IWANAMI SERIES IN MODERN MATHEMATICS

Translations of
MATHEMATICAL
MONOGRAPHS

Volume 195

Infinite-Dimensional Lie Algebras

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Preface

Over 30 years ago, in 1967 the so-called Kac-Moody algebras were discovered as infinite-dimensional Lie algebras. The representation theories of Lie algebras can be studied without taking account of their Lie groups. Starting with a simple finite-dimensional Lie algebra, we extend it to an infinite-dimensional Lie algebra as a loop algebra over the one-dimensional torus (i.e. the circle S^1), and its central extension is an affine Lie algebra. It has structures quite similar to those of simple finite-dimensional Lie algebras. Its Dynkin diagram is the diagram obtained by adding one vertex to the Dynkin diagram of a simple finite-dimensional Lie algebra, and its Weyl group is also the affine Weyl group of a simple finite-dimensional Lie algebra. All these had already appeared within the framework of theories of simple finite-dimensional Lie algebras, so these were not new. The character formulae of affine Lie algebras have the same form as in the case of simple finite-dimensional Lie algebras. Nevertheless, as soon as character formulae of highest weight modules over affine Lie algebras were discovered, the representation theories of affine Lie algebras made great progress, which one could not imagine from the case of simple finite-dimensional Lie algebras.

The fact that an affine Lie algebra is the infinite-dimensional extension of a simple finite-dimensional Lie algebra as a loop algebra over the circle S^1 gives rise to the following remarkable features.

i. An affine Lie algebra contains an infinite-dimensional Heisenberg Lie algebra as subalgebra; namely, a Cartan subalgebra of a simple finite-dimensional Lie algebra grows up to be a Heisenberg Lie algebra by the central extension of its loop algebra. In this way, a Heisenberg Lie algebra gets into an affine Lie algebra.

ii. The vector fields on S^1 act on the derived subalgebra of an affine Lie algebra as derivations.

(ii) is the reason why an affine Lie algebra works harmoniously with the Virasoro algebra. One might say that the above two features are the reason why representation theories of affine Lie algebras are so important.

As is well known, the irreducible representation of a Heisenberg algebra is equivalent to the Schrödinger representation on a space of functions. To construct the action of an affine Lie algebra in practice, one needs vertex operators in order to describe the action of elements not in the Heisenberg subalgebra, and by them representation theories of an affine Lie algebra become much richer than those of a Heisenberg algebra. One can see from an affine root system which sort of and how many vertex operators are necessary to construct a representation of an affine Lie algebra, and a character formula tells us how one can decompose a module over an affine Lie algebra as modules over a Heisenberg Lie algebra contained in the affine Lie algebra.

The Schrödinger representation of a Heisenberg algebra consists of creation and annihilation operators, and the former act freely. Therefore a representation of an affine Lie algebra contains free action with respect to that of the creation operators of a Heisenberg subalgebra. This will be the key of the modular invariance of characters of an affine Lie algebra. In this sense, I think that the heroes (behind the scenes) of the representation theories of affine Lie algebras are a Heisenberg Lie algebra and vertex operators.

But without such arguments, one can derive useful identities, called the Macdonald identities, by fiddling around with a character formula as an algebraic formula. When Macdonald [**Mac**] discovered the Macdonald identities by considering affine root systems, the "mysterious factors" arising as a stumbling block in the formulae are in fact the factors that have their origin in the free action of the creation operators of a Heisenberg Lie algebra.

What is interesting in an affine Lie algebra is that it may be non-trivial to apply general formulae to some concrete cases. For example, writing the denominator identity for $\hat{\mathfrak{sl}}(2, \mathbb{C})$ we obtain the Jacobi triple product identity, and writing hierarchies for $\hat{\mathfrak{sl}}(2, \mathbb{C})$ we obtain the KdV equation and the non-linear Schrödinger equation. Further, a representation theory describes not only equations but also their solutions. Namely, if one finds a solution, one can construct other solutions successively by letting vertex operators act on it. Since the action on the set of solutions is transitive, one can start from the easiest solution, i.e. the "trivial solution". But we do not touch upon these soliton equations in this book.

