Chapter 11

# The Lanczos Method

In the early 1940s Lark-Horovitz left the Purdue physics department. Both the educational and the research conditions became less favorable for Lanczos. He was seriously thinking about looking for a position at other institutions. During 1942, Lanczos was transferred to the aeronautical section of the mechanical department at Purdue, where he made computations on problems the Defense Department was interested in. He liked the work; however, it did not bring about the changes he sought in his work.

The year 1945 brought the solution. He writes in one of his letters:

> In the meantime my unfavorable situation here at the University has changed for the better. I have been in cooperation with Boeing Aircraft in Seattle, Washington for almost two years. Our relationship has developed in such a way that the company offered me a permanent position. It is somewhat paradoxical that I with my scientific interest can always get on as an applied mathematician. This, however, does not seek to be a complaint. I have given up my position here at Purdue and next week I leave for Seattle. [**121**]

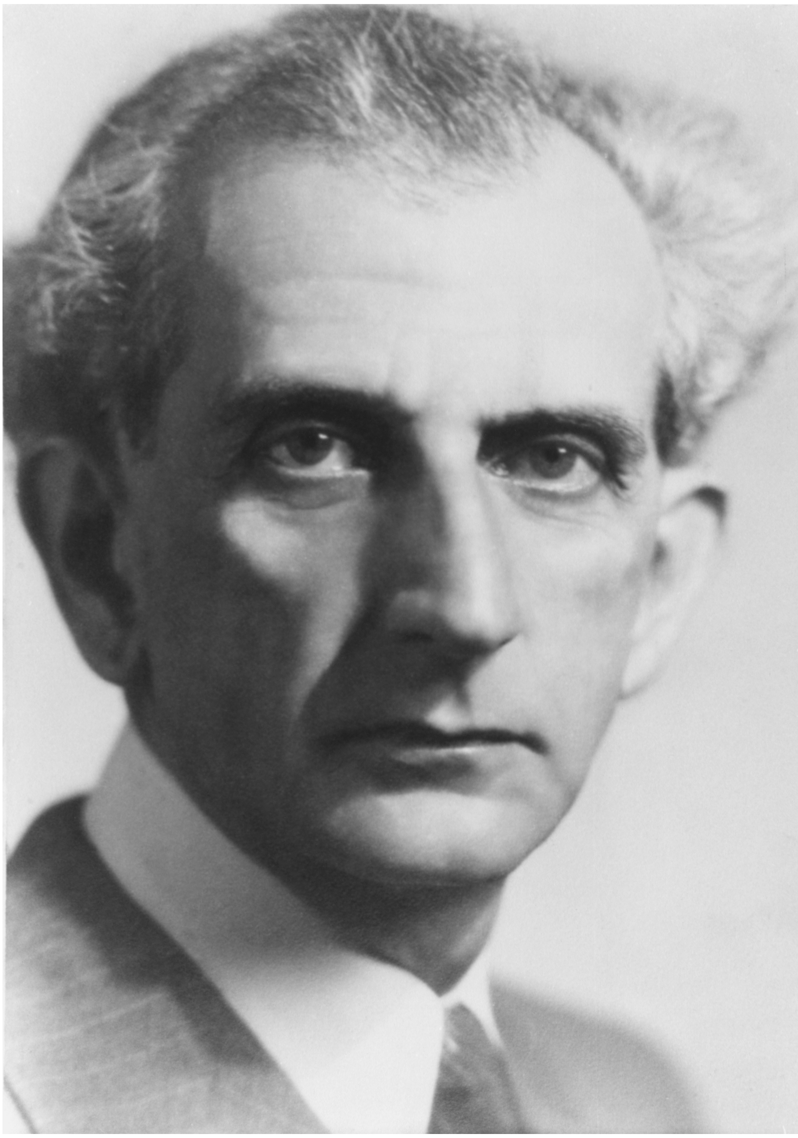

**Figure 4.** Cornelius Lanczos in 1945. The back of the original says, "To Sol and Eleanor Boyk with love Neils Lanczos." By permission of Eleanor Boyk.

