the mixed strategy $\frac{4}{7}A + \frac{3}{7}B$?
(f) What are the players' expected payoffs if Rose chooses A and Colin chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$?
(g) What are the players' expected payoffs if Rose chooses B and Colin chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$?
(h) What are the players' expected payoffs if Rose chooses the mixed strategy $\frac{3}{7}A + \frac{4}{7}B$ and Colin chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$?
(i) What is Rose's best response(s) to Colin choosing the mixed strategy $\frac{1}{2}A+\frac{1}{2}B$?
(j) What is the lowest payoff Colin could receive if he chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$?
(k) What are the players' expected payoffs if Rose chooses the mixed strategy $\frac{3}{7}A + \frac{4}{7}B$ and Colin chooses A?
(l) What are the players' expected payoffs if Rose chooses the mixed strategy $\frac{3}{7}A + \frac{4}{7}B$ and Colin chooses B?
(m) What are the players' expected payoffs if Rose chooses the mixed strategy $\frac{3}{7}A + \frac{4}{7}B$ and Colin chooses $\frac{1}{2}A + \frac{1}{2}B$?
(n) What is (are) Colin's best response(s) to Rose choosing the mixed strategy $\frac{3}{7}A + \frac{4}{7}B$?

(o) What is the lowest payoff Rose could receive is she chooses the mixed strategy $\frac{3}{7}A + \frac{4}{7}B$?
(p) Based on part (d), explain why $(\frac{3}{7}A + \frac{4}{7}B, \frac{4}{7}A + \frac{3}{7}B)$ is not a Nash equilibrium.
(q) Based on parts (i) and (n), explain why $(\frac{3}{7}A + \frac{4}{7}B, \frac{1}{2}A + \frac{1}{2}B)$ is a Nash equilibrium.
(r) Explain how Rose and Colin could implement the Nash equilibrium strategies described in part (q).
(s) If Rose and Colin use the Nash equilibrium strategies described in part (q) to play this game 100 times, roughly how many times will each outcome occur?
(t) Based on parts (e) and (j), explain why $\frac{1}{2}A+\frac{1}{2}B$ is not Colin's prudential strategy.
(u) Based on parts (e) and (k)–(m), explain why $\frac{4}{7}A + \frac{3}{7}B$ is a prudential strategy for Colin.
(v) Explain why A is prudential for Rose (even among mixed strategies).

(2) Consider the following game.

Cardinal Payoffs (Rose, Colin)		Colin A	B
	A	$(-3, 3)$	$(5, -5)$
Rose	B	$(-1, 1)$	$(3, -3)$
	C	$(2, -2)$	$(-2, 2)$
	D	$(3, -3)$	$(-6, 6)$

(a) Determine the pure prudential strategies, the dominated strategies, and the pure strategy Nash equilibria.
(b) Suppose Colin chooses the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$.
  (i) What is Rose's expected payoff if she chooses A?
  (ii) What is Rose's expected payoff if she chooses B?
  (iii) What is Rose's expected payoff if she chooses C?
  (iv) What is Rose's expected payoff if she chooses D?
  (v) What is Rose's best response(s) to Colin choosing $\frac{1}{2}A + \frac{1}{2}B$?
(c) What is Rose's best response(s) to Colin choosing the mixed strategy $\frac{5}{8}A+\frac{3}{8}B$?
(d) What is Colin's best response(s) to Rose choosing $\frac{1}{4}A + \frac{1}{4}B + \frac{1}{4}C + \frac{1}{4}D$?
(e) What is Colin's best response(s) to Rose choosing $\frac{1}{2}B + \frac{1}{2}C$?
(f) Is $(\frac{1}{4}A+\frac{1}{4}B+\frac{1}{4}C+\frac{1}{4}D, \frac{1}{2}A+\frac{1}{2}B)$ a Nash equilibrium? Why or why not?
(g) Is $(\frac{1}{4}A+\frac{1}{4}B+\frac{1}{4}C+\frac{1}{4}D, \frac{5}{8}A+\frac{3}{8}B)$ a Nash equilibrium? Why or why not?
(h) Is $(\frac{1}{2}B + \frac{1}{2}C, \frac{1}{2}A + \frac{1}{2}B)$ a Nash equilibrium? Why or why not?
(i) Is $(\frac{1}{2}B + \frac{1}{2}C, \frac{5}{8}A + \frac{3}{8}B)$ a Nash equilibrium? Why or why not?
(j) If Colin chooses the mixed strategy $\frac{1}{2}A+\frac{1}{2}B$, what is the lowest expected payoff he can obtain?
(k) If Colin chooses the mixed strategy $\frac{5}{8}A+\frac{3}{8}B$, what is the lowest expected payoff he can obtain?
(l) Based on parts (j) and (k), is $\frac{1}{2}A + \frac{1}{2}B$ a prudential strategy for Colin? Why or why not?
(m) If Rose chooses the mixed strategy $\frac{1}{2}B+\frac{1}{2}C$, what is the lowest expected payoff she can obtain?