In any case, many topics in affine Lie algebras are interesting enough when we specialize them to the case of $\hat{\mathfrak{sl}}(2, \mathbb{C})$ and just look at them. And in such a way, one may feel that one understands the topics. Conversely, once we find a phenomenon for one of the simplest affine Lie algebras such as $\hat{\mathfrak{sl}}(2, \mathbb{C})$, we can lift it up to general affine Lie algebras by making a fair copy of it in terms of Lie algebras, and similar phenomena will be mass-produced "at one stroke". Even if it is said that we must not do things "at one stroke" in these days, this "stroke" is quite exciting!

Thus in representation theories of affine Lie algebras, the basic tools for any purpose are character formulae, and we can say that they are the starting point of the representation theories.

So in this book, at first I will state the structure of Lie algebras (i.e. their root systems) and character formulae in Chapter 2. In this chapter, everything will be treated in a quite general setting, since one can expect that these facts will be developed and applied further. From Chapter 3 on, I will explain the modular invariance of the characters of affine Lie algebras. This is one of the most picturesque scenes, and I will sometimes pause to illuminate it further by concrete examples.

There is a book by Kac [**K4**] on Kac-Moody Lie algebras. Although the first edition was published 16 years ago, it still seems to me that this is the best book, both as an introductory book and as a side book. But I have heard that it is hard to start one's study with this book, because it begins from the theory of generalized Cartan matrices. Hence I started writing the present book as a guidebook to Kac's book. I hope this book will be an *hors d'œuvre* to the great feast of infinite-dimensional Lie algebras.

This book is not an encyclopedia. It does not contain everything that could be written, or even everything important, about infinite-dimensional Lie algebras, but only the facts needed for my development of the theory.

To ease the reader's way through this book, I will explain the derivation of each formula in full detail. The reader may mostly just follow along with his or her eyes, taking time out for occasional calculations. I can comfortably read this book like a novel. The facts appearing in mathematical nature are more beautiful and mysterious than any fiction. I am anxious about whether I have been able to depict them successfully.

Prof. Michio Jimbo encouraged me to write this book, and moreover he advised me, politely but strongly, on the manuscript. I would like to express my deepest gratitude to him. Dr. Kenji Iohara read through the manuscript, and pointed out and corrected lots of mistakes. Dr. Hiroyuki Tagawa has constructed the TEX environment in my office from the very beginning, and moreover he not only taught me how to use TEX and to draw figures personally but also produced some of the figures in this book himself. I also would like to thank the editors, who gave me much valuable advice on the description and on the contents, and those in the Iwanami publishing house who took care of me until I could complete the manuscript of this book.

The computation of the partition numbers shown in an example in §4.2 is carried out by the mathematical system"Reduce 3.6". I first became acquainted with "Reduce" in February 1986 when I had some computations to do. Thanks to the instruction of Prof. Seigo Okamoto, "Reduce" immediately became an important supplementary tool for my research. In addition, it was Prof. Okamoto who introduced me to Kac-Moody Lie algebras. I would like to express my heartiest gratitude to him for his great care.

Finally, I would like to express my appreciation to both of my parents, who let me achieve my dream of studying mathematics.

With gratitude to my wife, Yasuko

Minoru Wakimoto

Preface to the English Edition

This book was originally written in Japanese to provide an outlook and to serve as a guide to the theory of infinite-dimensional Lie algebras, which has grown up in quite recent decades in connection with various areas in mathematics and mathematical physics.

I tried to give an exposition with much detailed explanation, so that the theory may become more familiar to readers, though the topics treated in this book are quite limited for want of space. It will greatly please me if this book may raise its readers' interest in this area and be a help for further research and development of the theory.