## 11.1. An Applied Mathematician in Industry

The company news bureau reported about Lanczos's earlier stay at Boeing in 1944 in a special press release. It described Lanczos as "one of the nation's foremost authorities on mathematics . . . who in his office at Boeing Plant 1 . . . is busy with figures and mathematical formulae to develop shortcuts in solving problems of airplane design and manufacture" [**122**]. With regard to the need for accuracy, the report described Lanczos as "a no-slide-rule-man" because he never used one—they were not accurate enough for his work. Instead, he employed a desk calculator. His work mostly involved furthering the application of mathematics to airplane design.

Linear algebra, the system of equations and matrix theory, played an important role in the mathematical modeling Lanczos encountered at Boeing. In particular, the eigenvalue problems of matrices were profoundly important. Static phenomena such as the stability of a plane, or the problem of buckling, are equivalent to an eigenvalue problem. The eigenvalues of a certain matrix of the mathematical model of the physical problem determine the elastic vibrational frequencies, while the solutions of the system of equations form the eigenvectors or principal axes of the matrix and provide the vibration modes.

Calculating the inverse of a matrix proved to be a somewhat formidable task. To avoid matrix inversion, determining the matrix characteristic polynomial was a preferred method. The roots of this polynomial provided the eigenvalues. Special problems that Lanczos encountered included finding an arbitrary number of eigenvalues and eigenvectors without reducing the order of the matrix, preventing the accumulation of rounding errors in case of matrices with high dispersion of eigenvalues, and applying the method to solve eigenvalue problems of linear differential and integral operators.

Lanczos developed a progressive algorithm for the gradual construction of the characteristic polynomial even for cases when its relevant order is unknown (because some of the frequencies are missing). Starting from a trial vector and applying matrix transformations, Lanczos generated an iterated sequence of linearly independent vectors, each of them being a linear combination of the previous vectors. The procedure automatically comes to a halt when the proper degree of the polynomial has been reached. The coefficients of the final linear combination of the iterated vectors provide the coefficients of the characteristic polynomial. Dividing the polynomial by the root factors provides the true principal axes of the matrix.

The method, though indisputably an elegant one, had a serious limitation. In case of eigenvalues with considerable dispersion, the successive iterations will increase the gap, the large eigenvalues will monopolize the scene, and because of rounding errors the small eigenvalues begin to lose value. After a few iterations they will be practically drowned out. A certain kind of eigenvalue equality had to be established. Lanczos developed a modification of the method that protected the small eigenvalues by balancing the distribution of amplitudes in the most equitable fashion. To achieve this, the coefficients of the linear combination of the iterated vectors are established in such a way that the amplitude of the new vector should be minimal. The procedure operates in the case of $m$ eigenvalues with no more than $m + 1$ iterations (minimized iterations), preventing the accumulation of rounding errors. The generated vectors were orthogonal to each other (successive orthogonalization).

Lanczos demonstrated the power of the method by studying a vibration problem of large dispersion. For this purpose he investigated the lateral vibrational modes of an elastic bar that was clamped at one end and free at the other.

## 11.2. Walker Aimes, Lecturer

In addition to his research, Lanczos participated in education by giving a lecture course on "The Fourier Series and Its Applications", at the University of Washington in Seattle in the fall of 1947. His audience was composed of students of mathematics, physics, and engineering at both the undergraduate and graduate levels.

He remembered from his earlier lecture courses how much difficulty the problem of so-called "elementary" concepts presented to students. This meant that very often important concepts of mathematics, such as the idea of function, the meaning of a limit, uniform convergence, etc., were in fact “elementary” only because they were relegated to the undergraduate level of instruction, though their true significance cannot be properly grasped on that level. Lanczos composed the lecture material in a manner that proved to be stimulating to everybody. The subject of the lectures was developed from its early beginnings. It encouraged students to ask questions of a fundamental nature without fear of displaying ignorance of things that they should know.

The Fourier series as subject was particularly well suited to the process of developing an idea. A remarkable feature of the Fourier series was that it constantly led to a revision and a deepening of the fundamental concepts of higher mathematics. This made it possible to present the subject in close conjunction with its history, a thing Lanczos regarded as inseparable in every subject. The course was a great success. Its material was later published in the book *Discourse on Fourier Series* [**123**].

