## Contents

Foreword ..... xi
Introduction ..... xiii

1. Top-Tier Math Circles ..... xiii
2. Why, What, and for Whom? ..... xvi
3. Notation and Technicalities ..... XX
4. The Art of Being a Mathematician and Problem Solving ..... xxii
5. Acknowledgments ..... xxiii
Session 1. Geometric Re-Constructions. Part I ..... 1
6. Experimenting and Conjecturing ..... 1
7. A Triangle Workout ..... 5
8. Walking Along an Optimal Path ..... 10
9. Walking Along an Integer Grid ..... 14
10. To Prove or to Take for Granted? ..... 16
11. Hints and Solutions to Selected Problems ..... 20
Session 2. Rubik's Cube. Part II ..... 23
12. What Is a Group? ..... 23
13. Permutation Groups and Group Isomorphisms ..... 27
14. Properties of Groups and Their Subgroups ..... 31
15. Even and Odd Worlds ..... 35
16. How Many Cube Positions Can Be Reached? ..... 39
17. Conclusions ..... 45
18. Hints and Solutions to Selected Problems ..... 46
Session 3. Knotty Mathematics ..... 49
19. A Knot, or Not a Knot. That Is the Question. ..... 49
20. Reidemeister and Knot-Eating Machines ..... 53
21. Three Crayons Defeat an Army of Knots ..... 56
22. The Jones Polynomial ..... 64
23. Is This the End? ..... 70
24. Hints and Solutions to Selected Exercises ..... 70
Session 4. Multiplicative Functions. Part I ..... 79
25. Infinite Raffle: the Initial Setup ..... 79
26. What are Multiplicative Functions? ..... 82
27. Sum-Functions ..... 92
28. Hints and Solutions to Selected Problems ..... 96
Session 5. Introduction to Group Theory ..... 103
29. Puzzling It Out ..... 103
30. A Polynomial Prelude ..... 104
31. Action Groups ..... 105
32. General Groups ..... 110
33. Some More Examples of Groups ..... 112
34. Permutation (or Symmetric) Groups ..... 116
35. The 15-Puzzle Puzzled Out ..... 123
36. Hints and Solutions to Selected Problems ..... 126
Session 6. Monovariants. Part II ..... 141
37. Numerical Monovariants ..... 141
38. Constructive Activities ..... 149
39. Not Getting There ..... 153
40. Conway's Checkers ..... 158
41. Hints and Solutions to Selected Problems ..... 164
Session 7. Geometric Re-Constructions. Part II ..... 171
42. Optimal and Infinite Challenges ..... 171
43. A Pythagorean Path for the Intermediate ..... 173
44. Physics and Math Combine Forces ..... 176
45. Ptolemy's Lead into Trigonometry ..... 178
46. Hints and Solutions to Selected Problems ..... 185
Session 8. Complex Numbers. Part II ..... 189
47. Warning, "Teaser," and Strategy ..... 189
48. Conventions from the Past ..... 190
49. Complex Division ..... 190
50. The Triangle Inequality: No "Respect" for Addition? ..... 192
51. Integer Powers in $\mathbb{C}$ ..... 193
52. Roots in $\mathbb{C}$ ..... 196
53. Roots of Unity and Regular Polygons ..... 198
54. Geometric Promise Fulfilled ..... 200
55. Venturing Everywhere in the Plane ..... 202
56. Which are the "Closest" Lines ..... 205
57. Hints and Solutions to Selected Problems ..... 208
Session 9. Introduction to Inequalities. Part I ..... 211
58. The Language of Inequalities ..... 211
59. Arithmetic Mean - Geometric Mean Inequality ..... 212
60. Power Mean Inequality ..... 216
61. The Land of the Convex ..... 218
62. Applications of Convexity to Inequalities ..... 220
63. Geometry Leftovers and a Mean Summary ..... 223
64. Hints and Solutions to Selected Problems ..... 225
Session 10. Multiplicative Functions. Part II ..... 233
65. Dirichlet Product ..... 233
66. Möbius Inversion Formula ..... 237
67. The Euler Function $\boldsymbol{\phi}(\boldsymbol{n})$ ..... 243
68. The Taming of the ShrewD $\phi$ ..... 247
69. Hints and Solutions to Selected Problems ..... 252
Session 11. Monovariants. Part III ..... 263
70. The Balkan Roots Challenge ..... 263
71. Smoothing and Unsmoothing ..... 264
72. Rearranging Terms ..... 266
73. Convexity and Smoothing ..... 268
74. Random Fun with Smoothing ..... 275
75. Appendix on Limits and Endless Smoothing ..... 278
76. Hints and Solutions to Selected Problems ..... 281
Session 12. Geometric Re-Constructions. Part III ..... 287
77. Farmer-and-Cow via Inequalities and Calculus ..... 287
78. Optimal Bridge Located! ..... 292
79. Infinitely Many Angles and Infinite Series ..... 296
80. Historical Detour: from Today back to Archimedes? ..... 301
81. Hints and Solutions to Selected Problems ..... 302
Epilogue ..... 305
82. What Comes from Within ..... 305
83. The Culture of Circles ..... 306
84. Eastern European vs. USA Math Circles ..... 307
85. History and Power ..... 310
86. Does the U.S. Need Top-Tier Math Circles? ..... 314
Symbols and Notation ..... 321
Abbreviations ..... 325
Biographical Data ..... 327
Bibliography ..... 331
Credits ..... 335
Index ..... 337
