# Preface to the English Edition

This book is a translation of my book originally published in Japanese by Iwanami Shoten, Publishers. The aim of this book is to provide the reader with a concise introduction to stochastic analysis, in particular, the Malliavin calculus. I hope that the material of this book will reach more readers by this translation.

I would like to express my deep appreciation to the American Mathematical Society for publishing this translation, and to their staff for excellent support.

February 2004

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# Preface to the Japanese Edition

This book is an introduction to stochastic analysis. What stochastic analysis means is rather wide: here we roughly regard it as an analysis based on the Wiener space. It takes in various techniques of probability theory and is necessarily related to other parts of probability. In this book we concentrate on infinite dimensional analysis, in particular, the Malliavin calculus. Its main stage is really the Wiener space; to be more precise, our object is Wiener functionals. We have to analyze functionals on an infinite dimensional space, and we are forced to develop a calculus as in a finite dimensional space. As for integral, we have an abstract measure theory, which works efficiently even on an infinite dimensional space. We also have a theory of differentiation on an infinite dimensional space, but it does not match with the integration.

In the 1970's, Paul Malliavin made a breakthrough in this area. He presented a new calculus to realize a probabilistic approach to a question of hypoellipticity of Hörmander type. It was exactly a theory of differentiation for the Wiener space. The theory turned out to have applications not only to partial differential equations but also many other fields. Due to his contribution, the theory is usually called the Malliavin calculus. The theme of this book is a stochastic analysis which contains the Malliavin calculus as a main part. I have tried to make the description elementary, which at times may make this book rather redundant.

From the pioneering work of P. Malliavin this book seems to remain at a fundamental level. The research frontier is still far beyond this book, but I hope that it will prepare the reader to proceed to recent further topics. I am convinced that the reader will have enough tools at hand to do so after reading this book.

I wish to dedicate the book to Professor Shinzo Watanabe, who brought me to this area. I also wish to express sincere thanks to Professor Yoichiro Takahashi, who recommended that I write this book. Special thanks are due Professor Masanori Hino, who read the entire manuscript carefully and made numerous helpful suggestions. Finally, I would like to express my deep gratitude to all the editorial staff of Iwanami Shoten, Publishers, for their efforts.

August 1998

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# Outline of the Theory and the Objectives

The aim of this book is to give an introduction to stochastic calculus, in particular, the Malliavin calculus. The main aim of the Malliavin calculus is to analyze functionals defined on the Wiener space in which a Wiener process is realized.

In probability theory it is called a Wiener process, but it is usually called a Brownian motion. It originated in a discovery of Robert Brown in 1882, who found an extremely irregular movement of minute particles coming out of a pollen. Since then, this phenomenon has been much studied by physicists, and Norbert Wiener established it as a rigorous mathematical object. One reason for the importance of the Wiener process is that it describes various mathematical models. Among them, the most important is stochastic differential equations, developed by Kiyosi Itô. In fact, many concrete models, e.g., in physics, genetics, economics, etc., are described by stochastic differential equations. For the theory of stochastic differential equations, see, for instance, Ikeda and Watanabe [**7**], Karatzas and Shreve [**8**], Revuz and Yor [**22**], and others. In this book we are dealing with Wiener functionals such as those defined by stochastic differential equations.

Wiener functionals are realized on the Wiener space, which is an infinite dimensional space. What we need is a calculus on an infinite dimensional space, and consisting of two things: differentiation and integration. Integration is based on well-developed measure theory. It works even on infinite dimensional spaces. But differentiation is not as easy as integration. Of course we had a theory of Fréchet differentiation in Banach spaces. Many other attempts were also made, but they are not organically connected to the integration theory. For instance, differentiation is the inverse operation of integration in the one-dimensional case. Even in the multi-dimensional case, Stokes' formula is an exquisite combination of integration and differentiation. On the Wiener space, such a harmonious theory appeared only recently.

In 1976, an international symposium on stochastic differential equations was held at the Research Institute of Mathematical Sciences of Kyoto University. At that symposium, P. Malliavin mentioned, rather informally, a new theory of calculus on the Wiener space. The full details appeared in the proceedings of the symposium (see [**14**]). The preprint had been circulated a little earlier, and the author came to know it. It was really lucky for the author to learn the theory at such an early stage. In Japan, N. Ikeda and S. Watanabe noticed the importance of this work at once and started the study of this theory, which is now called the Malliavin calculus. Malliavin himself visited Japan many times and gave stimulation to Japanese probabilists. With such an opportunity, the study of the Malliavin calculus in Japan is still active. This is another example of the importance of international communication.

The Malliavin calculus is based on the Ornstein-Uhlenbeck operator, which is a second order differential operator. It is a counterpart of the Laplace operator in Euclidean space. Malliavin also captured the gradient operator through the square field operator (called operateur de carré du champ in French literature) associated with the Ornstein-Uhlenbeck operator. The gradient operator is connected to the notion of differentiation and nowadays is formulated in the framework of $H$-differentiation on the Wiener space. Here $H$ is the Cameron-Martin space, which we will explain later. The differentiation is considered only in a subspace $H$, and derivatives are extended by means of the completion in $L^p$ spaces. So to speak, derivatives are formulated in the sense of distributions. Through this procedure we could develop a flexible theory. The idea is simple, but what it brought is big.

