CHAPTER 1

Basic Definitions

1.1. C^*-algebras

For basic information about C^*-algebras we refer to the books [**31, 104, 123, 58**]. We present here only some results on C^*-algebras, which will be necessary for our purpose.

Recall that an involutive Banach algebra A is called a *C^*-algebra* if the equality

$$\|a^*a\| = \|a\|^2$$

holds for each element $a \in A$. Any C^*-algebra can be realized as a norm-closed subalgebra of the algebra of bounded operators $\mathcal{B}(H)$ on a Hilbert space H. We do not assume existence of the unit element in C^*-algebras. By A^+ we denote the C^*-algebra obtained from the C^*-algebra A by *unitalization.* As a linear space with involution, A^+ coincides with $A \oplus \mathbf{C}$ and multiplication in A^+ is given by the formula $(a, z)(b, w) = (ab + zb + aw, zw)$, $a, b \in A$, $z, w \in \mathbf{C}$. Any pair $(a, z) \in A^+$ defines an operator $A \to A$ by $b \mapsto ab + zb$ and the norm of (a, z) is the norm of this operator.

The *spectrum* of an element a of a unital C^*-algebra is the set $\mathrm{Sp}(a)$ of complex numbers z such that $a - z \cdot 1$ is not invertible. If a C^*-algebra A has no unit, then the spectrum of an element $a \in A$ is defined as its spectrum in the C^*-algebra $A^+ \supset A$. The spectrum is a compact subset of $\mathbf{C}$. An element $a \in A$ is called *positive* (we write $a \geq 0$) if it is *Hermitian*, i.e., if it satisfies the condition $a^* = a$, and if one of the following equivalent [**31**, 1.6.1] conditions holds:

(i) $\mathrm{Sp}(a) \subset [0, \infty)$;
(ii) $a = b^*b$ for some $b \in A$;
(iii) $a = h^2$ for some Hermitian $h \in A$.

The set of all positive elements $P^+(A)$ forms a closed convex cone in A and $P^+(A) \cap (-P^+(A)) = 0$. Among the Hermitian elements h defined in (iii) there exists a unique positive element, which is called the *positive square root* of a (we write $h = a^{1/2}$).

A linear functional $\varphi : A \longrightarrow \mathbf{C}$ is called *positive* if $\varphi(a) \geq 0$ for any positive element $a \in P^+(A)$. A positive linear functional is called a *state* if $\|\varphi\| = 1$. We have $\|a\| = \sup \varphi(a)$, where $a \geq 0$ and the supremum is taken over all states.

A C^*-homomorphism of an algebra A into the C^*-algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H is called a *representation*. A vector $\xi \in H$ is called *cyclic* for the representation $\pi : A \longrightarrow \mathcal{B}(H)$ if the set of all vectors of the form $\pi(a)\xi$, $a \in A$, is dense in H. The vector $\xi \in H$ is called *separating* for the representation $\pi : A \longrightarrow \mathcal{B}(H)$ if the equality $\pi(a)\xi = 0$ implies $a = 0$.

To each positive linear functional ω on a C^*-algebra A we can associate a unique (up to the unitary equivalence) representation π_ω of the algebra A on some

Hilbert space H_ω and a vector $\xi_\omega \in H_\omega$ such that $\omega(a) = (\pi_\omega(a)\xi_\omega, \xi_\omega)$ for all $a \in A$ and the vector ξ_ω is cyclic. The construction of such a representation is called the *GNS-construction.*

An *approximate unit* of a C^*-algebra A is an increasing net $e_\alpha \in A$, $\alpha \in \mathcal{A}$, such that $\|e_\alpha\| \le 1$ and $\lim \|a - ae_\alpha\| = 0$ for any $a \in A$. Each C^*-algebra has an approximate unit e_α such that $e_\alpha \ge 0$ and $e_\alpha \ge e_\beta$ for $\alpha \ge \beta$ [**31**].

DEFINITION 1.1.1. A C^*-algebra possessing a countable approximate unit is called *σ-unital.*

DEFINITION 1.1.2. An element $h \in A$ is called *strictly positive* if for any positive nonzero linear functional φ (or, equally, for any state) one has $\varphi(h) > 0$.

REMARK 1.1.3. Existence of a strictly positive element is equivalent to existence of a countable approximate unit. One can assume that $e_i \ge 0$. Then $h := \sum_i e_i/2^i$ is strictly positive. Conversely, $e_i := h^{1/i}$ is a countable approximate unit. Any separable C^*-algebra is σ-unital. The details can be found in [**104**].

We will often use the following statements.

LEMMA 1.1.4 ([**104**, Lemma 1.4.4]). *Let x, y and a be elements of a C^*-algebra A such that $a \ge 0$ and*

$$x^*x \le a^\alpha, \qquad yy^* \le a^\beta, \qquad \alpha + \beta > 1.$$

Then the sequence $u_n = x[(1/n) + a]^{-1/2} y$ is norm-convergent in A to an element u such that $\|u\| \le \|a^{(\alpha+\beta-1)/2}\|$.

PROOF. Put $d_{nm} := [(1/n) + a]^{-1/2} - [(1/m) + a]^{-1/2}$. Then

$$\begin{aligned}\|u_n - u_m\|^2 &= \|xd_{nm}y\|^2 = \|y^*d_{nm}x^*xd_{nm}y\| \\ &\le \|y^*d_{nm}a^\alpha d_{nm}y\| = \|a^{\alpha/2}d_{nm}y\|^2 \\ &= \|a^{\alpha/2}d_{nm}yy^*d_{nm}a^{\alpha/2}\| \le \|a^{\alpha/2}d_{nm}a^\beta d_{nm}a^{\alpha/2}\| = \|d_{nm}a^{(\alpha+\beta)/2}\|^2.\end{aligned}$$

Using, for example, the Dini theorem, we can see that the sequence of functions

$$[(1/n) + t]^{-1/2} t^{(\alpha+\beta)/2}, \qquad t \in \mathrm{Sp}(a),$$

is uniformly convergent to $t^{(\alpha+\beta-1)/2}$ on the spectrum of a, hence, $\|d_{nm}a^{(\alpha+\beta)/2}\| \to 0$. Therefore, by the Cauchy criterion, $\{u_n\}$ is norm-convergent to an element $u \in A$. Then, reasoning as above, we obtain

$$\|u_n\| = \|x[(1/n) + a]^{-1/2}y\| \le \|a^{\alpha/2}[(1/n) + a]^{-1/2}a^{\beta/2}\| \le \|a^{(\alpha+\beta-1)/2}\|.$$

Hence $\|u\| \le \|a^{(\alpha+\beta-1)/2}\|$. □

PROPOSITION 1.1.5 ([**104**, Prop. 1.4.5]). *Let x and a be elements of a C^*-algebra A such that $a \ge 0$ and $x^*x \le a$. For any $0 < \alpha < \frac{1}{2}$ there exists an element $u \in A$ such that $\|u\| \le \|a^{\frac{1}{2}-\alpha}\|$ and $x = ua^\alpha$.*

PROOF. Put $u_n := x[(1/n) + a]^{-\frac{1}{2}} a^{\frac{1}{2}-\alpha}$. By Lemma 1.1.4, $\{u_n\}$ is norm-convergent to an element $u \in A$ such that

$$\|u\| \le \|a^{\frac{1}{2}(1+1-2\alpha-1)}\| = \|a^{\frac{1}{2}-\alpha}\|.$$

Then

$$\begin{aligned}\|x - u_n a^\alpha\|^2 &= \|x(1 - [(1/n) + a]^{-1/2} a^{1/2})\|^2 \\ &\le \|a^{1/2}(1 - [(1/n) + a]^{-1/2} a^{1/2})\|^2 \longrightarrow 0\end{aligned}$$

as $n \longrightarrow \infty$, by the Dini theorem applied to the corresponding functions on the spectrum. Thus, $x = ua^\alpha$. □

1.2. Pre-Hilbert modules

Let $\mathcal{M}$ be a module over a C^*-algebra A. An action of an element $a \in A$ on $\mathcal{M}$ is denoted by $x \cdot a$, where $x \in \mathcal{M}$.

DEFINITION 1.2.1. A *pre-Hilbert A-module* is a (right) A-module $\mathcal{M}$ equipped with a sesquilinear form $\langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \longrightarrow A$ with the following properties:

(i) $\langle x, x \rangle \ge 0$ for any $x \in \mathcal{M}$;
(ii) $\langle x, x \rangle = 0$ implies that $x = 0$;
(iii) $\langle y, x \rangle = \langle x, y \rangle^*$ for any $x, y \in \mathcal{M}$;
(iv) $\langle x, y \cdot a \rangle = \langle x, y \rangle a$ for any $x, y \in \mathcal{M}$ and any $a \in A$.

The map $\langle \cdot, \cdot \rangle$ is called an *A-valued inner product.*

Here are some examples.

EXAMPLE 1.2.2. Let $J \subset A$ be a right ideal. Then J can be equipped with the structure of a pre-Hilbert A-module with the inner product of elements $x, y \in J$ defined by $\langle x, y \rangle := x^* y$.

EXAMPLE 1.2.3. Let $\{J_i\}$ be a countable set of right ideals of a C^*-algebra A and let $\mathcal{M}$ be the linear space of all sequences (x_i), $x_i \in J_i$, satisfying the condition $\sum_i \|x_i\|^2 < \infty$. Then $\mathcal{M}$ becomes a right A-module if the action of A is defined by $(x_i) \cdot a := (x_i a)$ for $(x_i) \in \mathcal{M}$, $a \in A$, and becomes a pre-Hilbert A-module if the inner product of elements $(x_i), (y_i) \in \mathcal{M}$ is defined by $\langle (x_i), (y_i) \rangle := \sum_i x_i^* y_i$.