(n) Based on parts (k) and (m), and the observation that the two players payoffs always sum to zero, explain why $\frac{1}{2}B + \frac{1}{2}C$ is a prudential strategy for Rose, and why $\frac{5}{8}A + \frac{3}{8}B$ is a prudential strategy for Colin.
(o) Is there any relationship between the prudential strategies and the Nash equilibrium strategies?
(p) How could you implement the mixed strategies $\frac{1}{2}B + \frac{1}{2}C$ and $\frac{5}{8}A + \frac{3}{8}B$?

(3) Consider the following game.

Cardinal Payoffs (Rose, Colin)		Colin A	Colin B	Colin C
Rose	A	$(50, 100)$	$(0, 30)$	$(30, 0)$
	B	$(80, 0)$	$(50, 20)$	$(20, 10)$
	C	$(0, 70)$	$(100, 40)$	$(50, 60)$

(a) If Rose chooses the mixed strategy $\frac{1}{2}B + \frac{1}{2}C$, what would Colin's best response(s) be?
(b) If Colin chooses the mixed strategy $\frac{3}{11}A + \frac{8}{11}C$, what would Rose's best response(s) be?
(c) Explain why the pair of strategies described in parts (a) and (b) form a Nash equilibrium, and how you would implement those strategies.
(d) If Rose chooses the mixed strategy $\frac{5}{11}B + \frac{6}{11}C$, what is Rose's minimum payoff?
(e) If Colin chooses the mixed strategy $\frac{3}{11}A + \frac{8}{11}C$, what is Rose's maximum payoff?
(f) Based on parts (d) and (e), explain why $\frac{3}{11}A + \frac{8}{11}C$ is a prudential strategy for Rose.
(g) If Colin chooses the mixed strategy B, what is Colin's minimum payoff?
(h) If Rose chooses the mixed strategy B, what is Colin's maximum payoff?
(i) Based on parts (g) and (h), explain why B is a prudential strategy for Colin.
(j) How would you expect Rose and Colin to play this game?

(4) For **Impromptu**, show that by using $\frac{1}{45}A + \frac{30}{45}B + \frac{14}{45}C$, Colin receives a payoff of 48 regardless of what Rose chooses to do. Further, show that if Rose chooses the mixed strategy $0.3A + 0.3B + 0.4C$, Colin will receive a payoff of 48 no matter what strategy he selects. Explain why this shows that $\frac{1}{45}A + \frac{30}{45}B + \frac{14}{45}C$ is prudential for Colin.

(5) Consider the cardinal payoff matrix obtained for **Risk-Neutral MARPS** in the dialogue, displayed again here.

Risk-Neutral MARPS Cardinal Payoffs		Colin ROCK	Colin PAPER	Colin SCISSORS
Rose	ROCK	$(0, 0)$	$(-1, 1)$	$(2, -2)$
	PAPER	$(1, -1)$	$(0, 0)$	$(-2, 2)$
	SCISSORS	$(-2, 2)$	$(2, -2)$	$(0, 0)$

(a) Explain why this assumes self-interested and risk-neutral players.
(b) Explain why (0.4ROCK + 0.4PAPER + 0.2SCISSORS, 0.4ROCK + 0.4PAPER + 0.2SCISSORS) is a Nash equilibrium.
(c) Explain why 0.4ROCK + 0.4PAPER + 0.2SCISSORS is a prudential strategy for each player.