## 11.3. Learning How to Use Computers

In the meantime, the digital computer gained more ground in numerical mathematics. To speed up the process, the government of the United States decided to focus on both financial and scientific resources. For this purpose, the Institute for Numerical Analysis (INA) of the National Bureau of Standards (NBS) was established and located on the campus of the University of California in Los Angeles. Its primary function was to conduct research and training in the field of mathematics pertinent to the efficient exploitation and further development of high-speed automatic digital computing machinery. Outstanding mathematicians were recruited not only from the United States but from abroad also. The usually liberal policy of the United States Civil Service Commission on hiring noncitizens greatly contributed to the success. Among the noncitizens were John Todd, Olga Taussky-Todd, Ottó Szász, and A. M. Ostrowski, a Russian-born mathematician of the University of Basel, all of whom had experience in numerical mathematics.

The years 1947–1950 were transitional years as the state of the art moved from hand and punch card equipment to the new scientific electronic automatic computers (SEAC). These machines had to be designed and built, and the mathematicians had to learn how to use them.

> It was necessary to develop a philosophy for the solution of massive problems. It was, roughly speaking, that of a controlled computational experiment—that is, to compare the theoretical results of academic problems with the experimental ones obtained by the use of appropriate algorithms on academic problems. [**124**]

John Todd became the chief of the Computational Laboratory and Olga Taussky-Todd became a mathematical consultant in the Division. They had an active interest in the development of the INA.

At the invitation of J. H. Curtiss, Chief of the NBS at that time, Lanczos joined the staff of the INA. The invitation could not have come at a better time. Boeing had provided Lanczos with practical problems in the airline industry. While working at Boeing, he defined the mathematical problem, constructed the necessary algorithm for its solution, and then developed and generalized the mathematical method. Completion of the latter task required academic circumstances with appropriate computational apparatus. All of these were waiting for him in Los Angeles. In January 1949 he joined the staff of the INA. He liked the organization of the work at the institution. Free from teaching duties, he could devote his time entirely to developing an effective computational application of his method. The result was published in a seminal paper [**125**].

While Lanczos was working on his paper, A. M. Ostrowski, who also worked for some time at the INA, pointed out to Lanczos that his eigenvalue method paralleled the earlier work of some Russian scientists. A. N. Krylov's paper of 1931 was entitled "On the numerical solution of the equation by which, in technical problems, the frequencies of small oscillations of material systems are determined". Lanczos checked the relevant reviews in the reference journal Zentralblatt and informed Ostrowski that the literature available to him showed no evidence that the eigenvalue method and the results he obtained have been found earlier. In footnote 4 of his seminal paper [**126**] Lanczos says, "On the basis of the reviews of these papers in the Zentralblatt the author believes that the two methods [Lanczos's and Krylov's] coincide only in the point of departure. The author has not, however, read these Russian papers." (Lanczos did not know Russian.) Using matrix transformation, Krylov created a sequence of consecutive vectors that had the smallest set of consecutive iterates that are linearly dependent. The coefficients of a vanishing combination are the coefficients of a divisor of the characteristic polynomial of the matrix. The space these vectors determine is called the Krylov subspace [**127**].

The Krylov method proved to be quite general; it has found its real applicability through its various implementations. Alston S. Householder discussed the original method of Krylov and several other methods that involve the formation, at least implicitly, of a Krylov sequence [**128**].

## 11.4. Exceptional History of an Exceptional Method

Krylov's iterative solver generated a huge class of approximate methods, among which the Lanczos Method was one of the most widely applied. The algorithm generates a set of orthogonal vectors which satisfy a recurrence relation. It connects three consecutive vectors with the result that each newly generated vector is orthogonal to all of the previous ones. The numerical constants of the relation are determined during the process from the condition that the length of each newly generated vector should be minimal. After a minimal number of iterations (minimized iterations) in view of the Hamilton-Cayley identity, the last vector must become a linear combination of the previous vectors.

When Lanczos discovered his method, its every deficiency could not be discovered. Digital computers were just coming on the scene. Testing the method for large-scale linear systems still had to be waited for. Though the order of the matrices Lanczos used was no more than 20, the method works for big matrices, provided there is a fast enough computer to use. His seminal paper was not easy ro read even for professionals because of the scientific idioms Lanczos used. In the last interview of his life speaking on his scientific career, Lanczos said:

> Recently I read in a paper that mathematicians call me not a mathematician but a "mathemagician" because I express myself not in the idioms they regard to be the proper ones nowadays. However, though my idioms are unusual, what I do is correct and applicable. I am convinced that what I write is just as correct and rigorous as if I had expressed them in the usual idioms. [**130**] (The paper referred to is [**129**].)