Let $\mathcal{K}$ be a right A-module equipped with a sesquilinear map $[\cdot, \cdot] : \mathcal{K} \times \mathcal{K} \longrightarrow A$ satisfying all properties of Definition 1.2.1 except (ii). Put

$$N := \{x \in \mathcal{K} : [x, x] = 0\}.$$

For each positive linear functional φ on the C^*-algebra A the map $(x, y) \mapsto \varphi([x, y])$ is a (degenerate) inner product on $\mathcal{K}$, hence the set $N_\varphi = \{x \in \mathcal{K} : \varphi([x, x]) = 0\}$ is a linear subspace in $\mathcal{K}$. By taking the intersection of all such subspaces we see that $N = \bigcap_\varphi N_\varphi$ is also a linear subspace in $\mathcal{K}$. It follows from properties (iii) and (iv) of Definition 1.2.1 that $N \cdot A \subset N$. Therefore N is a submodule in $\mathcal{K}$. The quotient module $\mathcal{M} = \mathcal{K}/N$ is equipped with the obvious structure of a pre-Hilbert A-module with the inner product $\langle x + N, y + N \rangle := [x, y]$.

Let $\mathcal{M}$ be a pre-Hilbert A-module, $x \in \mathcal{M}$. Put $\|x\|_{\mathcal{M}} := \|\langle x, x \rangle\|^{1/2}$. We usually skip the subscript $\mathcal{M}$ when it does not lead to confusion of norms.

PROPOSITION 1.2.4 ([**100**]). *The function $\|\cdot\|_{\mathcal{M}}$ is a norm on $\mathcal{M}$ and satisfies the following properties:*

(i) $\|x \cdot a\|_{\mathcal{M}} \le \|x\|_{\mathcal{M}} \cdot \|a\|$ *for any* $x \in \mathcal{M}$, $a \in A$;
(ii) $\langle x, y \rangle \langle y, x \rangle \le \|y\|_{\mathcal{M}}^2 \langle x, x \rangle$ *for any* $x, y \in \mathcal{M}$;
(iii) $\|\langle x, y \rangle\| \le \|x\|_{\mathcal{M}} \|y\|_{\mathcal{M}}$ *for any* $x, y \in \mathcal{M}$.

PROOF. For any positive linear functional φ on A, the function $x \mapsto \varphi(\langle x, x\rangle)^{1/2}$ defines a seminorm on $\mathcal{M}$. For each $x \in \mathcal{M}$,

$$\|x\|_{\mathcal{M}} = \|\langle x, x\rangle\|^{1/2} = \sup\{\varphi(\langle x, x\rangle)^{1/2}\},$$

where the supremum is taken over all states φ on A. Therefore $\|\cdot\|_{\mathcal{M}}$ is a seminorm and, by property (ii) of Definition 1.2.1, $\|\cdot\|_{\mathcal{M}}$ is a norm on $\mathcal{M}$. Statement (i) follows from the equality

$$\|x \cdot a\|^2_{\mathcal{M}} = \|\langle x \cdot a, x \cdot a\rangle\| = \|a^* \langle x, x\rangle a\| \leq \|a\|^2 \|\langle x, x\rangle\| = \|x\|^2_{\mathcal{M}} \|a\|^2 .$$

To prove (ii) we take $x, y \in \mathcal{M}$ and a positive linear functional φ on A. Applying the Cauchy–Bunyakovskii inequality for the (degenerate) inner product $\varphi(\langle\cdot,\cdot\rangle)$ on $\mathcal{M}$ we obtain

$$\begin{aligned}\varphi(\langle x, y\rangle\langle y, x\rangle) &= \varphi(\langle x, y \cdot \langle y, x\rangle\rangle)\\ &\leq \varphi(\langle x, x\rangle)^{1/2} \cdot \varphi(\langle y \cdot \langle y, x\rangle, y \cdot \langle y, x\rangle\rangle)^{1/2}\\ &= \varphi(\langle x, x\rangle)^{1/2} \cdot \varphi(\langle x, y\rangle\langle y, y\rangle\langle y, x\rangle)^{1/2}\\ &\leq \varphi(\langle x, x\rangle)^{1/2} \cdot \|\langle y, y\rangle\|^{1/2} \cdot \varphi(\langle x, y\rangle\langle y, x\rangle)^{1/2}.\end{aligned}$$

Thus, for any positive linear functional φ, we have $\varphi(\langle y, x\rangle\langle x, y\rangle) \leq \|y\|^2_{\mathcal{M}} \cdot \varphi(\langle x, x\rangle)$, so statement (ii) is proved. This evidently implies statement (iii). □

We call inequality (ii) (and also its consequence, inequality (iii)) of Proposition 1.2.4 the *Cauchy–Bunyakovskii inequality* for Hilbert C^*-modules.

REMARK 1.2.5. For any C^*-pre-Hilbert module, or more precisely, for any sesquilinear form $\langle\cdot,\cdot\rangle$, the *polarization equality*

$$4\langle y, x\rangle = \sum_{k=0}^{3} i^k \langle x + i^k y, x + i^k y\rangle$$

is obviously satisfied for all x, $y \in \mathcal{M}$.

1.3. Hilbert C^*-modules

DEFINITION 1.3.1. An A-module $\mathcal{M}$ that is at the same time a Banach space with a norm $\|\cdot\|$ satisfying the inequality $\|x \cdot a\| \leq \|x\| \, \|a\|$, $x \in \mathcal{M}$, $a \in A$, is called a *Banach* A-module.

DEFINITION 1.3.2. A pre-Hilbert A-module $\mathcal{M}$ is called a *Hilbert C^*-module* if it is complete with respect to the norm $\|\cdot\|_{\mathcal{M}}$.

If $\mathcal{M}$ is a pre-Hilbert A-module, then the action of the C^*-algebra A and the A-valued inner product on $\mathcal{M}$ can be extended to the completion $\widetilde{\mathcal{M}}$, which thus becomes a Hilbert C^*-module. Consider some examples.

EXAMPLE 1.3.3. If $J \subset A$ is a right ideal, then the pre-Hilbert module J is complete with respect to the norm $\|\cdot\|_J = \|\cdot\|$. In particular, the C^*-algebra A itself is a *free* Hilbert A-module with one generator.

EXAMPLE 1.3.4. If $\{\mathcal{M}_i\}$ is a finite set of Hilbert A-modules, then one can define the direct sum $\oplus\mathcal{M}_i$. The inner product on $\oplus\mathcal{M}_i$ is given by the formula $\langle x, y\rangle := \sum_i \langle x_i, y_i\rangle$, where $x = (x_i), y = (y_i) \in \oplus\mathcal{M}_i$. We denote the direct sum of n copies of a Hilbert module $\mathcal{M}$ by $\mathcal{M}^n$ or $L_n(\mathcal{M})$.

Here and further we denote by $\oplus$ the direct sum of orthogonal submodules and by $\widetilde{\oplus}$ the direct sum of Banach subspaces without orthogonality.

EXAMPLE 1.3.5. If $\{\mathcal{M}_i\}$, $i \in \mathbf{N}$, is a countable set of Hilbert A-modules, then one can define their direct sum $\oplus\mathcal{M}_i$. On the A-module $\oplus\mathcal{M}_i$ of all sequences $x = (x_i) : x_i \in \mathcal{M}_i$, such that the series $\sum_i \langle x_i, x_i\rangle$ is norm-convergent in the C^*-algebra A, we define the inner product by

$$\langle x, y\rangle := \sum\nolimits_i \langle x_i, y_i\rangle \quad \text{for } x, y \in \oplus\mathcal{M}_i.$$

Let us check that this series converges. Since the series $\sum_i \langle x_i, x_i\rangle$ and $\sum_i \langle y_i, y_i\rangle$ are convergent, for any $\varepsilon > 0$ there exists a number N such that for all $n > 0$ we have

$$\left\| \sum_{i=N}^{N+n} \langle x_i, x_i\rangle \right\| < \varepsilon, \quad \left\| \sum_{i=N}^{N+n} \langle y_i, y_i\rangle \right\| < \varepsilon.$$

Then

$$\left\| \sum_{i=N}^{N+n} \langle x_i, y_i\rangle \right\| \le \left\| \sum_{i=N}^{N+n} \langle x_i, x_i\rangle \right\| \cdot \left\| \sum_{i=N}^{N+n} \langle y_i, y_i\rangle \right\| < \varepsilon^2.$$

This proves that the inner product is well defined.

