Translations of
MATHEMATICAL MONOGRAPHS

Volume 192

C^*-Algebras and Elliptic Operators in Differential Topology

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Preface

The theorem on the index of an elliptic operator proved by Atiyah and Singer at the beginning of the sixties has initiated a quickly developing field of research, which can be briefly described as the study of topological invariants of manifolds using methods of functional analysis and the theory of differential operators. Some classical results such as the Hodge theory of harmonic forms and the Riemann–Roch theorem in the Hirzebruch formulation have become part of it in a natural way.

In the subsequent three decades research in this area has led to a series of remarkable results. At the same time, the variety of functional methods used here has increased considerably. In particular, Banach algebras and noncommutative differential geometry were used intensively. The main part of the theory has acquired rather precise outlines; moreover, the variety of its external connections and applications expanded, including even some areas of mathematical physics.

The main goal of our book is to introduce the reader to some of these methods, and also to the concrete topological problems solved with their help. An exhaustive description of functional methods in the topology of manifolds would deal with a large number of mathematical theories and would take several volumes. The present work has more modest purposes: it is devoted only to an introduction to the subject and can serve as the first step leading to the study of original works and specialized monographs.

The limited volume of the book has forced us to exclude the presentation of a series of interesting problems. Some of them were entirely omitted others are only discussed briefly. Nevertheless, we hope that what remains is enough to give a connected picture of the subject, at least of its most important parts.

Now we briefly describe the contents of the book.

In the first chapter K-theory, in terms of which the basic index theorem will be formulated, is described. In Section 1.1 some necessary information from homology algebra is presented. In Section 1.2, Hilbert modules are defined. They play the role of Hilbert spaces, in particular, Sobolev spaces, in C^*-index theory. Further, the corresponding analogs of compact operators are introduced, and their basic properties are proved. Finally, a technical theorem about averaging by means of a compact Lie group G acting on the space of operators for a Hilbert module is proved. Most of the results of this section were obtained in the papers [**134, 133, 92, 96, 69, 34**], while the basic information is contained in [**107, 78, 38**]. Section 1.3 is devoted to equivariant K-theory, defined from a projective module over a unital C^*-algebra A and from the corresponding bundles. After recalling the basic information about Clifford algebras, we define K-groups following the ideas of [**57**] and derive their main properties. An important computational tool of equivariant K-theory is the use of spectral sequences. A description of K-groups via classifying spaces is obtained. For this purpose the theory of Fredholm operators is developed, and an appropriate analog of the Kuiper theorem is applied. Finally, the theorem on the Thom isomorphism in our K-theory is proved. The papers [**96, 134, 132, 135**] were the principal source of our presentation. In the last section of this chapter, two approaches to KK-theory are described: the original approach of Kasparov [**70**] and the approach of Cuntz [**27**].

The second chapter is devoted to the proof of the equivariant index theorem for C^*-elliptic operators [**135**]. In the first section the necessary calculus of pseudodifferential operators over an algebra A and analytical index theory are developed. The proof of the theorem itself is contained in Section 2.2. In the last section, the proof of the classical Atiyah–Singer theorem is presented. It becomes a special case of the C^*-theorem (when $A = \mathbb{C}$).

The third chapter is devoted to one of the central problems of differential topology: the description of all homotopy invariant rational Pontryagin classes of smooth non-simply-connected manifolds. The classification of smooth structures on a non-simply-connected manifold of given homotopy type created in the mid-sixties by S. P. Novikov and C. T. C. Wall has allowed us to prove that homotopy invariant rational Pontryagin numbers of a non-simply-connected manifold can be only rather special characteristic numbers, which are called higher signatures. In the papers [**103, 104**] S. P. Novikov conjectured the homotopy invariance of higher signatures. Among various approaches to the proof of this conjecture, the direction based on methods of functional analysis, finite-dimensional and infinite-dimensional representations of the fundamental group developed rather quickly. In a series of papers [**88, 89, 90**] A. S. Mishchenko created the theory of algebraic surgery and introduced an important homotopy invariant of non-simply-connected manifolds, known as the symmetric signature. This theory has become a necessary part of the most successful studies of the problem of higher signatures.