After its discovery, Lanczos's method was forgotten for two decades before it captured the attention of scientists, but it has kept their interest since then. Those who used the method felt free to improve, adjust, correct, or modify it. Lanczos's successors produced effective numerical methods from eigenvalue computation to matrix tridiagonalization for large-scale matrices on high-speed digital computers [**131**]. The Lanczos Method today is one of the most frequently used numerical methods in matrix computations and in its present form is the common product of an international community of applied mathematicians.

In 2000, the editors of the journal *Computing in Science and Engineering*, while putting together the January/February 2000 issue, composed a list of 10 algorithms that exerted the greatest influence on the development and practice of science and engineering in the 20th century. For each algorithm a person or a group received credit for inventing or discovering the method [**132**]. One of the "Top 10 Algorithms" is the "Krylov Subspace Iteration". The list includes the most widely employed implementations of the Krylov iteration solver, among them the "Minimized Iteration Method for the Solution of Systems of Linear Equations". Cornelius Lanczos received credit for its discovery [**133**]. He was nominated and accepted into the NIST portrait gallery for the paper. The nomination committee observed that the result is undoubtedly the single most influential result of original research in mathematics in the history of the NBS/NIST and surely ranks among the top NBS contributions in any field of science.

## 11.5. Accused of Disloyalty

While working at the INA, Lanczos entered into correspondence with Erwin Schrödinger, who was the director of the Dublin Institute for Advanced Studies (DIAS), Ireland. Schrödinger was interested in working on unified field theory and Lanczos's paper "Matter Waves and Electricity" captured his attention. He offered Lanczos a visiting professorship of one year at the DIAS. Lanczos accepted and occupied the position in October of 1952.

While he was in Dublin, Senator Joseph McCarthy's anti-Communist campaign, which gave the period from 1950–1954 the name McCarthyism, had an extremely negative effect upon the INA. Even though the inquiries turned up nothing, and all staff were cleared, the institution could not avoid the security investigations and their corrosive effect on morale. The continued harassment of Edward U. Condon, the Director of the NBS, contributed to his decision to resign. Lanczos himself was accused of disloyalty, though the details were unknown to him. He went back to the United States to find out the particulars of the accusations.

For half a year up to the end of 1953, he was a specialist in computing with North American Aviation in Los Angeles. At that time the FBI officially contacted him. The details of the visit can be found in one of Lanczos's letters:

> Yesterday two FBI agents visited me—for a sort of interrogation. It is strange, after all, that because one formerly belonged to the Wallace party and made some pacifist and (previously regarded to be) Russian-sympathizing statements, one remains a somewhat suspicious character until the end of one's life. Otherwise the people were very friendly, and I have no objection against the organization. However, this "file" can be terribly abused. [**134**]

Later in the same letter Lanczos bitterly remarked, "Though my work was highly esteemed at North American Aviation, they did not regard me worthy of a permanent position." As with his previous workplace, the NBS Operation was greatly reduced, and the INA in

Los Angeles was closed. It was obvious to Lanczos that he should leave the United States.

## 11.6. Back to Europe

In May 1954, at the invitation of Eamon de Valera, Prime Minister of the Republic of Ireland, Lanczos accepted the post of senior professor in the School of Theoretical Physics of the DIAS. Though he was pleased with the development, his future post worried him. During the previous 10–15 years he had dealt almost exclusively with numerical mathematics and felt that he was far away from physics. Nonetheless, he stated in his letter, "I still have the feeling that everything will turn out well in Ireland, and I am very thankful to my destiny for this new development" [**135**].

Before Lanczos occupied his position, his private life also underwent a change. His son Elmar, who was close to 20, decided to remain in the United States and wanted to live in Seattle. Lanczos agreed to it. He himself married Ilse Hildebrand, a highly educated German lady whom he knew from his years in Germany. Having the wedding abroad, Lanczos and his wife arrived in Dublin in May 1954, where Lanczos occupied his position as senior professor of the School of Theoretical Physics of the DIAS.

**Figure 5.** Ilse Lanczos, Cornelius Lanczos's second wife, December 1954. Courtesy of Ilse Camacho, Cornelius Lanczos's step-daughter.