Let us verify completeness of the module $\oplus\mathcal{M}_i$. Let $x^{(n)} = (x_i^{(n)}) \in \oplus\mathcal{M}_i$ be a Cauchy sequence. Then for any $\varepsilon > 0$ there exists a number N such that

$$\left\| \sum_i \langle x_i^{(n)} - x_i^{(m)}, x_i^{(n)} - x_i^{(m)}\rangle \right\| < \varepsilon \tag{1.1}$$

for all $n, m \ge N$. Since all summands in (1.1) are positive, the inequality

$$\left\| \langle x_i^{(n)} - x_i^{(m)}, x_i^{(n)} - x_i^{(m)}\rangle \right\| < \varepsilon$$

holds for each number i separately. But then the sequences $x_i^{(n)} \in \mathcal{M}_i$ are Cauchy sequences, hence they converge to the limits $x_i = \lim x_i^{(n)} \in \mathcal{M}_i$. Let us verify that the series $\sum_i \langle x_i, x_i\rangle$ is norm-convergent in A. Let us fix $\varepsilon > 0$. There exists a number $n > N$ such that the estimate (1.1) holds. Let us choose a number K such that

$$\left\| \sum_{i=K}^{\infty} \langle x_i^{(n)}, x_i^{(n)}\rangle \right\| < \varepsilon.$$

Then, for any $k > 0$, we have

$$\left\| \sum_{i=K}^{K+k} \left(\langle x_i^{(m)}, x_i^{(m)}\rangle + \langle x_i^{(n)} - x_i^{(m)}, x_i^{(m)}\rangle + \langle x_i^{(m)}, x_i^{(n)} - x_i^{(m)}\rangle + \langle x_i^{(n)}, x_i^{(n)}\rangle \right) \right\|$$
$$= \left\| \sum_{i=K}^{K+k} \langle x_i^{(n)} - x_i^{(m)}, x_i^{(n)} - x_i^{(m)}\rangle \right\| \le \left\| \sum_{i=1}^{\infty} \langle x_i^{(n)} - x_i^{(m)}, x_i^{(n)} - x_i^{(m)}\rangle \right\| < \varepsilon.$$

Therefore

$$\left\|\sum_{i=K}^{K+k}\langle x_i^{(m)},x_i^{(m)}\rangle\right\| < 2\varepsilon + \left\|\sum_{i=K}^{K+k}\langle x_i^{(n)}-x_i^{(m)},x_i^{(m)}\rangle\right\| + \left\|\sum_{i=K}^{K+k}\langle x_i^{(m)},x_i^{(n)}-x_i^{(m)}\rangle\right\|$$
$$\leq 2\varepsilon + 2\left\|\sum_{i=K}^{K+k}\langle x_i^{(n)}-x_i^{(m)},x_i^{(n)}-x_i^{(m)}\rangle\right\|^{1/2}\left\|\langle x_i^{(m)},x_i^{(m)}\rangle\right\|^{1/2}$$
$$\leq 2\varepsilon + 2\varepsilon^{1/2}\left\|\langle x_i^{(m)},x_i^{(m)}\rangle\right\|^{1/2}.$$

Now, by solving the quadratic inequality, we obtain that

$$\left\|\sum_{i=K}^{K+k}\langle x_i^{(m)},x_i^{(m)}\rangle\right\| < (1+\sqrt{3})^2\varepsilon < 8\varepsilon. \tag{1.2}$$

Passing to the limit $m\to\infty$ in the inequality (1.2), we obtain that

$$\left\|\sum_{i=K}^{K+k}\langle x_i,x_i\rangle\right\| < 8\varepsilon.$$

This proves that the series $\sum_i\langle x_i,x_i\rangle$ is norm-convergent.

The direct sum of a countable number of copies of a Hilbert module $\mathcal{M}$ is denoted by $l_2(\mathcal{M})$ or $H_{\mathcal{M}}$. The Hilbert C^*-module $l_2(A)$ (another notation is H_A) is called the *standard* Hilbert C^*-module over A. If the C^*-algebra is unital, then the Hilbert module H_A possesses the standard basis $\{e_i\}$, $i\in\mathbf{N}$, where $e_i = (0,\dots,0,1,0,\dots,0,\dots)$ with the unit being the i-th entry.

Example 1.3.6. Let $B\subset A$ be a C^*-subalgebra of a C^*-algebra A. Let A and B be unital with the common unit. Assume that there exists a linear map $E: A\longrightarrow B$ that is a projection (i.e., $E^2=E$) of norm ≤ 1. Such a map is called a *conditional expectation* from A to B. Conditional expectation is a positive map, i.e., $E(a^*a)\geq 0$ for all $a\in A$, and it satisfies the equality

$$E(b_1ab_2) = b_1E(a)b_2 \quad \text{for } a\in A, b_1,b_2\in B$$

(see [**123**]). A conditional expectation is called *faithful* if, for any positive element $a\in P^+(A)$, the equality $E(a)=0$ implies $a=0$. When the conditional expectation is faithful, one can introduce the structure of a pre-Hilbert B-module on the C^*-algebra A by

$$\langle x,y\rangle = E(x^*y),\quad x,y\in A.$$

We will give a condition for this module to be a Hilbert C^*-module (i.e., to be complete) in Section 4.5.

Let $\mathcal{N}\subset\mathcal{M}$ be a closed submodule of a Hilbert C^*-module $\mathcal{M}$. We define the *orthogonal complement* $\mathcal{N}^\perp$ by the formula

$$\mathcal{N}^\perp = \{y\in\mathcal{M} : \langle x,y\rangle = 0 \quad \text{for all } x\in\mathcal{N}\}.$$

Then $\mathcal{N}^\perp$ is a closed submodule of the Hilbert C^*-module $\mathcal{M}$ too. However, the equality $\mathcal{M}=\mathcal{N}\oplus\mathcal{N}^\perp$ does not always hold, as the following example shows.

Example 1.3.7. Let $A=C[0,1]$ be the C^*-algebra of all continuous functions on the segment $[0,1]$. Consider, in the Hilbert A-module $\mathcal{M}=A$, the submodule $\mathcal{N}=C_0(0,1)$ of functions that vanish at the end points of the segment. Then, obviously, $\mathcal{N}^\perp = 0$.

If $\mathcal{M}$ is a Hilbert A-module, then we denote by $\mathcal{M}\cdot A$ the closure in $\mathcal{M}$ of the linear span of all the elements of the form $x\cdot a$, $x\in\mathcal{M}$, $a\in A$.

LEMMA 1.3.8. $\mathcal{M}\cdot A=\mathcal{M}$.

PROOF. Let $e_\alpha\in A$ be an approximate unit. Then, for any $x\in\mathcal{M}$,

$$\begin{aligned}\|x-x\cdot e_\alpha\|^2 &= \|\langle x-x\cdot e_\alpha, x-x\cdot e_\alpha\rangle\| \\ &\le (1+\|e_\alpha^*\|)\,\|\langle x,x\rangle-\langle x,x\rangle e_\alpha\|\to 0,\end{aligned}$$

hence the elements of the form $x\cdot e_\alpha$ are dense in $\mathcal{M}$. □

We will often use the following statement.

LEMMA 1.3.9. *For any* $x\in\mathcal{M}$

$$x=\lim_{\varepsilon\to 0}x\langle x,x\rangle(\langle x,x\rangle+\varepsilon)^{-1}.$$

PROOF. Let $\langle x,x\rangle=a$. Then

$$\begin{aligned}&\|x\langle x,x\rangle(\langle x,x\rangle+\varepsilon)^{-1}-x\|^2\\ &\quad=\|\langle x(\langle x,x\rangle(\langle x,x\rangle+\varepsilon)^{-1}-1),x(\langle x,x\rangle(\langle x,x\rangle+\varepsilon)^{-1}-1)\rangle\|\\ &\quad=\|a(a^2(a+\varepsilon)^{-2}-2a(a+\varepsilon)^{-1}+1)\|\\ &\quad=\|a^3(a+\varepsilon)^{-2}-2a^2(a+\varepsilon)^{-1}+a\|\longrightarrow 0,\end{aligned}$$

since the following inequalities hold under the condition $t\ge 0$:

$$\begin{aligned}|t^3(t+\varepsilon)^{-2}-t| &= \left|t\left(\left(\frac{t}{t+\varepsilon}\right)^2-1\right)\right| = \left|t\left(\frac{-\varepsilon^2-2\varepsilon t}{(t+\varepsilon)^2}\right)\right|\\ &=\varepsilon\left|\frac{\varepsilon t+2t^2}{(t+\varepsilon)^2}\right|\le\varepsilon\left(\frac12+2\right)=\frac32\,\varepsilon\end{aligned}$$

and

$$|t^2(t+\varepsilon)^{-1}-t|=\left|\frac{t\varepsilon}{t+\varepsilon}\right|<\varepsilon.$$

□

The following statement is an analog of the polar decomposition for Hilbert C^*-modules. We will see below that, similarly to the case of C^*-algebras, the exact polar decomposition exists only in the case of Hilbert C^*-modules over W^*-algebras.