In the Malliavin calculus, there appear two fundamental operators: one is the Ornstein-Uhlenbeck $L$ and the other is $H$-differentiation $D$. In accordance with them, we can define two kinds of Sobolev spaces. They are closely related to each other, and in fact we can prove equivalence in the $L^p$ $(p > 1)$ setting. This is due to P. A. Meyer and brought a neat basis in the Malliavin calculus. We discuss Meyer's result in Chapters 3 and 4. At present, two kinds of proof are known; we give a proof along the original idea of Meyer in the framework of Littlewood-Paley theory. The proof is probabilistic, and it is an good example that shows the power of martingale theory and Itô's formula.

To develop a calculus on an infinite dimensional space, it is a natural and effective way to extend results from finite dimensional space. As a matter of fact, there are some analogies between them.

We list some of them below; rigorous definitions will be given later. Our object is the Wiener space, and a typical finite dimensional model is the Euclidean space.

| Euclidean space | Wiener space |
|---|---|
| Lebesgue measure $dx$ | Wiener measure $\mu$ |
| Brownian motion | Ornstein-Uhlenbeck process |
| Laplace operator $\Delta$ | Ornstein-Uhlenbeck operator $L$ |
| gradient operator $\nabla$ | $H$-differentiation $D$ |
| $H^{r,p} = (1-\Delta)^{-r/2}L^p(dx)$ | $W^{r,p} = (1-L)^{-r/2}L^p(\mu)$ |
| Sobolev inequality | logarithmic Sobolev inequality |

We make some remarks on Sobolev spaces. In finite dimensional space, better differentiability improves the degree of integrability. But in the Wiener space, better differentiability improves the integrability only by the logarithmic order. We cannot expect more than this, but it is still powerful. Applications of this fact are not given in this book, but its importance is stressed in recent literature not only on the Wiener space but also on the general finite dimensional spaces. Historically E. Nelson first proved the hypercontractivity of the Ornstein-Uhlenbeck semigroup, and then L. Gross found the equivalence between hypercontractivity and the logarithmic Sobolev inequality. Incidentally, a primitive notion of $H$-differentiation is due to Gross.

In this framework, we can build a theory of differential calculus on the Wiener space, e.g., the chain rule of the composite function, integration by parts formula. On the other hand, the powerful method of the Fourier transform is available in Euclidean space. There is no correspondence on the Wiener space. This remains as a future problem. Along the way, there are some aspects of the Fourier transform. One is that it gives a spectral decomposition. From this viewpoint, the spectrum of the Ornstein-Uhlenbeck operator is completely known, and in fact, the eigenspaces are exactly the spaces of multiple Wiener integrals. We discuss this topic in §1.2.

If we consider the Hilbert transform in connection with the Fourier transform, its $L^p$ theory corresponds to Meyer's equivalence. In fact, Meyer proved his equivalence by using martingale theory, which has its origin in Fourier analysis. The spirit of analysis, including classical Fourier analysis, flows here as well.

The central part of this book is Chapter 6, where we discuss the issue of hypoellipticity of Hörmander type. Malliavin originally built his theory to give a probabilistic proof to this problem. His work was followed up by S. Kusuoka and D. Stroock. They investigated

Malliavin's work and sharpened it to its present form. We discuss this problem following the Kusuoka-Stroock method. They piled up enormous estimates towards the non-degeneracy of Malliavin's covariance matrix. The reader will find here the real taste of analysis.

**Some Frequently-used Notation.**

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the set of natural numbers, integers, rational numbers, real numbers, complex numbers, respectively. A suffix $+$ refers to non-negative numbers. For instance, $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.
- $\forall$ means "for all" and $\exists$ means "there exist."
- $C^n(\mathbb{R}^d)$ stands for the space of all functions on $\mathbb{R}^d$ of class $C^n$. A suffix $b$ refers to bounded functions, a suffix 0 to functions with compact support and a suffix $+$ to non-negative functions. In general, functions are real valued. To specify the space of values, we write, e.g., $C^n(\mathbb{R} \to \mathbb{R}^k)$. $L^p$ stands for the space of all $p$-th integrable functions, and we write $L^p(\mu)$ if we need to specify the measure $\mu$. If the measure is clear in the context, we sometimes write, e.g., $L^p([0, \infty))$. $L^p$ functions are usually real valued. To specify a space of values, we write, e.g., $L^p(\mu; K)$.
- A point in $\mathbb{R}^d$ is denoted by $x = (x^1, \dots, x^d)$ with superscript. A partial derivative of function $f$ is denoted by $\frac{\partial f}{\partial x^j}$ or simply by $\partial_j f$.
- $\delta_{ij}$ is Kronecker's delta.
- $a \wedge b$ stands for the minimum of $a$ and $b$, $a \vee b$ for the maximum.