Introduction

One of the most beautiful mathematical topics I encountered as a student was functional equations: the study of functions which satisfy given equations, such as

$$f(x + y) = f(x) + f(y).$$

Functional equations arise in all areas of mathematics and even more so in science, engineering, and the social sciences. They appear at all levels of mathematics: from the elementary definition of an even function as one that satisfies the functional equation

$$f(x) = f(-x),$$

to the forefront of research. For example, during the last quarter century, the celebrated Yang–Baxter functional equation has been at the heart of many different areas of mathematics and theoretical physics, such as lattice integrable systems, factorized scattering in quantum field theory, braid and knot theory, and quantum groups, to name a few. The Yang–Baxter equation is a system of N^6 functional equations for the $(N^2 \times N^2)$-matrix $\boldsymbol{R}(x)$ whose entries are functions of x:

$$\sum_{\alpha,\beta,\gamma=1}^{N} \boldsymbol{R}_{jk}^{\alpha\beta}(x-y)\, \boldsymbol{R}_{i\alpha}^{\ell\gamma}(x)\, \boldsymbol{R}_{\gamma\beta}^{mn}(y) = \sum_{\alpha,\beta,\gamma=1}^{N} \boldsymbol{R}_{ij}^{\alpha\beta}(y)\, \boldsymbol{R}_{\beta k}^{\gamma n}(x)\, \boldsymbol{R}_{\alpha\gamma}^{\ell m}(x-y),$$

where i, j, k, ℓ, m, n take the values $1, 2, \ldots, N$. Amazingly, although this equation appears to impose more constraints (N^6) than the number of unknowns (N^4), it has a rich set of solutions. The study of the Yang–Baxter equation is well beyond our scope; however, the reader will hopefully come to appreciate the role that functional equations play in mathematics and science over the course of reading this book.

Despite being such a rich subject, this topic does not fit perfectly into any of the conventional areas of mathematics and thus a systematic way of studying functional equations is not found in the traditional curriculum. Several good books on the topic exist, but are either relatively hard to find [40] or quite advanced [8, 9, 28]. While this book was in the making, a nice, simple book [39] appeared, which has considerable overlap with, and is complementary to, this book.

We should point out in advance that there is no universal technique for
solving functional equations, as will become obvious once the reader solves the
problems throughout this book. At the current time, thorough knowledge of
many areas of mathematics, lots of imagination, and, perhaps, a bit of luck
are the best methods for solving such equations.

The core of the book is the result of a series of lectures I presented to
the Putnam team at University of Central Florida soon after my arrival there.
My personal belief is that the training of a math team should be based on a
systematic approach for each subject of interest for the mathematical compe-
titions (e.g., functional equations, inequalities, finite and infinite sums, etc.)
instead of randomly solving hard problems to get experience. Therefore, this
book is an attempt to present the fundamentals of the subject in a pedagog-
ical manner, accessible to students who have a background on the theory of
functions up to differentiability. Some parts of the book use additional ideas
from calculus, but they may be omitted on a first reading.

In particular, the book should be useful to high school students who par-
ticipate in math competitions and have an interest in International Math
Olympiads (IMO). Many of the problems I have included I solved as a high
school student[1] in preparation for the national math exam of my home coun-
try, Greece. I still enjoy them as much as I enjoyed them then. In fact, I
now appreciate their beauty much more, as I have grown to have a better
understanding of the subject and its importance.

College students with an interest in the Putnam Competition should also
find the book useful, as standard texts do not provide a thorough coverage
of functional equations. Finally, we hope that any person with interest in
mathematics will find in this book something to his or her interest.

However, a piece of advice is in order here for all readers: The majority of
the problems discussed in this book are related to mathematical competitions
which target ingenuity and insight, so they are not easy. Approach them with
caution. They should serve as a source of enjoyment, not despair!

And a final comment: The interested reader should, perhaps, read the
book simultaneously with Smítal's [40] and Small's [39] books, which are at the
same level. I cover more functional equations but Smítal presents beautifully
the topic of iterations and functional equations of one variable.[2] Similarly,
Small's book [39] is a very enjoyable, well-written book and focuses on the
most essential aspects of functional equations. Once the reader is done with
these three books, he may read Aczél's and Kuczma's authoritative books

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[1]Yes, I still have many of my notes! As a result, some of the problems I present might be
taken from national or local competitions without reference. I would appreciate a note from
readers who discover such omissions or other typos and mistakes.

