Preface

The goals of this text are threefold.

- *Content*: *Euclidean Geometry.* Give students a working knowledge of the basic content of Euclidean geometry that they can use and apply.
- *Structure*: *Axiomatic Development.* Present geometry through an axiomatic development that begins with a small set of intuitive axioms from which the entire subject is derived.
- *Pedagogy*: *Guided Inquiry.* Guide students to solve problems and prove theorems on their own while the instructor serves as a mentor, thereby empowering them with the ability to draw logical conclusions from given information.

Existing geometry texts appear to either present the required content at the cost of a focus on reasoning, or to offer an axiomatic development at the cost of essential content. Those that present geometry through guided inquiry are decidedly rare. This book is, to the best of our knowledge, the only available introductory geometry text that gives an axiomatic development of the required content in a guided inquiry format. In this preface we will justify each of these goals, outline the impediments to achieving all three simultaneously.

The basic topics of *Euclidean geometry* will be required by any student pursuing further study of mathematics or science. Geometry is particularly important for those preparing for K-12 teaching. The content presented here follows the guidelines given by Hung-Hsi Wu of U. C. Berkeley in "The Mis-Education of Mathematics Teachers" [**17**]. Wu's thesis is that, before we do anything else, our primary obligation to pre-service teachers is to give them a sound knowledge of the topics they will actually need to teach.

In order to justify the need for an *axiomatic development*, we must ask ourselves why it is that, for so many centuries, Euclidean geometry has been viewed as an essential component of a sound education. Not just for aspiring mathematicians, scientists and engineers, but for everyone. To answer this question we need to go back in time and review the history of geometry.

Geometry as we now understand it was first organized into a systematic whole by Euclid, the great scholar of Alexandria, who wrote his *Elements of Geometry*[**6**] around the year 300 B.C. Euclid drew on the work of ancient Babylon, Egypt, India and perhaps elsewhere dating back as much as 3000 years earlier. During this intervening period geometry had been viewed as an empirical science: the science of earth measure. This meant that the facts of geometry were established by conducting experiments and taking measurements of the results, as we still do in the sciences today. Facts established in this way were and remain vulnerable to two forces of uncertainty. One is that experiments can only be done on a few samples. If the samples are not truly representative of the whole, the conclusions can be erroneous. The other is that every measurement is at best an approximation. The result was that geometry prior to Euclid consisted of a grab bag of rules and clever tricks that had been gathered over centuries, some of which were correct and some of which were at best reasonable approximations. (See Problem 161.)

The significance of Euclid's *Elements* lies not only in the practical value of the geometric facts it provides, but more importantly in the method it uses to establish those facts. Euclid presented a small list of facts, called *axioms*, which he asked his readers to experimentally verify as thoroughly as they were able. By logical reasoning, he then argued that an amazingly complex collection of other facts could be deduced from those axioms alone. The advantage of Euclid's method was that, provided the axioms were true, we could have full confidence in conclusions drawn from them by correct logical arguments.

This method of gaining reliable knowledge has had an enormous influence on the development of Western thought. It was reflected and reinforced in 1687 when Isaac Newton published his *Principia*. In this three volume work Newton laid out his laws of physics; a small set of axioms from which he deduced the entire theory of classical mechanics while inventing the calculus along the way. In a quite different domain, Thomas Jefferson justified the Declaration of Independence by carefully laying out his assumptions and then deducing from them their inevitable consequence.

> *"We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain unalienable Rights,. . . "*

Because of its historically demonstrated power to establish truth, the method of Euclid was a central pillar of our educational system over two millennia. According to Howard Eves [**7**], “No work, except The Bible, has been more widely used....” Euclidean geometry offered a prototype of an exact system of assumptions and deductions that served as a model for less exacting areas of study. In 1911 David Smith said that we teach geometry to share the joy of its unique access to ultimate truth [**15**]. While our knowledge of other subjects is necessarily inexact, he observed that

> *“before the world was fashioned the square of the hypotenuse was equal to the sum of the squares of the two sides of a right triangle, and it will be so after it is dead and the inhabitant of Mars, if he exists, probably knows its truth as we know it.”*

The logical development of Euclidean geometry led students to understand how conclusions follow from given information, a skill they would need in all walks of life. It fostered the growth of critical thinking, certainly an attribute required of the citizens of any functioning democracy.