The first section of the third chapter contains the necessary information on the theory of characteristic classes of smooth manifolds. In the second section, higher signatures are introduced and the Novikov conjecture is formulated. The third section contains the necessary facts on Hermitian K-theory, the theory of algebraic Poincaré complexes, and Fredholm representations of algebras with involution. Central in the chapter is the fourth section, devoted to the proof of the generalized Hirzebruch formulas due to Mishchenko. In the fifth section, the Mishchenko theorem about the homotopy invariance of higher signatures is proved for fundamental groups with the following property: the classifying spaces of these groups are complete Riemannian manifolds with metric of nonpositive curvature. In the fourth and fifth sections we follow Mishchenko's paper [**91**]. In the sixth section the homotopy invariance of higher signatures is proved for fundamental groups isomorphic to discrete subgroups of linear algebraic groups over local locally compact fields [**121, 122**]. The seventh section of the chapter is devoted to the approach of G. G. Kasparov to the problem of higher signatures. We consider, in particular, another proof of the basic theorem in the fifth section, mainly following the presentation in [**65**]. In the final section, some applications of the Dirac operator to the problem of positive curvature of spinor manifolds are considered [**113, 135**].

In the first section of the fourth chapter the fundamentals of noncommutative differential geometry and the theory of homology with inner symmetries, including cyclic and dihedral homology, are presented. Here we principally follow [**22, 80, 123**]. In the second section this theory and the C^*-index theorem are applied to constructing the theory of generalized Lefschetz numbers [**136**]. In the final section, applying the same results, we give a sketch of the proof by Connes and Moscovici [**24**] of the Novikov conjecture for a class of groups including the hyperbolic groups of Gromov.

Now a few words about topics not included in the book. First of all, these are various approaches to the proof of the index theorem. Our proof of the index theorem for equivariant C^*-elliptic operators follows the classical lines of the work of Atiyah and Singer.

The set of problems related to the approach based on the heat equation is not considered in the book. The detailed presentation of these problems can be found in [**41, 111, 8**].

It is possible to study one more proof, which uses methods of probability theory, given in the work of Bismut [**9**].

Some interesting generalizations of the index theorem to elliptic operators on noncompact manifolds have appeared in the last decade. They use either so-called bounded geometry or some new cohomology theories (L^2 cohomology and "coarse cohomology"). It is possible to find the principal results obtained in this direction in [**36, 35, 112, 47**]. One more approach to noncompact theory of the index is based on the use of foliations with noncompact leaves [**99, 22, 25**].

There is a long list of literature devoted to generalizations of the index theorem to nonsmooth manifolds [**129, 130, 128, 131, 117**]. The works on the index for orbifolds and similar objects follow the same purpose (see the survey [**140**]).

The theory of the index is intensively developing now in the so-called algebraic direction, essentially using the formalism of cohomology theories of cyclic type obtained in a series of interesting papers of Cuntz and Quillen [**29, 31, 30**]. Their results have led Nistor to a new proof of the theorem of Connes–Moscovici [**102**]. Among the other results of this direction, let us emphasize the work of Nest and Tsygan on the algebraic index theorem (see [**101**] and the bibliography therein).

Our book essentially contains only an introduction to the KK-theory of Kasparov. A detailed exposition of this theory and its various applications can be found in [**71, 120, 54**]. Various aspects of K-theory and homology of Banach algebras is contained in [**10, 50, 139**]. Many parts of the theory presented here were developed in the context of real algebras [**49**].

Two volumes of the recently published transactions of the conference in Oberwolfach [**37**] can serve as an excellent survey, devoted to the modern state of the Novikov conjecture on higher signatures, in which, in particular, a rather complete bibliography on this problem is presented.

The preparation of the book was a difficult and long matter, and many specialists have helped us in this work. First of all we are indebted to our teacher, Professor A. S. Mishchenko, for making us familar with functional methods in differential topology and stimulating our research in this field. We had a number of useful contacts on problems considered in the book with I. K. Babenko, V. V. Belokurov, Yu. G. Borisovich, L. G. Brown, J. Cuntz, A. T. Fomenko, T. Friedrich, E. A. Gorin, M. L. Gromov, A. Ya. Helemskii, A. A. Irmatov, J. Kaminker, M. Karoubi, G. G. Kasparov, V. M. Manuilov, A. V. Mikhalev, A. I. Nemytov, R. Nest, V. Nistor, M. M. Postnikov, A. Ranicki, J. Rosenberg, E. T. Shavgulidze, G. Skandalis, B. L. Tsygan, A. Valette, I. A. Volodin, and S. Weinberger.

A significant part of the book is based on lectures of the second author at the University of Greifswald (Germany), where he enjoyed the hospitality of Professor J. Eichhorn with the support of the German Service of Academic Interchange (DAAD).