PROPOSITION 1.3.10 ([**71**]). *Let* $\mathcal{M}$ *be a Hilbert* A*-module,* $x\in\mathcal{M}$*, and* $0<\alpha<1/2$*. Then there exists an element* $z\in\mathcal{M}$ *such that* $x=z\cdot\langle x,x\rangle^\alpha$*.*

PROOF. For $n\in\mathbf{N}$ put

$$g_n(\lambda)=\begin{cases}n^{-\alpha/2}, & \text{if } \lambda\le 1/n,\\ \lambda^{\alpha/2}, & \text{if } \lambda>1/n.\end{cases}$$

Then, by the spectral theorem,

$$\begin{aligned}\|x\cdot(g_n(\langle x,x\rangle)-g_m(\langle x,x\rangle))\| &= \left\|\langle x,x\rangle(g_n(\langle x,x\rangle)-g_m(\langle x,x\rangle))^2\right\|^{1/2}\\ &=\sup\{|\lambda(g_n(\lambda)-g_m(\lambda))|:\lambda\in\mathrm{Sp}(\langle x,x\rangle)\}.\end{aligned}$$

Therefore the sequence $x \cdot g_n(\langle x,x\rangle)$ is a Cauchy sequence, so it has a limit $z \in \mathcal{M}$. Then

$$\begin{aligned}\|z\langle x,x\rangle^\alpha - x\| &= \lim_{n\to\infty} \|x \cdot g_n(\langle x,x\rangle)\langle x,x\rangle^\alpha - x\| \\ &= \lim_{n\to\infty} \|x(g_n(\langle x,x\rangle)\langle x,x\rangle^\alpha - 1)\| \\ &= \lim_{n\to\infty} \sup\{|\lambda^{1/2}(g_n(\lambda)\lambda^\alpha - 1)| : \lambda \in \mathrm{Sp}\,\langle x,x\rangle\} = 0.\end{aligned}$$

This completes the proof. □

1.4. The standard Hilbert module H_A

DEFINITION 1.4.1. A Hilbert C^*-module $\mathcal{M}$ is called *finitely generated* if there exists a finite set $\{x_i\} \subset \mathcal{M}$ such that $\mathcal{M}$ equals the linear span (over $\mathbf{C}$ and A) of this set. A Hilbert C^*-module $\mathcal{M}$ is called *countably generated* if there exists a countable set $\{x_i\} \subset \mathcal{M}$ such that $\mathcal{M}$ equals the norm-closure of the linear span (over $\mathbf{C}$ and A) of this set.

THEOREM 1.4.2 (Kasparov stabilization theorem, [**63**]). *Let A be a C^*-algebra and $\mathcal{M}$ a countably generated Hilbert A-module. Then $\mathcal{M} \oplus H_A \cong H_A$.*

PROOF. We start by proving the theorem for the case where A is unital. It is convenient here to use the procedure of almost orthogonalization [**36**]. An element x of the Hilbert C^*-module $\mathcal{N}$ is called *nonsingular* if the element $\langle x,x\rangle \in A$ is invertible. The set $\{x_i\} \in \mathcal{N}$ is called *orthonormal* if $\langle x_i, x_j\rangle = \delta_{ij}$. It is called a *basis* of the module $\mathcal{N}$ if finite sums of the form $\sum_i x_i \cdot a_i$, $a_i \in A$, are dense in $\mathcal{N}$.

LEMMA 1.4.3 ([**36**]). *Let $\mathcal{N}$ be a Hilbert A-module that contains the orthonormal elements $e_1, \dots, e_n$, $x \in \mathcal{N}$, $\varepsilon > 0$. If an element $y \in \mathcal{N}$ satisfies $\langle y,y\rangle = 1$ and $y \perp \{x, e_1, \dots, e_n\}$, then there exists an element $e_{n+1} \in \mathcal{N}$ such that*

(i) *the elements $e_1, \dots, e_n, e_{n+1}$ are orthonormal,*
(ii) $e_{n+1} \in \mathrm{Span}_A(e_1, \dots, e_n, x, y)$,
(iii) $\mathrm{dist}(x, \mathrm{Span}_A(e_1, \dots, e_{n+1})) \le \varepsilon$.

PROOF. Let

$$x' = x - \sum_{i=1}^n e_i\langle e_i, x\rangle, \qquad x'' = x' + \varepsilon y.$$

Then

$$\langle x'', x''\rangle = \langle x', x'\rangle + \varepsilon^2 \ge \varepsilon^2 > 0.$$

Therefore the element x'' is nonsingular. Put $e_{n+1} = x'' \cdot \langle x'', x''\rangle^{-1/2}$. Then

$$e_{n+1} \in \mathrm{Span}_A(x', y) \perp \{e_1, \dots, e_n\}.$$

Therefore the elements $e_1, \dots, e_n, e_{n+1}$ are orthonormal. Since we have taken $x' \in \mathrm{Span}_A(x, e_1, \dots, e_n)$ and $e_{n+1} \in \mathrm{Span}_A(x', y)$, we obtain (ii). Finally, put

$$w = e_{n+1}\langle x'', x''\rangle^{1/2} + \sum_{i=1}^n e_i\langle e_i, x\rangle \in \mathrm{Span}_A(e_1, \dots, e_{n+1}).$$

Then the equality $\|w - x\| = \|x'' - x'\| = \|\varepsilon y\| = \varepsilon$ proves (iii). □

We return now to the proof of Theorem 1.4.2. Let $\{y_n\}$ be the sequence of all generators of module $\mathcal{M}$. By $\{e_n\}$ we denote the standard basis of the module H_A. Let $\{x_n\} \subset \{e_n\} \cup \{y_n\}$ be a sequence, in which one meets each element e_n and each element y_n *infinitely* many times. Then the set $\{x_n\}$ is generating for the module $\mathcal{M} \oplus H_A$. We will prove the theorem by induction. Let us assume that the orthonormal elements $\overline{e}_1, \dots, \overline{e}_n \in \mathcal{M} \oplus H_A$ and the number $m(n) \geq n$ are already constructed in such a way that

(i) $\{\overline{e}_1, \dots, \overline{e}_n\} \subset \operatorname{Span}_A(x_1, \dots, x_n, e_1, \dots, e_{m(n)})$,
(ii) $\operatorname{dist}(x_k, \operatorname{Span}_A(\overline{e}_1, \dots, \overline{e}_k)) \leq \frac{1}{k}$, $1 \leq k \leq n$.

Since each element x_i is equal to e_j or y_k, one can find a number $m' > m(n)$ such that $e_{m'} \perp \{x_1, \dots, x_{n+1}\}$. Since $e_{m'} \perp \{e_1, \dots, e_{m(n)}\}$, it follows from (i) that

$$e_{m'} \perp \{x_{n+1}, \overline{e}_1, \dots, \overline{e}_n\}.$$

By Lemma 1.4.3, there exists an element

$$\overline{e}_{n+1} \in \operatorname{Span}_A(\overline{e}_1, \dots, \overline{e}_n, x_{n+1}, e_{m'}) \tag{1.3}$$

such that the elements $\overline{e}_1, \dots, \overline{e}_n, \overline{e}_{n+1}$ are orthonormal and

$$\operatorname{dist}(x_{n+1}, \operatorname{Span}_A(\overline{e}_1, \dots, \overline{e}_{n+1})) \leq \frac{1}{n+1}.$$

It follows from (1.3) and from condition (i) that

$$\{\overline{e}_1, \dots, \overline{e}_{n+1}\} \subset \operatorname{Span}_A(x_1, \dots, x_{n+1}, e_1, \dots, e_{m'}).$$

By setting $m(n+1) = m'$, we complete the induction step. Thus, an orthonormal sequence $\overline{e}_n$ satisfying properties (i) and (ii) has been constructed. But property (ii) means that this sequence generates the whole module $\mathcal{M} \oplus H_A$, so $\mathcal{M} \oplus H_A \cong H_A$.

Thus, Theorem 1.4.2 is proved for unital C^*-algebras. Let A be a nonunital C^*-algebra and let A^+ be its unitalization. Defining the action of A^+ on the Hilbert A-module $\mathcal{M}$ by the formula $x \cdot (a, \lambda) := x \cdot a + x\lambda$, $x \in \mathcal{M}$, $(a, \lambda) \in A^+$, $\lambda \in \mathbf{C}$, we equip $\mathcal{M}$ with the structure of a Hilbert A^+-module. Consider the A^+-module H_{A^+} and denote by $H_{A^+}A$ the closure of the linear span of all the elements of the form $x \cdot a$, $x \in H_{A^+}$, $a \in A$, in H_{A^+}. It is easy to see that $H_{A^+}A = H_A$. The isomorphism $\mathcal{M} \oplus H_{A^+} \cong H_{A^+}$ implies the isomorphism

$$\mathcal{M} \oplus H_A = \mathcal{M}A \oplus H_{A^+}A = (\mathcal{M} \oplus H_{A^+})A \cong H_{A^+}A = H_A.$$

□

DEFINITION 1.4.4. A Hilbert A-module $\mathcal{M}$ is called a *finitely generated projective* A-module if there exists a Hilbert A-module $\mathcal{N}$ such that $\mathcal{M} \oplus \mathcal{N} \cong L_n(A)$ for some n.

The following two theorems of Dupré and Fillmore show that finite-dimensional projective submodules lie in Hilbert C^*-modules in the simplest way.