But a logical development of geometry itself can be a boring exercise of questionable benefit to students unless they are actively engaged in it themselves. The call for active learning in some kind of guided inquiry format goes well beyond geometry, and even mathematics. By “guided inquiry learning” we refer to any methodology that replaces traditional lectures and textbooks with some form of student centered activities. Robert L. Moore founded the school of “Inquiry-Based Learning” (IBL) in mathematics, well documented in his biography [**13**], that forms the basis for the present text. Nobel laureate Carl Wieman has spent years developing a form of guided inquiry teaching of large sections of physics [**5**]. David Hanson founded the NSF funded and widely used program POGIL (“Process Oriented Guided Inquiry Learning”) for teaching chemistry with students working in small groups. Paulo Freire’s 1968 book *Pedagogy of the Oppressed* [**8**] has led to the growing movement toward “Critical Pedagogy” in sociology and political science.

We are now beginning to see statistically significant controlled studies that demonstrate the effectiveness of these methods. A report in *Science* [**5**] documents a study led by Carl Wieman of learning outcomes in two very large first year physics classes, one taught “by 3 hours of traditional lecture given by an experienced highly rated instructor” and the other “by a trained but inexperienced instructor” using guided inquiry. They “found increased student attendance, higher engagement, and more than twice the learning" in the guided inquiry section. Sandra Laursen led a three year study of the efficacy of guided inquiry learning in mathematics as her group witnessed it practiced by the Universities of California at Santa Barbara, Chicago, Michigan and Texas[**11**]. They concluded, "The approaches implemented at the IBL Mathematics Centers benefited students in multiple, profound, and perhaps lasting ways. Learning gains and attitudinal changes were especially positive for groups that are often under-served by traditional lecture-based approaches, including women and lower-achieving students." These findings appears to support Paulo Freire's thesis in "Pedagogy of the Oppressed".

In geometry "guided inquiry" means that instructors serve as mentors, presenting problems and theorems to the students, listening to their ideas, reading their work, and generally giving them just enough information to allow them to learn by solving problems and proving theorems on their own. What is needed is a carefully crafted sequence of problems to solve and theorems to prove that will lead them through the core ideas. The steps in the sequence need to be big enough to offer a challenge but small enough to be done by the students. In the last section of this introduction we give details as to how this book fulfills that need. For an excellent account of the use of guided inquiry learning in mathematics see the text of Coppin, Mahavier, May and Parker [**4**].

For these reasons we conclude that an *axiomatic development* of geometry from a small set of assumptions is an essential component of a beginning geometry text. Because we want to prepare students to participate in this logical process, we want to see it taught through some form of *guided inquiry.* And of course we must provide the basic content of *Euclidean geometry* needed by those who will later use or teach it.

These justifications of our three goals seems clear enough. So why are we unable to find a single modern beginning geometry text that meets these goals? We will describe three pedagogical dilemmas that appear to have confounded the teaching of geometry in recent years. But first we need to complete our review of the history of geometry.

Hilbert's Geometry

As mathematics began to mature after the Renaissance, mathematicians looked more carefully at Euclid's development and discovered that it failed to meet his stated objectives in three ways - each of which required fundamentally new advances in mathematics in order to rectify.

First, the proofs of his theorems contained hidden assumptions that were not explicitly stated in his axioms. For centuries mathematicians attempted to resolve this problem by filling in the missing assumptions. This proved to be a devilishly difficult task, as it was unclear exactly what constituted a correct mathematical proof or what we know about physical space that we can use to create a theory describing it.

