Preface

Hilbert C^*-modules provide a natural generalization of Hilbert spaces arising when the field of scalars **C** is replaced by an arbitrary C^*-algebra. This generalization, in the case of commutative C^*-algebras, appeared in the paper [**59**] of I. Kaplansky; however the noncommutative case seemed too complicated at that time. The general theory of Hilbert C^*-modules (i.e., for an arbitrary C^*-algebra serving as 'scalars') appeared 25 years ago in the pioneering papers of W. Paschke [**100**] and M. Rieffel [**108**]. This theory has proved to be a very convenient tool in the theory of operator algebras, allowing one to obtain information about C^*-algebras by studying Hilbert C^*-modules over them. In particular, a series of results about some classes of C^*-algebras (like AW^*-algebras and monotone complete C^*-algebras) was obtained in this way [**38**]. An important notion of Morita equivalence for C^*-algebras was also formulated in terms of Hilbert C^*-modules [**109, 18**]. This notion also has applications in the theory of group representations. It turned out to be possible to extract information on group actions from Hilbert C^*-modules arising from these actions [**102, 110**]. Some results about conditional expectations of finite index [**5, 133**] and about completely positive maps of C^*-algebras [**2**] were also obtained using Hilbert C^*-modules.

The theory of Hilbert C^*-modules may also be considered as a noncommutative generalization of the theory of vector bundles [**33, 69**]. This was the reason Hilbert modules became a tool in topological applications — namely in index theory of elliptic operators, in K- and KK-theory [**92, 90, 62, 63, 64, 65, 128**] and in noncommutative geometry as a whole [**24, 29**].

Among other applications, one should emphasize the theory of quantum groups [**135, 136**], unbounded operators as a tool for Kasparov's KK-theory [**3, 4**] (also Section 8.4) and some physical applications [**72, 80**].

Alongside these applications, the theory of Hilbert C^*-modules itself has been developed as well. A number of results about the structure of Hilbert modules and about operators on them have been obtained [**74, 42, 88, 77, 81, 127**]. Besides these results, an axiomatic approach in the theory of Hilbert modules based on the theory of operator spaces and tensor products was developed [**12, 11**].

A detailed bibliography of the theory of Hilbert C^*-modules can be found in [**43**].

Some results presented here were only announced in the literature or the proofs were discussed rather briefly. We have tried to fill such gaps. We could not discuss all the aspects of the theory of Hilbert C^*-modules here, but we tried to explain in detail the basic notions and theorems of this theory, a number of important examples, and also some results related to the authors' interest.

A major part of this book formed the content of the lecture course presented by the authors at the Department of Mechanics and Mathematics of Moscow State University in 1997.

We are grateful to A. S. Mishchenko for introducing the theory of Hilbert C^*-modules to us. Together with Yu. P. Solovyov, he has acquainted us with the circle of problems related to its applications in topology.

While working in this field and in the process of writing the present text, a significant influence on us was made by our friend and co-author M. Frank.

We have discussed a number of problems of the Hilbert C^*-module theory with L. Brown, A. A. Irmatov, G. G. Kasparov, R. Nest, G. K. Pedersen and W. Paschke. Some applications were considered also with J. Cuntz, A. Ya. Helemskii, J. Kaminker, V. Nistor, J. Rosenberg, K. Thomsen, B. L. Tsygan and others.

Our research was partially supported by a series of subsequent Russian Foundation of Basic Research grants.