THEOREM 1.4.5 (Dupré – Fillmore, [**36**]). *Let A be a unital C^*-algebra and let $\mathcal{M}$ be a finite-dimensional projective A-submodule in the standard Hilbert A-module H_A. Then*

(i) *the nonsingular elements of the module $\mathcal{M}^\perp$ are dense in $\mathcal{M}^\perp$;*
(ii) *$H_A = \mathcal{M} \oplus \mathcal{M}^\perp$;*
(iii) *$\mathcal{M}^\perp \cong H_A$.*

Proof. We begin the proof of the theorem with the case where $\mathcal{M} \cong L_n(A)$. Let $g_1, \dots, g_n$ be an orthonormal basis in $\mathcal{M}$. Fix $\varepsilon > 0$. For each m put

$$e'_m = e_m - \sum_{i=1}^{n} g_i \langle g_i, e_m \rangle.$$

Then $e'_m \in \mathcal{M}^\perp$ and

$$\langle e'_m, e'_m \rangle = 1 - \sum_{i=1}^{n} \langle e_m, g_i \rangle \langle g_i, e_m \rangle.$$

Since $\langle x, e_m \rangle \to 0$ for each $x \in H_A$, we conclude that $\langle e'_m, e'_m \rangle \to 1$. Therefore there exists a number m_0 such that for any $m \geq m_0$, the element e'_m is nonsingular. Then one can define

$$e''_m = e'_m \langle e'_m, e'_m \rangle^{-1/2}$$

with $\langle e''_m, e''_m \rangle = 1$. Let $x \in \mathcal{M}^\perp$. Then

$$\langle e''_m, x \rangle = \langle e'_m, e'_m \rangle^{-1/2} \langle e'_m, x \rangle = \langle e'_m, e'_m \rangle^{-1/2} \langle e_m, x \rangle \to 0.$$

Choose a number $m \geq m_0$ such that $\| \langle e''_m, x \rangle \| < \varepsilon$ and set

$$x' = x + \varepsilon \, e''_m.$$

It is easy to see that

$$\|x' - x\| = \varepsilon. \tag{1.4}$$

Let us check that the element x' is nonsingular. Put

$$u = x - e''_m \langle e''_m, x \rangle, \qquad v = e''_m (\langle e''_m, x \rangle + \varepsilon \, 1).$$

Then $u \perp v$ (since $u \perp e''_m$) and $x' = u + v$. Therefore

$$\langle x', x' \rangle = \langle u, u \rangle + \langle v, v \rangle = \langle u, u \rangle + (\langle e''_m, x \rangle + \varepsilon \, 1)^* (\langle e''_m, x \rangle + \varepsilon \, 1), \tag{1.5}$$

and the right-hand side of equality (1.5) is invertible since $\| \langle e''_m, x \rangle \| < \varepsilon$. Therefore $\langle x', x' \rangle$ is invertible too. Together with estimate (1.4), this proves statement (i).

Let $\{x_n\}$ be a sequence in which each element e_m is repeated *infinitely* many times. Put $x = x_1 - \sum_{i=1}^{n} g_i \langle g_i, x_1 \rangle$. Then (taking $\varepsilon = 1$) one can find an element $g_{n+1} \in \mathcal{M}^\perp$ such that $\langle g_{n+1}, g_{n+1} \rangle = 1$, $\operatorname{dist}(x, g_{n+1} A) \leq 1$. Therefore $\operatorname{dist}(x_1, \operatorname{Span}_A(g_1, \dots, g_{n+1})) \leq 1$. At the next step we replace the module $\mathcal{M}$ by $\operatorname{Span}_A(g_1, \dots, g_{n+1})$, x_1 by x_2, and $\varepsilon = 1$ by $\varepsilon = 1/2$. Going on with this procedure, we obtain an orthonormal basis $\{g_k\}$, $k \in \mathbf{N}$, extending the basis $g_1, \dots, g_n$ of submodule $\mathcal{M}$, and the remaining part $\{g_k : k > n\}$ is a basis of the module $\mathcal{M}^\perp$. This proves statements (ii) and (iii).

We pass now to the case of an arbitrary finitely generated projective module $\mathcal{M}$. Let $\mathcal{M} \oplus \mathcal{N} \cong L_n(A)$. By Theorem 1.4.2, $\mathcal{N} \oplus H_A \cong H_A$, hence

$$L_n(A) \cong \mathcal{N} \oplus \mathcal{M} \subset \mathcal{N} \oplus H_A \cong H_A.$$

Therefore, if $\mathcal{K}$ is the orthogonal complement to the submodule $\mathcal{N} \oplus \mathcal{M}$ in the module $\mathcal{N} \oplus H_A$, then $\mathcal{K} \cong H_A$ and $\mathcal{N} \oplus \mathcal{M} \oplus \mathcal{K} = \mathcal{N} \oplus H_A$. But $\mathcal{K} = \mathcal{M}^\perp$ is obviously the orthogonal complement to the submodule $\mathcal{M}$ in the module H_A. □

Theorem 1.4.6 ([**36**]). *Let A be a unital C^*-algebra and let $\mathcal{M}$ be a finitely generated projective Hilbert submodule in an arbitrary Hilbert A-module $\mathcal{N}$. Then $\mathcal{N} = \mathcal{M} \oplus \mathcal{M}^\perp$.*

PROOF. As in the previous theorem, the proof can be reduced to the case where $\mathcal{M}$ is a free module, $\mathcal{M} = L_n(A)$. If $\{g_1, \dots, g_n\}$ is the standard basis of $\mathcal{M}$, then put $x' = x - \sum_{i=1}^n g_i\langle g_i, x\rangle$ for $x \in \mathcal{N}$. Then $x' \in \mathcal{M}$ and $x - x' \in \mathcal{M}^\perp$, hence $\mathcal{N} = \mathcal{M} \oplus \mathcal{M}^\perp$. □

For a Hilbert A-module $\mathcal{M}$ denote by $\langle \mathcal{M}, \mathcal{M}\rangle \subset A$ the closure of the linear span of all $\langle x, x\rangle$, $x \in \mathcal{M}$. The set $\langle \mathcal{M}, \mathcal{M}\rangle$ is obviously a closed two-sided involutive ideal in the C^*-algebra A.

DEFINITION 1.4.7. A Hilbert A-module is called *full* if $\langle \mathcal{M}, \mathcal{M}\rangle = A$.

One can always consider any Hilbert module as a full Hilbert module over the C^*-algebra $\langle \mathcal{M}, \mathcal{M}\rangle$.

The standard Hilbert C^*-module is obviously full.

1.5. Hilbert C^*-bimodules and strong Morita equivalence

In this section we briefly discuss the case where a module has two C^*-module structures over two C^*-algebras. Such C^*-bimodules were studied by Rieffel in [**109**] and we follow these papers here.

Let A and B be two C^*-algebras. Let $\mathcal{M}$ be a right pre-Hilbert C^*-module over A with the inner product $\langle\cdot,\cdot\rangle_A$ and a left pre-Hilbert C^*-module over B with the inner product $\langle\cdot,\cdot\rangle_B$. The latter means that the sesquilinear form $\langle\cdot,\cdot\rangle_B$ is conjugate linear in the first variable.

DEFINITION 1.5.1. The module $\mathcal{M}$ is called a *pre-Hilbert A-B-bimodule* if the following conditions hold:

(i) $\langle x, y\rangle_B z = x\langle y, z\rangle_A$ for any $x, y, z \in \mathcal{M}$;
(ii) $\langle bx, bx\rangle_A \le \|b\|^2\langle x, x\rangle_A$ and $\langle xa, xa\rangle_B \le \|a\|^2\langle x, x\rangle_B$.

LEMMA 1.5.2. *One has $\|\langle x, x\rangle_A\| = \|\langle x, x\rangle_B\|$ for any $x \in \mathcal{M}$.*

PROOF. Denote $\langle x, x\rangle_A = a$, $\langle x, x\rangle_B = b$. Then $bx = xa$ for any $x \in \mathcal{M}$. Since

$$a^3 = \langle xa, xa\rangle_A = \langle bx, bx\rangle_A \le \|b\|^2 a,$$

$\|a\|^3 \le \|b\|\|a\|^2$, hence $\|a\| \le \|b\|$. Similarly one obtains $\|b\| \le \|a\|$. □

Thus we see that $\|x\| = \|\langle x, x\rangle_A\|^{1/2} = \|\langle x, x\rangle_B\|^{1/2}$ defines a norm on $\mathcal{M}$.

DEFINITION 1.5.3. A pre-Hilbert A-B-bimodule $\mathcal{M}$ is called *an A-B-equivalence bimodule* if it is complete with respect to the norm and if it is full both as a right and as a left Hilbert C^*-module.

LEMMA 1.5.4. *One has $\langle bx, y\rangle_A = \langle x, b^*y\rangle_A$ and $\langle xa, y\rangle_B = \langle x, ya^*\rangle_B$ for any $x, y \in \mathcal{M}$, $a \in A$, $b \in B$.*

PROOF. It suffices to prove the first statement since the second is similar. Since the bimodule $\mathcal{M}$ is full as a B-module, we can assume without loss of generality that $b = \langle z, t\rangle_B$ for some $z, t \in \mathcal{M}$. Then

$$\begin{aligned}\langle bx, y\rangle_A &= \langle z\langle t, x\rangle_A, y\rangle_A = \langle t, x\rangle_A^*\langle z, y\rangle_A \\ &= \langle x, t\rangle_A\langle z, y\rangle_A.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle x, b^* y\rangle_A &= \langle x, \langle t, z\rangle_B y\rangle_A = \langle x, t\langle z, y\rangle_A\rangle_A \\ &= \langle x, t\rangle_A \langle z, y\rangle_A.\end{aligned}$$

□

DEFINITION 1.5.5. Two C^*-algebras A and B are called *strongly Morita equivalent* if there exists an A-B-equivalence bimodule.