(d) How would you play a single round of **Risk-Neutral MARPS**? Several rounds of **Risk-Neutral MARPS**?

(6) Consider the following modified cardinal payoff matrix for **MARPS**.

Risk-Varying MARPS Cardinal Payoffs		Colin ROCK	PAPER	SCISSORS
	ROCK	$(0,0)$	$(-\frac{1}{2},\frac{1}{2})$	$(2,-2)$
Rose	PAPER	$(\frac{1}{2},-\frac{1}{2})$	$(0,0)$	$(-2,2)$
	SCISSORS	$(-2,2)$	$(2,-2)$	$(0,0)$

(a) Explain why this assumes self-interested players who are risk adverse for losses (which is why people buy property insurance) and risk loving for gains (which is why people buy lottery tickets).

(b) Explain why ($\frac{4}{9}$ROCK + $\frac{4}{9}$PAPER + $\frac{1}{9}$SCISSORS, $\frac{4}{9}$ROCK + $\frac{4}{9}$PAPER + $\frac{1}{9}$SCISSORS) is a Nash equilibrium.

(c) Explain why $\frac{4}{9}$ROCK + $\frac{4}{9}$PAPER + $\frac{1}{9}$SCISSORS is a prudential strategy for each player.

(d) How would you play a single round of **Risk-Varying MARPS**? Several rounds of **Risk-Varying MARPS**?

(7) **RPS**. Consider the following modified cardinal payoff matrix for **MARPS**:

RPS Cardinal Payoffs		Colin ROCK	PAPER	SCISSORS
	ROCK	$(0,0)$	$(-1,1)$	$(1,-1)$
Rose	PAPER	$(1,-1)$	$(0,0)$	$(-1,1)$
	SCISSORS	$(-1,1)$	$(1,-1)$	$(0,0)$

(a) Explain why this assumes that the players only care about a win versus a tie versus a loss.

(b) Explain why ($\frac{1}{3}$ROCK + $\frac{1}{3}$PAPER + $\frac{1}{3}$SCISSORS, $\frac{1}{3}$ROCK + $\frac{1}{3}$PAPER + $\frac{1}{3}$SCISSORS) is a Nash equilibrium.

(c) Explain why $\frac{1}{3}$ROCK + $\frac{1}{3}$PAPER + $\frac{1}{3}$SCISSORS is a prudential strategy for each player.

(d) How would you play a single round of **RPS**? Several rounds of **RPS**?

(e) Gintis [**19**, page 67] describes the side-blotched lizard *Uta stansburiana* as having three distinct male types. The orange-throats are violently aggressive, keep large harems of females, and defend large territories. The blue-throats are less aggressive, keep small harems, and defend small territories. The yellow-stripes are very docile but they look like females, so they can infiltrate another male's territory and secretly copulate with the females. So, in pairwise competitions for passing on their genes, orange-throats beat out blue throats who beat out yellow-stripes who beat out orange-throats. Explain how this scenario can be modeled by **RPS**. What should the population distribution of male *Uta stansburiana* be?

(8) **Bacteria**. Pairs of *Simplicio illustratum* often compete for food. Three behavioral patterns have been observed: AGGRESSIVE, ASSERTIVE, and COMPLIANT. An AGGRESSIVE *illustratum* will immediately attack its competitor and will fight until it wins the food or is injured. A COMPLIANT *illustratum* will only posture in hopes that this will scare off its opponent. An ASSERTIVE *illustratum* will posture first but eventually make a brief attack