[2]One can detect some influence from Smítal's book on my presentation in Chapters 16
and 17. I highly recommended this book to any serious student.

[8, 9, 28], which contain a vast amount of information. Beyond that level, the reader will be ready to immerse himself in the current literature on the subject, and perhaps even conduct his own research in the area.

More on How to Approach This Book

Mathematics has a reputation of being a very stiff subject that only follows a pre-determined pattern: definitions, axioms, theorems, proofs. Any deviations from this pre-established pattern are neither welcome nor wise. In this respect, mathematics appears to be very different from the sciences and even from its closest neighbor, theoretical physics. For mathematics uses **demonstrative reasoning** and the sciences use **plausible reasoning**.[3]

As mentioned previously, the majority of the problems in this book are taken from mathematical competitions. As such, they are precious not only for their originality and beauty, but also for the lessons they teach the students regarding mathematical discovery. Without a doubt, the final solutions of these problems must be presented as demonstrative reasoning. However, to reach this stage, a student must first go through the stage of plausible reasoning. Solving a problem first involves guessing the solution. Proving the solution first involves visualization of the proof. Polishing the proof requires trying these steps over and over. Therefore demonstrative reasoning and plausible reasoning are not two distinct, isolated methods; they are just the two faces of the process of discovering new results in mathematics. So, in agreement with Pólya's motto, the problems in this book should be used to "let us learn proving, but also *let us learn guessing.*"

Unfortunately, due to lack of time and space, I have omitted the plausible reasoning which can be experienced by attending Math Circles, and I have presented only the demonstrative reasoning in what I believe to be a polished way. However, students should be assured that the solutions presented could not have been found without hunches and guesses of some kind. The beginning student should thus be neither discouraged nor disappointed if his reasoning, either at the plausible or at the demonstrative stage, fails to be as good as those of experienced solvers. Paul Erdős, one of the most prolific publishers of papers in mathematical history and an extraordinary problem solver, has said: "Nobody blames a mathematician if the first proof of a new theorem is clumsy." Therefore, solve these problems as many times as necessary to improve your solutions. With every new solution you discover, you gain invaluable knowledge that becomes your new weapon to attack new problems. This is not really a new idea; it has been quite familiar to anyone who has tried problem solving. Here is René Descartes' testimony: "Each problem that

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[3]The terms are discussed in G. Pólya's classic two-volume work [32] which teaches students the role of guessing in rigorous mathematics and how to become an effective guesser.

I solved became a rule which served afterwards to solve other problems." Any reader who places considerable amount of effort on the problems will benefit from them, even if he does not completely solve all the problems.

Of course, all this assumes that you see beauty in these problems. I cannot define mathematical beauty, but I know it when I see it. It is really love at first sight. Upon completion of the first draft of the manuscript, I showed it to some good students who had not attended my lectures. Given their high academic performance in the standard curriculum, I expected them to make favorable comments about the choice of problems, but there was one student who declared them to be trick problems! He appeared unimpressed and, I think, completely uninterested in them. I had always assumed, even after many years of teaching, that if a student stands well above the crowd in mathematical ability, he will also see well beyond the crowd. However, this incident made me realize that even some students with mathematical talent, when they have been educated in a dry and boring traditional system, often by unmotivated teachers, have never developed mathematical aesthetics and a deep understanding of how the process of mathematical discovery really happens. As a skillful but blind diamond cutter, looking at the diamond, never sees its sparkle and so cannot appreciate the fine cuts, so a student who has never been shown true mathematics but is otherwise a skillful solver may not appreciate the extraordinary beauty of math problems.

Therefore, it is my obligation as I present this book to students to declare: Trick mathematics is the norm. Straightforward proofs are the exceptions. There are patterns, but these are the products of experience and hard work of the giants before us. To unveil a pattern, you must first produce many trick proofs. The trickier the proof, the greater the hidden beauty. Such is the case, for example, with Sharkovskii's theorem, stated in Chapter 16—a fine jewel of modern mathematics.

Appreciating and solving the problems given in mathematical competitions builds strong intuition, develops mathematical ingenuity, and promotes deep understanding for trick proofs. It leads the way to creating new generations of outstanding mathematicians. Dear reader, please approach this book with this spirit in mind.

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July 2010
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