For the publishing of this English edition, I thank Iwanami Shoten Publishers, the editors of the AMS, the translator and, in particular, Professor Katsumi Nomizu for a lot of kind attention.

July 2000, in Fukuoka

Overview

When we consider the structure of a simple finite-dimensional Lie algebra $\mathfrak{g}$, we first choose a Cartan subalgebra and make the root space decomposition. Next we classify the roots into positive and negative ones. The minimal roots among the positive roots with respect to the decomposition (that is, the positive roots which cannot be decomposed into a sum of roots) are called **simple roots**. Denote them by $\alpha_1, \cdots, \alpha_l$. Then the matrix of inner products $\left(\dfrac{2\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_i, \alpha_i\rangle}\right)$ is called the **Cartan matrix** of $\mathfrak{g}$. To reconstruct a Lie algebra from the Cartan matrix, we have only to impose conditions that can be extracted from the Cartan matrix on the free Lie algebra generated by $2l$ letters, say $e_1, \cdots, e_l; f_1, \cdots, f_l$. In this way, we can reproduce the original Lie algebra, and we have the one-to-one correspondence:

$$\text{Simple Finite-Dimensional Lie Algebras} \quad \longleftrightarrow \quad \text{Cartan Matrices.}$$

Namely, if two Lie algebras are isomorphic, then their Cartan matrices are the same; otherwise their Cartan matrices differ. Therefore the simple finite-dimensional Lie algebras are classified in terms of their Cartan matrices. This is the well-known Cartan-Killing theory, and it appeared about one hundred years ago. But after that, the Cartan-Killing theory on simple finite-dimensional Lie algebras fell asleep peacefully as a completed theory, i.e. as a "theory for textbooks".

But after two thirds of this century has passed, two princes suddenly approached this "Sleeping Beauty". The outrageous princes who kissed the forbidden lips were Kac and Moody. Since Cartan matrices classify the simple finite-dimensional Lie algebras,

> Relaxing conditions on Cartan matrices a bit, one might be able to obtain infinite-dimensional Lie algebras!

We call such modified Cartan matrices generalized Cartan matrices (GCM for short). A new class of infinite-dimensional Lie algebras, i.e., **Kac-Moody Lie algebras**, was born. From its origin, one can argue similarly as in the case of finite dimension. What is necessary to explicate its structure is a representation theory of $\mathfrak{sl}(2,\mathbb{C})$ only. Furthermore, when a GCM is symmetrizable, the corresponding Lie algebra possesses an inner product, and one can define the Casimir element. As an application, character formulae can be proved. They have the same form as in the finite-dimensional case. The points that are different from the finite-dimensional cases are

i. the Weyl group becomes an infinite group, and
ii. imaginary roots appear.

Nonetheless, these two facts have great consequences.

The most lovely class of infinite-dimensional Lie algebras is the affine Lie algebras. An affine Lie algebra is the central extension of the loop algebra of a simple finite-dimensional Lie algebra, so we know the structure of its root system very well, and its Weyl group is the semi-direct product of the Weyl group of a simple finite-dimensional Lie algebra and a $\mathbb{Z}$-lattice. In the character formulae the summation over the $\mathbb{Z}$-lattice gives us theta functions, and hence the character of integrable representations over an affine Lie algebra is a modular function. I will explain this in Chapter 4, and in Chapter 5 I will explain fusion coefficients as a topic related to the transformation matrices of modular transformation of the characters.

After this brief awakening, the 'Beauty' went to sleep again. This 'Beauty' sleeps well. The next prince who disturbed her sleep was Borcherds. Anyone who is woken from a first sleep will be offended. In addition, the task of hauling Cartan matrices in is so daring that it might change the properties of simple roots. But the 'Beauty' is also sweet-tempered. She never loses her temper. Hence Borcherds built up a fine theory.