(before making a hasty retreat) in hopes of scaring off its opponent. The cardinal payoff matrix assumes that the food is worth 60, posturing costs 10, a brief attack costs 20, and a long fight costs 60. So, when two AGGRESSIVE *illustrati* compete, they have a long fight and end up splitting the food, resulting in each player receiving $\frac{1}{2}60 - 60 = -30$. When an AGGRESSIVE competes with an ASSERTIVE, there is a brief attack that the AGGRESSIVE wins resulting in $60 - 20 = 40$ for the AGGRESSIVE and $0 - 20$ for the ASSERTIVE. When an AGGRESSIVE competes with a COMPLIANT, the immediate attack results in an immediate win for the AGGRESSIVE without any substantive effort on either *illustratum*'s part, resulting in 60 for the AGGRESSIVE and 0 for the COMPLIANT. When two ASSERTIVE *illustrati* compete, they have a brief attack and then split the food, resulting in each *illustratum* receiving $\frac{1}{2}60 - 20 = 10$. When an ASSERTIVE competes with a COMPLIANT, both posture and the ASSERTIVE eventually attacks and wins the food, resulting in $60 - 10 = 50$ for the ASSERTIVE and $0 - 10 = -10$ for the COMPLIANT. Finally, when two COMPLIANT *illustrati* compete, they both posture and eventually split the food, resulting in each player receiving $\frac{1}{2}60 - 10 = 20$.

Cardinal Payoffs	AGGRESSIVE	ASSERTIVE	COMPLIANT
AGGRESSIVE	$(-30, -30)$	$(40, -20)$	$(60, 0)$
ASSERTIVE	$(-20, 40)$	$(10, 10)$	$(50, -10)$
COMPLIANT	$(0, 60)$	$(-10, 50)$	$(20, 20)$

(a) Explain why each player using

$$\frac{7}{12}\text{AGGRESSIVE} + \frac{1}{12}\text{ASSERTIVE} + \frac{4}{12}\text{COMPLIANT}$$

forms a Nash equilibrium.

(b) What does the Nash equilibrium tell us about the original scenario?

(9) Construct a new strategic game in which it would be wise for Rose to play the pure strategy $(0, 1, 0)$.

(10) In this exercise, we will prove that $((0.6 - t)A + tB + 0.4C, 0.5A + 0.5C)$ for all t between 0 and 0.3 are the Nash equilibria for **Impromptu**.

(a) What is Rose's best response if Colin chooses $0.5A + 0.5C$?

(b) What is Colin's best response if Rose chooses $(0.6 - t)A + tB + 0.4C$? Your answer will depend on whether $t < 0.3$, $t = 0.3$, or $t > 0.3$.

(c) Based on parts (a) and (b), explain why $((0.6 - t)A + tB + 0.4C, 0.5A + 0.5C)$ is a Nash equilibrium for all t between 0 and 0.3.

(d) Suppose that in a Nash equilibrium, Rose uses a pure strategy. Find a contradiction to this supposition. Thus, in a Nash equilibrium, Rose cannot use a pure strategy.

(e) Suppose that in a Nash equilibrium, Rose uses a strategy of the form $(1-p)A + pB$ where $0 < p < 1$. Find a contradiction to this supposition. Thus, in a Nash equilibrium, Rose cannot use a strategy of this form.

(f) Suppose that in a Nash equilibrium, Rose uses a strategy of the form $(1-p)B + pC$ where $0 < p < 1$. Find a contradiction to this supposition. Thus, in a Nash equilibrium, Rose cannot use a strategy of this form.

(g) Suppose that in a Nash equilibrium, Rose uses a strategy of the form $(1-p)A + pC$ where $0 < p < 1$. Prove that $p = 0.4$ and Colin is using the strategy $0.5A+0.5C$, which corresponds to the Nash equilibrium described above with $t = 0$.

(h) Suppose that in a Nash equilibrium, Rose uses a strategy of the form $(1 - p - q)A + pB + qC$, where $0 < p$, $0 < q$, and $p + q < 1$. Prove that $q = 0.4$, $0 < p < 0.3$, and Colin is using the strategy $0.5A + 0.5C$, which correspond to the Nash equilibria described above with $t = p$.