LEMMA 1.5.6. *Strong Morita equivalence is an equivalence relation.*

PROOF. Reflexivity of this equivalence relation is clear, symmetry follows from considering the conjugate module $\widetilde{X}$, which consists of the same elements, but the bimodule structure is given by $a\widetilde{x} = x\alpha^*$ and $\widetilde{x}b = b^*x$, $x \in \mathcal{M}$, $a \in A$, $b \in B$. So we have to check transitivity. Let $\mathcal{M}$ be an A-B-equivalence bimodule and let $\mathcal{N}$ be a B-C-equivalence bimodule, where C is one more C^*-algebra. Then $\mathcal{N} \otimes_B \mathcal{M}$ is a left C-module and a right A-module. For finite sums $\sum_i n_i \otimes m_i$, $\sum_j n'_j \otimes m'_j$ in $\mathcal{N} \otimes_B \mathcal{M}$ define an A-valued inner product by

$$\langle \sum_i n_i \otimes m_i, \sum_j n'_j \otimes m'_j\rangle_A = \sum_{ij} \langle \langle n_i, n'_j\rangle_B m_i, m'_j\rangle_A$$

and similarly define a C-valued inner product. It is easy to check that these inner products satisfy all the necessary properties, except, possibly, nondegeneracy. Let $\mathcal{L} \subset \mathcal{N} \otimes_B \mathcal{M}$ be the (maybe empty) set of all finite sums $l = \sum_i n_i \otimes m_i \in \mathcal{N} \otimes_B \mathcal{M}$ with $\langle l, l\rangle_A = 0$. Then the completion of $\mathcal{N} \otimes_B \mathcal{M}/\mathcal{L}$ with respect to the norm obtained from any of the two inner products gives us an A-C-equivalence bimodule (for example, fullness follows from Lemma 1.3.8). □

EXAMPLE 1.5.7. Consider the standard right Hilbert A-module H_A and endow it with the left module structure over the C^*-algebra $A \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the C^*-algebra of compact operators on a separable Hilbert space. Elements of $A \otimes \mathcal{K}$ can be written as infinite matrices of the form (a_{ij}), $i, j \in \mathbf{N}$, with entries in A. The left action of such a matrix (a_{ij}) on a sequence $(x_i) \in H_A$ results in the sequence $(\sum_j a_{ij} x_j)$. Define the $A \otimes \mathcal{K}$-valued inner product on H_A by

$$\langle (x_i), (y_j)\rangle_{A\otimes\mathcal{K}} = (x_i y_j^*) \in A \otimes \mathcal{K}$$

(it is easy to check that the matrix $(x_i y_j^*)$ defines an element from $A \otimes \mathcal{K}$ because both sequences (x_i) and (y_j) vanish at infinity). All properties of an A-$A \otimes \mathcal{K}$-equivalence bimodule can be easily checked and thus we see that A is strongly Morita equivalent to $A \otimes \mathcal{K}$. Similarly, taking A^n instead of H_A, we obtain the strong Morita equivalence between A and the C^*-algebra $M_n(A)$ of $n \times n$-matrices with entries from A.

This example and Lemma 1.5.6 imply the following statement.

COROLLARY 1.5.8. *Let A and B be C^*-algebras. If the C^*-algebras $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ are isomorphic, then A and B are strongly Morita equivalent.*

A criterion for the strong Morita equivalence was obtained in [**15, 18**]. Since the method of proof is far removed from the Hilbert C^*-module technique, we skip the proof.

THEOREM 1.5.9 ([**15, 18**]). *Let A and B be strongly Morita equivalent C^*-algebras. If they both have countable approximate identities, then the C^*-algebras $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ are isomorphic.*

CHAPTER 2

Operators on Hilbert Modules

2.1. Bounded and adjointable operators

Let $\mathcal{M}, \mathcal{N}$ be Hilbert C^*-modules over a C^*-algebra A. A bounded $\mathbf{C}$-linear A-homomorphism from $\mathcal{M}$ to $\mathcal{N}$ is called an *operator* from $\mathcal{M}$ to $\mathcal{N}$. Let $\mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$ denote the set of all operators from $\mathcal{M}$ to $\mathcal{N}$. If $\mathcal{N} = \mathcal{M}$, then $\mathrm{End}_A(\mathcal{M}) = \mathrm{Hom}_A(\mathcal{M}, \mathcal{M})$ is obviously a Banach algebra. However, we shall soon see that there is no natural involution on this algebra. Let $T \in \mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$. We say that T is *adjointable* if there exists an operator $T^* \in \mathrm{Hom}_A(\mathcal{N}, \mathcal{M})$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle$$

for all $x \in \mathcal{M}$, $y \in \mathcal{N}$.

Lemma 2.1.1. *Let $\mathcal{M}$ be a Hilbert A-module and let $T : \mathcal{M} \to \mathcal{M}$ and $T^* : \mathcal{M} \to \mathcal{M}$ be maps such that*

$$\langle x, Ty\rangle = \langle T^*x, y\rangle$$

for all $x, y \in \mathcal{M}$. Then T is a bounded $\mathbf{C}$-linear A-homomorphism (*and T^* is as well*).

Proof. For any $x, y, z \in \mathcal{M}$, $w \in \mathbf{C}$ and $a \in A$ one has

$$\begin{aligned}
\langle z, T(x+y)\rangle &= \langle T^*z, x+y\rangle = \langle T^*z, x\rangle + \langle T^*z, y\rangle \\
&= \langle z, Tx\rangle + \langle z, Ty\rangle = \langle z, Tx + Ty\rangle, \\
\langle z, Twx\rangle &= \langle T^*z, x\rangle w = \langle z, Tx\rangle w = \langle z, wTx\rangle, \\
\langle z, T(xa)\rangle &= \langle T^*z, xa\rangle = \langle T^*z, x\rangle a = \langle z, Tx\rangle a = \langle z, (Tx)a\rangle.
\end{aligned}$$

Since z is an arbitrary element, it follows that

$$T(x+y) = Tx + Ty, \qquad T(wx) = wTx, \qquad T(xa) = (Tx)a,$$

and linearity properties hold.

To prove the continuity of T we should verify that its graph is closed. Let $x_\alpha \to x$, $T(x_\alpha) \to y$ in $\mathcal{M}$, and let $z \in \mathcal{M}$ be an arbitrary element. Then

$$\begin{aligned}
0 &= \langle T^*(y - Tx), x_\alpha\rangle - \langle T^*(y - Tx), x_\alpha\rangle \\
&= \langle y - Tx, T(x_\alpha)\rangle - \langle T^*(y - Tx), x_\alpha\rangle \\
&\longrightarrow \langle y - Tx, y\rangle - \langle T^*(y - Tx), x\rangle = \langle y - Tx, y - Tx\rangle. \qquad \square
\end{aligned}$$

We show now that there exist nonadjointable operators.

EXAMPLE 2.1.2. Let A be a unital C^*-algebra. As above, the standard basis of the Hilbert module H_A consists of the elements $e_i = (0, \dots, 0, 1, 0, \dots)$, where 1 is the i-th entry. To each operator $T \in \mathrm{End}_A(H_A)$ one can associate an infinite matrix with respect to this basis,

$$\|t_{ij}\|, \qquad t_{ij} = \langle e_i, Te_j \rangle.$$

Then the adjoint operator has the matrix $\|t^*_{ji}\|$.

Let $A = C([0,1])$ and let the functions $\varphi_i \in A$, $i = 1, 2, \dots$, be defined by the formula

$$\varphi_i = \begin{cases} 0 & \text{on } [0, \frac{1}{i+1}] \text{ and } [\frac{1}{i}, 1], \\ 1 & \text{at the point } x_i = \frac{1}{2}\left(\frac{1}{i} + \frac{1}{i+1}\right), \\ \text{is linear} & \text{on } [\frac{1}{i+1}, x_i] \text{ and } [x_i, \frac{1}{i}]. \end{cases}$$

Let an operator $T \in \mathrm{End}_A(H_A)$ have the matrix

$$\begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

(actually it is an operator from the module H_A to A, i.e., an A-functional). It is easy to verify that T is bounded. But the operator T^* is not well defined since it should have the matrix

$$\begin{pmatrix} \varphi^*_1 & 0 & 0 & \dots \\ \varphi^*_2 & 0 & 0 & \dots \\ \varphi^*_3 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

and the image of the basis element e_1 should be an element of H_A having the first column as its coordinates and it has to be an element of H_A, which is impossible since the series $\sum \varphi_i \varphi^*_i$ is not norm-convergent in the C^*-algebra A.

Denote by $\mathrm{Hom}^*_A(\mathcal{M}, \mathcal{N})$ the set of all adjointable operators from $\mathcal{M}$ to $\mathcal{N}$. The algebra $\mathrm{End}^*_A(\mathcal{M}) = \mathrm{Hom}^*_A(\mathcal{M}, \mathcal{M})$ is an involutive Banach algebra. Moreover, it is a C^*-algebra; this follows from the estimate

$$\|T^*T\| \geq \sup_{x \in B_1(\mathcal{M})} \{\langle T^*Tx, x \rangle\} = \sup_{x \in B_1(\mathcal{M})} \{\langle Tx, Tx \rangle\} = \|T\|^2,$$

where $B_1(\mathcal{M})$ denotes the unit ball of the module $\mathcal{M}$.

We will use the following statement frequently without special reference.