Needless to say, Borcherds started his research on infinite-dimensional Lie algebras not from the viewpoint of Cartan matrices. He discovered the vertex operator algebras from his research on representations of the Monster simple group. From the simple roots of the infinite-dimensional Lie algebra associated to a vertex operator algebra, he had an insight into the fact that the following situation is significant:

> We do not have to mind even if some strangers
> (i.e. simple imaginary roots) mix up within the set
> of simple roots.

A Cartan matrix allowing the existence of such simple roots is called a **GKM matrix** (generalized Kac-Moody matrix), and the corresponding infinite-dimensional Lie algebra is called a **GKM Lie algebra**. Even in this setting, analogous arguments work well to prove character formulae. In this case, the forms of character formulae are not entirely the same as before, but slightly more complicated, and they have a new effect. The formula obtained by specializing character formulae to the trivial representation is called the **denominator identity**. This is a formula of the form "Infinite Product = Infinite Sum". Since imaginary simple roots create a new environment in the case of GKM Lie algebras, their denominator identities show us new equalities for a much wider class of modular functions than those arising from Kac-Moody Lie algebras.

But concerning GKM algebras, the only formula that has some applications up to now is the denominator identity, and the character formulae themselves of the other integrable representations have not been studied yet. Character formulae are too good to apply only to the trivial representation. I would like to see some nice applications.

In this book, for the reason to be explained in §1.4, we use the term BKM algebra instead of GKM algebra. The character formulae for BKM Lie algebras will be important for applications. For this purpose, in Chapter 2, I will describe in detail the root system and the character formulae for BKM algebras and BKM superalgebras in a general setting.

Therefore the main theme of this book are in Chapter 2, "Character formulae for BKM Lie superalgebras", and in Chapter 4, "Modular transformation of characters of affine Lie algebras". Representation theories of affine Lie algebras have applications to many fields. One could explain each topic little by little, but I have preferred to restrict myself to the theme on modular transformations of affine Lie algebras after Chapter 4. But when I looked back after writing up till Chapter 5, I felt something was insufficient. For example, without any description of how one can use the matrix elements of modular transformations presented as examples in Chapter 4, they are simply enumerations of data and may not show the readers their beauty. After I began writing the manuscript of this book, for a while, I was going to omit the Virasoro algebra, because of the rich variety of affine Lie algebras. Representation theories of the Virasoro algebra are so ample that I cannot mention them briefly. It is impossible to fit them into this book. But representation theories of affine Lie algebras without the Virasoro algebra became dreary, like stale beer. So Chapter 6 was added, to give a brief sketch of how representation theories of the Virasoro algebra work (the chapter entitled "In Lieu of Postscript").

Although representation theories of affine Lie algebras are related to several fields, I cannot write about them, owing to the lack of space and of my knowledge. These other fields include, in no particular order, conformal field theory and KZ equations, lattice models, soliton equations, and so on. The first two are covered in the book *Field Theory and Topology* in this series.

It seemed to me that Prof. Ryogo Hirota's use of the theory of bilinear differential operators to find the exact solutions is a marvelous method, like magic. Through investigating it, the group of Prof. Mikio Sato, Prof. Etsuro Date, Prof. Michio Jimbo, Prof. Masaki Kashiwara and Prof. Tetsuji Miwa in the Research Institute of Mathematical Science, Kyoto University, discovered that the symmetry hidden inside the non-linear differential equations called soliton equations is an affine Lie algebra. In May 1982, when the lecture course taught by Kac for the spring semester drew to a close, their theory was introduced in his lecture as a "Work of Japanese School". I got into the lecture-hall without permission or payment for this lecture, and the theory developed on the board was so beautiful that my heart burst into flames. Everyone who is alone in a foreign country will be nationalistic. In addition I am timid. It was the Kyoto school that gave me the motivation to leap at infinite-dimensional Lie algebras, forgetting everything I had been working on before. There are some explanations of the theory, e.g. in [**Sa**].