Secondly, there was general agreement that Euclid's axioms themselves were assertions that we know to be true from experimental evidence – with one exception. This was Euclid's Parallel Postulate (our Axiom 5): *For every line ℓ and every point P not on ℓ, there is at most one line containing P that is parallel to ℓ.* Euclid himself was very conservative in the use of this axiom, which he avoided as long as possible. For centuries mathematicians attempted to justify the Parallel Postulate by proving it from Euclid's other axioms, but these efforts remained fruitless.

Finally "geometry", from "earth measure" in Greek, was intended to refer to a system of numerical measurement of physical objects. Yet the early Greeks had access to only rational numbers, and had no effective means to describe the area or volume of an irregular shape. They knew from Pythagoras's theorem that there were many lengths that did not have a rational length. This fact led Euclid to favor an axiomatic formulation of geometry over a numerical formulation. The later books of the *Elements* illustrate Euclid's awkward struggle with segments whose lengths were not in a rational ratio.

The late nineteenth century saw a number of fundamental advances in mathematics that led to a resolution of these problems. Boole, Frege and others began to clarify and formalize logic and set theory, which underly all of mathematics and help to delineate what constitutes a mathematical proof. Beltrami, Klein and Poincaré finally demonstrated that the desired proof of the Parallel Postulate was impossible. They did this by showing that any model of Euclidean geometry could be used to construct a model of *non-Euclidean geometry*, that is, a model satisfying all of Euclid's axioms *except* the parallel axiom. Subsequently Dedekind (1872) discovered the real number continuum, allowing him to assign a positive length to every geometric segment. A geometric line could then be viewed as a real number line, and the resulting Cartesian coordinate plane was now indeed a model of Euclid's axioms. This model of Euclidean geometry gave rise, via the work of Beltrami, Klein and Poincaré, to models of non-Euclidean geometry as well. Both were revealed to be fully valid mathematical systems. Altogether these discoveries showed that the truth or falsity of the Parallel Postulate is not a mathematical question at all; it is an empirical question about the nature of physical space. Working in a different direction Riemann, Jordan and Lebegue developed a theory of measure that made it possible to specify which regions could be assigned an area or a volume and to describe methods to compute them.

Drawing on this new foundational understanding of mathematics, the German mathematician David Hilbert fully resolved the problems of Euclid's system in his 1899 *Grundlagen der Geometrie*[**10**]. In this work he provided a new system of axioms for Euclidean geometry and demonstrated that all of Euclid's theorems logically follow from his system. (See Appendix C.) From these theorems also followed a full theory of measurement of lengths, areas and angles. A version of these axioms is listed at the end of this introduction. They include the parallel axiom, upon which much of Euclidean geometry depends. From Hilbert's now standard view of geometry, "point", "line", "between" and "congruent" are taken to be primitive (undefined) terms. They only acquire a meaning in a particular model of geometry, such as physical space or the Cartesian coordinate plane.

The Demise of Euclidean Geometry

Over the past 40 years three pedagogical dilemmas have conspired to phase Euclidean geometry, as an axiomatic study, out of our national curriculum. Two of these dilemmas are serious concerns that are within our grasp to resolve. The third is somewhat speculative, but has recently been at last rectified.

The first of these dilemmas is a direct outcome of the mathematical advancements we have discussed. Hilbert's 1899 work was a monumental contribution to twentieth century mathematics. Resolving the problems inherited from Euclid not only put geometry solidly on a foundation of set theory and logic; it also supplied a blueprint for similar axiomatic developments in other fields of mathematics as well. But the question as to how Hilbert's new insights were to be incorporated into mathematics curricula became a perplexing dilemma. A rigorous development of axiomatic geometry from Hilbert's axioms is a lengthy and sophisticated process requiring considerable time and a serious commitment to abstract mathematics. The responses to this fact were radically different in high schools and in universities.