PROPOSITION 2.1.3. *For an operator $T : \mathcal{M} \to \mathcal{M}$, the following conditions are equivalent:*

(i) *T is a positive element of the C^*-algebra* $\mathrm{End}^*(\mathcal{M})$*;*
(ii) *for any $x \in \mathcal{M}$ the inequality $\langle Tx, x \rangle \geq 0$ is fulfilled, i.e., is positive in the C^*-algebra A.*

PROOF. The first condition is equivalent to the equality $T = S^*S$ for some $S \in \mathrm{End}^*(\mathcal{M})$. Therefore

$$\langle Tx, x \rangle = \langle Sx, Sx \rangle \geq 0 \qquad \text{for any } x \in \mathcal{M}.$$

Now let $\langle Tx, x \rangle \geq 0$ for all $x \in \mathcal{M}$. Then

$$\langle Tx, x \rangle = \langle Tx, x \rangle^* = \langle x, Tx \rangle \qquad \text{for all } x \in \mathcal{M}.$$

The map $(x, y) \mapsto \langle Tx, y\rangle$ defines a sesquilinear form on $\mathcal{M}$. Therefore, by the polarization equality 1.2.5, $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in \mathcal{M}$. By Lemma 2.1.1, this means that $T \in \mathrm{End}^*(\mathcal{M})$ and $T = T^*$. So T is a selfadjoint element of the C^*-algebra $\mathrm{End}^*(\mathcal{M})$, so (see [**31**, 1.6.5]) it can be represented as the difference $T = T_+ - T_-$ of two elements of $\mathrm{End}^*(\mathcal{M})$, where $T_+ \geq 0$, $T_- \geq 0$ and $T_+T_- = T_-T_+ = 0$. Then $\langle T_-y, y\rangle \leq \langle T_+y, y\rangle$ for any $y \in \mathcal{M}$. In particular,

$$\langle T_-^3 x, x\rangle = \langle T_-^2 x, T_-x\rangle \leq \langle T_+T_-x, T_-x\rangle = 0.$$

On the other hand, $T_- \geq 0$ and $T_-^3 \geq 0$, hence $\langle T_-^3 x, x\rangle \geq 0$ (because the statement in this direction is already proved). So the only possibility left is $\langle T_-^3 x, x\rangle = 0$ for any $x \in \mathcal{M}$. By the polarization equality, this implies $\langle T_-^3 x, y\rangle = 0$ for all $x, y \in \mathcal{M}$, hence $T_-^3 = 0$, $T_- = 0$. Thus, $T = T_+ \geq 0$. □

THEOREM 2.1.4 ([**100**]). *Let $\mathcal{M}$ and $\mathcal{N}$ be Hilbert A-modules and let $T : \mathcal{M} \to \mathcal{N}$ be a linear map. Then the following conditions are equivalent:*

(i) *the operator T is bounded and A-linear, i.e., $T(x \cdot a) = Tx \cdot a$ for all $x \in \mathcal{M}$, $a \in A$;*
(ii) *there exists a constant $K \geq 0$ such that the inequality $\langle Tx, Tx\rangle \leq K\langle x, x\rangle$ holds in A for all $x \in \mathcal{M}$.*

PROOF. To obtain the second statement from the first, let us assume that $T(x \cdot a) = Tx \cdot a$ and that $\|T\| \leq 1$. If the C^*-algebra A is not unital, then we consider $\mathcal{M}$ and $\mathcal{N}$ as modules over the C^*-algebra A^+ obtained from A by unitalizing. For $x \in \mathcal{M}$ and $n \in \mathbf{N}$ put

$$a_n = \left(\langle x, x\rangle + \frac{1}{n}\right)^{-1/2}, \qquad x_n = x \cdot a_n.$$

Then $\langle x_n, x_n\rangle = a_n^*\langle x, x\rangle a_n = \langle x, x\rangle\big(\langle x, x\rangle + \frac{1}{n}\big)^{-1} \leq 1$. Therefore $\|x_n\| \leq 1$, hence $\|Tx_n\| \leq 1$. Then the inequality $\langle Tx_n, Tx_n\rangle \leq 1$ holds in A (or in A^+) for all $n \in \mathbf{N}$. But

$$\langle Tx, Tx\rangle = a_n^{-1}\langle Tx_n, Tx_n\rangle a_n^{-1} \leq a_n^{-2} = \langle x, x\rangle + \frac{1}{n}. \tag{2.1}$$

Passing to the limit as $n \to \infty$ in the inequality (2.1), we obtain $\langle Tx, Tx\rangle \leq \langle x, x\rangle$.

To derive the first statement from the second, assume that the inequality $\langle Tx, Tx\rangle \leq \langle x, x\rangle$ holds for all $x \in \mathcal{M}$. It clearly follows that T is bounded, $\|T\| \leq 1$. Let $x \in \mathcal{M}$, $y \in \mathcal{N}$. Define the map $r : A^+ \longrightarrow A^+$ by the formula

$$r(a) = \langle y, T(x \cdot a)\rangle.$$

Then

$$\begin{aligned} r(a)^*r(a) &= \langle T(x \cdot a), y\rangle\langle y, T(x \cdot a)\rangle \leq \|y\|^2 \langle T(x \cdot a), T(x \cdot a)\rangle \\ &\leq \|y\|^2 \langle x \cdot a, x \cdot a\rangle = \|y\|^2 a^*\langle x, x\rangle a \\ &\leq \|y\|^2 \|x\|^2 a^*a. \end{aligned}$$

To complete the proof we use the following statement.

LEMMA 2.1.5 ([**56, 100**]). *Let A be a unital C^*-algebra and let $r : A \longrightarrow A$ be a linear map such that for some constant $K \geq 0$, the inequality $r(a)^*r(a) \leq Ka^*a$ holds for all $a \in A$. Then $r(a) = r(1)a$ for all $a \in A$.*

Thus $r(a) = r(1)a$, i.e.,

$$\langle y, T(x \cdot a)\rangle = \langle y, Tx\rangle a = \langle y, Tx \cdot a\rangle$$

for all $y \in \mathcal{N}$, $x \in \mathcal{M}$. This implies the first statement of the theorem. □

COROLLARY 2.1.6. *Let $\mathcal{M}, \mathcal{N}$ be Hilbert A-modules, $T \in \mathrm{End}_A(\mathcal{M}, \mathcal{N})$. Then*

$$\|T\| = \inf\left\{K^{1/2} : \langle Tx, Tx\rangle \le K\langle x, x\rangle \quad \forall x \in \mathcal{M}\right\}.$$

EXAMPLE 2.1.7. Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$ be a decomposition into an orthogonal direct sum of Hilbert modules. We define the operator $P : \mathcal{M} \longrightarrow \mathcal{M}$ to be the projection onto the submodule $\mathcal{N}$ along the submodule $\mathcal{L}$. Then P is bounded, $\|P\| = 1$, and $P^* = P$, hence $P \in \mathrm{End}^*_A(\mathcal{M})$.

2.2. Compact operators in Hilbert modules

Let $\mathcal{M}, \mathcal{N}$ be Hilbert A-modules, $x \in \mathcal{N}$, $y \in \mathcal{M}$. Define the operator $\theta_{x,y} : \mathcal{M} \longrightarrow \mathcal{N}$ by its action on an element $z \in \mathcal{M}$ by the formula

$$\theta_{x,y}(z) := x\langle y, z\rangle. \tag{2.2}$$

Operators of the form (2.2) are called *elementary* operators. They clearly satisfy the equalities

(i) $(\theta_{x,y})^* = \theta_{y,x}$;
(ii) $\theta_{x,y}\theta_{u,v} = \theta_{x\langle y,u\rangle,v} = \theta_{x,v\langle u,y\rangle}$ for $u \in \mathcal{M}$, $v \in \mathcal{N}$;
(iii) $T\theta_{x,y} = \theta_{Tx,y}$ for $T \in \mathrm{Hom}_A(\mathcal{N}, \mathcal{L})$;
(iv) $\theta_{x,y}S = \theta_{x,S^*y}$ for $S \in \mathrm{Hom}^*_A(\mathcal{L}, \mathcal{M})$.

We denote the closed linear span of the set of all elementary operators by $\mathcal{K}(\mathcal{M}, \mathcal{N})$. The elements of $\mathcal{K}(\mathcal{M}, \mathcal{N})$ are called *compact* operators. In the case $\mathcal{N} = \mathcal{M}$ the equalities (i)–(iv) mean that the algebra $\mathcal{K}(\mathcal{M}) = \mathcal{K}(\mathcal{M}, \mathcal{M})$ is a closed two-sided ideal in the C^*-algebra $\mathrm{End}^*_A(\mathcal{M})$. Compact operators acting on Hilbert modules are not compact operators in the usual sense, when one considers them as operators from one Banach space to another. However, they are a natural generalization of compact operators on a Hilbert space.

PROPOSITION 2.2.1. *Let H_A be the standard Hilbert module over a unital C^*-algebra A and let $L_n(A) \subset H_A$ be the free submodule generated by the first n elements of the standard basis. An operator $K \in \mathrm{End}_A(H_A)$ is compact if and only if the norms of restrictions of K onto the orthogonal complements $L_n(A)^\perp$ of the submodules $L_n(A)$ vanish as $n \to \infty$.*

PROOF. Denote by p_n the projection in H_A onto the submodule $L_n(A)^\perp$. Then, for any $z \perp L_n(A)$, one has

$$\begin{aligned}\|\theta_{x,y}(z)\|^2 = \|\langle\theta_{x,y}(z), \theta_{x,y}(z)\rangle\| &= \left\|\langle y, z\rangle^*\langle x, x\rangle\langle y, z\rangle\right\| \\ &\le \|x\|^2 \|\langle y, z\rangle\|^2 = \|x\|^2 \|\langle p_n y, z\rangle\|^2 \\ &\le \|x\|^2 \cdot \|p_n y\|^2 \cdot \|z\|^2 .\end{aligned}$$

Since $\|p_n y\|$ tends to zero, the same is true for the norm of the restriction of the operator $\theta_{x,y}$ to the submodule $L_n(A)^\perp$, hence, for the norm of any compact operator. Let us assume now that for some operator K, one has $\left\|K|_{L_n(A)^\perp}\right\| \to 0$.