It took considerable time after Hilbert for this dilemma to be fully recognized at the level of high school mathematics. In the preface of the 1960 geometry text of Brumfiel, Eicholz and Shanks [**1**], the authors refer to Hilbert's rectification of the omissions in Euclid:

> *"Means to remedy these gaps have been known for about sixty years, but strangely enough a mathematically adequate and yet elementary treatment of plane geometry in the spirit of Euclid has not appeared in print. This text represents an earnest effort to do just this."*

Subsequent experience showed that this text erred on the side of "mathematically adequate" rather than "elementary", and as a result found a very limited following at the high school level. This left secondary schools in a serious bind. They could neither teach geometry from the much too advanced standpoint of Hilbert, nor were they willing to continue offering courses based on the now publicly discredited work of Euclid. In the seventies this dilemma was finally resolved by shifting to the analytic geometry of the coordinate plane. Geometry was integrated with algebra, and focused on the practical geometric tools required in the sciences. Virtually nothing remained of the focus on logical reasoning that had previously been provided, with admitted flaws, by Euclid. This meant that the primary rationale for teaching geometry at the high school level was effectively lost.

Most colleges and universities have continued to teach geometry as an axiomatic system. The 1913 edition of the widely used *Plane and Solid Geometry* by G. Wentworth and D.E. Smith [**16**] still adhered to the gentler approach of Euclid and made no attempt to incorporate Hilbert's new approach. The first of many fine expositions of Hilbert's system was George Halsted's 1904 text *Rational Geometry: A Text-Book for the Science of Space* [**9**]. These texts have largely replaced Euclid in university courses. But from a pedagogical standpoint Hilbert's system is a mixed blessing. It finally did provide students of mathematics with a long sought after axiomatic development of geometry that was complete and correct. However that development came at the cost of covering very little of the normal content of plane geometry in a single course. A readily available example is "Hilbert Geometry"[**2**], from which the author taught for many years before writing the present text. After extended foundational work on betweenness and intersections, students only have time left to do essentially the content of our current pages 13–16 in Chapter 2.

The impact of these events on secondary schools and universities are intimately intertwined, as pre-service secondary school teachers form a primary audience for university courses in geometry. These two opposing alternatives to Euclid's geometry have been particularly disadvantageous for pre-service teachers. The geometry they now learn at the university has almost nothing to do with the geometry they will teach themselves. Wu [**17**] likens this situation to one in which we prepare pre-service French teachers by teaching them Latin at the university and then asking them to make the necessary adjustments when their own students arrive to learn French.

Growing pressure for accountability in our schools through standardized testing, forcefully implemented by the No Child Left Behind Act of 2001, has subsequently resulted in a second pedagogical dilemma. Educators required to show quantitative evidence of student achievement are naturally pressed to teach material that can readily provide that evidence. Unfortunately high level reasoning skills and intellectual creativity do not fit this requirement, whereas large quantities of factual content and memorized proofs do. Modern authors fulfill this demand with texts ranging from 600 to 800 pages while educational research repeatedly bemoans a growing concern about how little students have to show for their efforts. This trend has only reinforced the abandonment of any logical reasoning in our geometry courses. Other subject areas have suffered similar effects when over simplified presentations have distorted the essential content of the ideas. See, for example, James Loewen's analysis of the teaching of history in *Lies My Teacher Told Me* [**12**].

A third pedagogical dilemma arose from an unexpected corner. Recall that the discoveries of non-Euclidean geometries by Beltrami, Klein and Poincaré demonstrated that there is no proof of the Parallel Postulate from Euclid's other axioms. We obtain a perfectly good mathematical theory if we replace this Postulate with either the Hyperbolic Parallel Postulate (*... there are at least two lines containing P that are parallel to ℓ.*) or the Elliptic Parallel Postulate (*... there is no line containing P that is parallel to ℓ.*) But the non-Euclidean revolution did not address the question as to which, if any, of these postulates is true in physical space. It only told us that the nature of parallel lines is not a question of mathematics but rather a question of empirical science.

During the eighties and nineties this question suddenly came to the forefront. Advances in astronomy made it possible to see deeply into distant space, and theoretical work in cosmology allowed us to draw conclusions about what we saw. The question as to whether space was elliptic (positive curvature), hyperbolic (negative curvature) or Euclidean (zero curvature) was directly tied to the question of whether cosmic expansion would eventually halt and lead to cosmic collapse, would expand forever beyond some positive rate or would expand at a rate that asymptotically approached zero. This connection drew growing public interest as speculation moved from one option to another with every new observation. We lived each day with the prospect of reading headlines announcing that space was now determined to be elliptic, hyperbolic or Euclidean. In this context Euclidean geometry in university courses is often truncated to make room for a non-Euclidean unit.

A Moment of Opportunity

The curvature of space within the theoretically visible universe was finally measured by the Wilkinson Microwave Anisotropy Probe (WMAP), launched by NASA in 2001, which returned a full sky photograph of the Microwave Background Radiation. This data finally demonstrated that space has curvature zero within the visible universe. The measurement itself is confirmed by the fact that any non-zero curvature in the early universe would have moved rapidly away from zero as the universe expanded, creating an easily seen non-zero curvature today. As a result we at last have scientific proof that Euclidean geometry accurately models the theoretically visible universe.

Van Hiele Levels

Before attempting to design a revised Euclidean geometry course, we need merge our understanding of geometry itself with an understanding of a student's development of geometric thinking. Developmental levels of geometric thinking are commonly classified using the hierarchy developed by Pierre and Dina van Hiele, beginning in the late 1950s. The presumption of the five van Hiele levels is that successful learning of skills above the first level require some reasonable mastery of all lower level skills. We briefly describe these five levels.

- H1. **Visualization.** Identify shapes and other geometric configurations according to their appearance. (For example, given a collection of quadrilaterals with the rhombuses identified, correctly identify the rhombuses in another collection where they are not previously identified.)
- H2. **Description/Analysis.** Recognize and characterize shapes by their properties. (For example, recognize the properties that distinguish a rhombus.)

H3. **Abstractions/Relations.** Form abstract definitions and distinguish between necessary and sufficient sets of conditions for a concept. (For example, articulate definitions of "square" and "rhombus", recognizing a square as a rhombus with additional properties.)

H4. **Formal Deduction.** Prove theorems within a formal axiomatic system. (For example, prove that the diagonals of a rhombus are perpendicular.)

H5. **Rigor/Metamathematics.** Understand and describe relations between different formal axiomatic systems. (For example, explain the significance of non-Euclidean geometries and of Hilbert's new system of axioms.)

This book is primarily written to develop an H4 level knowledge of geometry. However instructors need to bear in mind that lower Van Hiele levels of understanding, and particularly level H3, may not be reliably in place. Lack of appreciation for these foundational levels of understanding among educators can lead to subsequent frustration and failure, as is well documented by the NCTM [**3**]. For this reason it is important introduce definitions through a class discussion that climbs up through the Van Hiele levels to the extent that students need this. Understanding that words can have precise and exact definitions in particular contexts, and using them correctly in those contexts, is a major educational step that many never master. Geometry provides an ideal opportunity to do this by solidifying and H2 level understanding of a concept and then asking students to formulate an H3 level definition. Time for these discussions is well spent, as it grounds students in the foundational concepts before launching into the H4 level abstractions.

Resolution

The third pedagogical dilemma was resolved by WMAP. Resolution of the second will require a change in public policy based on a reassessment of the purpose of teaching geometry. This text offers a resolution of the first: how can we guide students through an axiomatic development of geometry that is mathematically sound and still covers the content they need in a single course?

We will adopt the perspective of Euclid, but now with the modern insight afforded us by the advances that have been made in mathematics since Euclid's time. We will present here a new axiomatic development of plane geometry. In contrast to Hilbert, we will begin with a foundational base large enough to reach our content goals in a single course. In contrast to Euclid, our foundational base will be sufficient to yield our content goals through tight rigorous proofs. Our foundational base will add to a standard naive understanding of logic and set theory three **foundational principles**, ten **axioms** and three unproven **limit theorems**.

Our three foundational principles are intended to be freely used by students without necessarily being stated explicitly, just as the foundational principles of naive logic and set theory are used without being stated in most of mathematics. This means that here we have to act as a vigilant jury when proofs are given, making certain that we agree with each application of these principles.

Non-triviality. *Two points determine a line, every line contains at least two points, and there exist three non-collinear points.*

[These are Hilbert's Axioms 1, 2 and 3.]

Betweenness. *Properties of betweenness that hold for all points and lines in the coordinate plane hold here as well.*

[Examples: If A, B, C and D are four points on a line with B between A and C and C between A and D, then B is between D and A. Every point on a line is between two other points on that line. No line contains four points A, B, C, X with X between each two of the others. Every line separates the points not on it into two sides in such a way that two points are on the same side if and only if no point of the line is between them.]

Intersections. *Points of intersection of lines and/or circles exist if, when constructed with a straightedge and compass in a coordinate plane, they necessarily cross.*

[Examples: For every circle $\mathbf{C}$, every line or other circle connecting a point inside $\mathbf{C}$ with a point outside $\mathbf{C}$ must intersect $\mathbf{C}$. A ray emanating from a point inside $\mathbf{C}$ must intersect $\mathbf{C}$. A line containing a point inside $\mathbf{C}$ intersects $\mathbf{C}$ in exactly two places. The diagonals of a rectangle intersect.]

Our second and third foundational principles differ from axioms because they are essentially subjective; neither being precisely specified. We have left both to the good intuition and personal experience of our students, and have found that this rarely raises any questions. If questions do arise, the instructor should state the relevant fact clearly and reassure the students that they are at liberty to use it as they need it. In the unlikely case that further questions arise, students can be told that these facts will be proven in a more advanced course using a more fundamental set of axioms. The following proof illustrates typical applications of these principles.

Theorem 35. *Every segment has a midpoint.*

Proof. Given the segment AB, let $\mathbf{C}_A$ be the circle with center A passing through B, and let $\mathbf{C}_B$ be the circle with center B passing through A. These circles intersect [Intersection Principle] at two points X and Y on opposite sides of $\overleftrightarrow{AB}$. [Non-triviality Principle: the notation "$\overleftrightarrow{AB}$" itself incorporates the fact that there is a unique line containing A and B.] Let M be the point of intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{XY}$. Then M is between A and B and is between X and Y [Betweenness Principle]. By Axiom 2 we have $\triangle AXY \cong \triangle BXY$. Consequently $\angle AXY \cong \angle BXY$. By Axiom 3 this gives us $\triangle AXM \cong \triangle BXM$. Therefore $AM \cong MB$, and M is the midpoint of segment AB. $\square$

In this context we do not see our addition of these foundational principles to Hilbert's foundation of logic and set theory qualitatively different from Hilbert's foundation of logic and set theory itself. Recall that Gödel's Completeness Theorem says that every semantic consequence of a set of first order sentence can be derived from those sentences with the standard rules of deduction. It is well known (and easy to prove) that there is no complete set of rules of deduction for second order logic. Hilbert's Axiom 14 quantifies over sets of points, and consequently his system is at least second order. As a result, there is no known completeness theorem to identify correct proofs in Hilbert's system. This means that here as well we have to act as a vigilant jury when proofs are given, making certain that we agree with the underlying reasoning.

Each of our axioms is presented in the text at the point that it is first needed, and all ten are listed together in Appendix A. Axioms 1, 6 and 7 describe measurement, drawing on an intuitive familiarity with lines and planes that all students have but that was not fully understood prior to the nineteenth century work of Dedekind, Riemann, Jordan and Lebesgue. In addition to our foundational principles and axioms, we will state and use three limit theorems whose complete proofs would be a serious digression from this text: the Rectangle Area Theorem, the Scaling Theorem and The Theorem Pi. In each case the students will prove approximating cases of the theorem to provide compelling evidence that it is true as stated. Instructors are urged to read Appendix B: "Guidelines for the Instructor" and the *Instructor Supplement* at $\langle www.ams.org/bookpages/mcl-9.\rangle$