Then, since $\sum_{m=1}^{n} Ke_m\langle e_m, z\rangle = 0$ for any $z \perp L_n(A)$, for $\|z\| \le 1$ and $z \perp L_n(A)$ one has

$$\sup_z \left\| Kz - \sum_{m=1}^{n} Ke_m\langle e_m, z\rangle \right\| = \sup_z \|Kz\| \longrightarrow 0 \tag{2.3}$$

as $n \to \infty$. If $z \in L_n(A)$, then $Kz = \sum_{m=1}^{n} Ke_m\langle e_m, z\rangle$. This means that (2.3) still holds if the supremum is taken over the unit ball of the whole module H_A. Therefore the operator K is the norm limit of the operators $K_n = \sum_{m=1}^{n} \theta_{Ke_m, e_m}$. □

Note that, in the case of modules over nonunital C^*-algebras, the statement of 2.2.1 is not valid.

Let $\mathcal{K}$ denote the C^*-algebra of compact operators on a separable Hilbert space H. Since the algebra $\mathcal{K}$ is nuclear [**70**], there is a unique C^*-seminorm on the algebraic tensor product of $\mathcal{K}$ by any C^*-algebra A and we denote its completion with respect to this seminorm by $\mathcal{K} \otimes A$. Denote by $M_n(A) = M_n \otimes A$ the C^*-algebra of all $n\times n$-matrices with entries from A, where M_n is the algebra of complex $n\times n$-matrices.

PROPOSITION 2.2.2. *There exist natural isometric isomorphisms:*

(i) $\mathcal{K}(A) \cong A$;
(ii) $\mathcal{K}(L_n(A)) \cong M_n(A)$;
(iii) $\mathcal{K}(H_A) \cong \mathcal{K} \otimes A$.

PROOF. If a C^*-algebra is unital, then statement (i) is clear. In the general case consider the map $\varphi : \mathrm{Span}_{\mathbf{C}}(\theta_{a,b} : a, b \in A) \longrightarrow A$ defined by the formula

$$\varphi\left(\sum_{i=1}^{n} \lambda_i \theta_{a_i,b_i}\right) = \sum_{i=1}^{\infty} \lambda_i a_i b_i^*.$$

Let us verify that this map is well defined. If $\sum_i \lambda_i \theta_{a_i,b_i} = \sum_j \mu_j \theta_{c_j,d_j}$, then $\sum_i \lambda_i a_i b_i^* x = \sum_j \mu_j c_j d_j^* x$ for any $x \in A$. Therefore $\sum_i \lambda_i a_i b_i^* = \sum_j \mu_j c_j d_j^*$. The map φ is multiplicative and involutive,

$$\varphi(\theta_{a,b}\ \theta_{c,d}) = \varphi(\theta_{ab^*,dc^*}) = \varphi(\theta_{a,b})\varphi(\theta_{c,d}); \qquad \varphi(\theta_{a,b}^*) = \varphi(\theta_{b,a}) = \varphi(\theta_{a,b})^*.$$

Surjectivity of φ follows from the possibility of the factorization $a = u(a^*a)^{1/4}$ for any $a \in A$ (see 1.1.5). If (u_α), $\alpha \in \mathcal{A}$, is an approximate unit of the C^*-algebra A, then

$$\lim_\alpha \left\| \sum_{i=1}^{n} \lambda_i \theta_{a_i,b_i}(u_\alpha) \right\| = \left\| \sum_{i=1}^{n} \lambda_i a_i b_i^* \right\|.$$

Therefore $\|\varphi(k)\| \le \|k\|$ for $k = \sum_{i=1}^{n} \lambda_i \theta_{a_i,b_i}$. This means that the map φ can be extended by continuity up to a map defined on the whole algebra $\mathcal{K}(A)$. The estimate

$$\left\| \sum_{i=1}^{\infty} \lambda_i \theta_{a_i,b_i} \right\| = \sup_{\|x\|\le 1} \left\| \sum_{i=1}^{\infty} \lambda_i a_i b_i^* x \right\| \le \left\| \sum_{i=1}^{\infty} \lambda_i a_i b_i^* \right\| = \left\| \varphi\left(\sum_{i=1}^{\infty} \lambda_i \theta_{a_i,b_i} \right) \right\|$$

shows that the map φ is an isometry, so statement (i) is proved. Statement (ii) can be proved in a similar way using the map

$$\varphi_n : \theta_{a_1\oplus\cdots\oplus a_n, b_1\oplus\cdots\oplus b_n} \longmapsto \begin{pmatrix} a_1 b_1^* & \dots & a_1 b_n^* \\ \vdots & & \vdots \\ a_n b_1^* & \dots & a_n b_n^* \end{pmatrix}.$$

Finally, since there exists an isometric map from the linear space $\bigcup_n \mathcal{K}(L_n(A))$ to the linear space $\bigcup_n M_n(A)$ and since these spaces are dense in C^*-algebras $\mathcal{K}(H_A)$ and $\mathcal{K}\otimes A$, respectively, statement (iii) follows. $\square$

LEMMA 2.2.3. *For any $x \in \mathcal{M}$ there exists $z \in \mathcal{M}$ and $k = \theta_{u,v} \in \mathcal{K}(\mathcal{M})$ such that $x = kz$.*

PROOF. Put

$$u := v := z := \lim_{\varepsilon\to 0} x\,(\varepsilon + \langle x, x\rangle^{1/3})^{-1}.$$

Since $s^2(\varepsilon + s)^{-1}$ is uniformly convergent to s on bounded sets, in order to prove that u is well defined, note that for $t = \langle x, x\rangle$, one has

$$\begin{aligned}\langle x\,(\varepsilon+&\langle x,x\rangle^{1/3})^{-1} - x\,(\mu + \langle x,x\rangle^{1/3})^{-1}, x\,(\varepsilon + \langle x,x\rangle^{1/3})^{-1} - x\,(\mu + \langle x,x\rangle^{1/3})^{-1}\rangle \\ &= [(\varepsilon + t^{1/3})^{-1} - (\mu + t^{1/3})^{-1}]t[(\varepsilon + t^{1/3})^{-1} - (\mu + t^{1/3})^{-1}] \\ &= [(\varepsilon + t^{1/3})^{-1} - (\mu + t^{1/3})^{-1}]^2 (t^{1/3})^4.\end{aligned}$$

The same arguments show that $x = kz$. $\square$

Note that we have also proved that $\mathcal{M}\langle\mathcal{M},\mathcal{M}\rangle = \mathcal{M}$.

THEOREM 2.2.4. *A Hilbert A-module $\mathcal{M}$ is countably generated if and only if the C^*-algebra $\mathcal{K}(\mathcal{M})$ is σ-unital.*

PROOF. Let $\mathcal{K}(\mathcal{M})$ be σ-unital and let $æ_n$ be a countable approximate unit for it. Then

$$x = \lim_{n\to\infty} æ_n x \quad \text{for any } x \in \mathcal{M}. \tag{2.4}$$

Indeed, by Lemma 2.2.3, $x = kz$ holds for some $k \in \mathcal{K}(\mathcal{M})$, $z \in \mathcal{M}$. Since $æ_n k \longrightarrow k$ with respect to the norm, one has $æ_n x = æ_n kz \longrightarrow kz = x$.

By definition, any compact operator can be approximated by a linear combination of elementary ones. Hence, for each $æ_n$ there exist elements x_i^n and y_i^n in $\mathcal{M}$ such that

$$\left\| \sum_{i=1}^{s(n)} \theta_{x_i^n, y_i^n} - æ_n \right\| < \frac{1}{n}, \qquad n = 1, 2, \dots.$$

Let us show that the countable set x_i^n, $i = 1, \dots, s(n)$, $n = 1, 2, \dots$, generates the module $\mathcal{M}$. Consider an arbitrary element $x \in \mathcal{M}$ and a number $\varepsilon > 0$. By (2.4) one can find a sufficiently large n such that

$$\|x - æ_n x\| < \frac{\varepsilon}{2} \qquad \text{and} \qquad \frac{1}{n} < \frac{\varepsilon}{2}.$$

Then

$$\left\| x - \sum_{i=1}^{s(n)} x_i^n \cdot \langle y_i^n, x\rangle \right\| \le \|x - æ_n(x)\| + \left\| æ_n(x) - \sum_{i=1}^{s(n)} \theta_{x_i^n, y_i^n}(x) \right\|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Now let the module $\mathcal{M}$ be countably generated. It can be viewed as a module over the algebra A^+ obtained by unitalizing A (if it was not unital) with respect to the action $x \cdot (a, \mu) := x \cdot a + \mu x$, $x \in \mathcal{M}$, $a \in A$, $\mu \in \mathbf{C}$. If it were countably generated over A, then it should be countably generated over A^+ as well. Since in the definition of elementary compact operators only the A-valued inner product is involved, one has $\mathcal{K}_A(\mathcal{M}) = \mathcal{K}_{A^+}(\mathcal{M})$. Thus we can restrict ourselves to the case where A is unital.

So let $\mathcal{M}$ be a countably generated Hilbert module over a unital algebra A. By the Kasparov stabilization theorem, $\mathcal{M} \oplus H_A \cong H_A$. Let $\iota : \mathcal{M} \to H_A$ be the corresponding inclusion and let $\pi : H_A \to \mathcal{M}$ be the corresponding selfadjoint projection. Let $\{e_i\}$ denote the standard basis of H_A. Recall that for a C^*-algebra, the property of being σ-unital is equivalent to that of having a strictly positive element. Consider

$$æ := \sum_{n=1}^{\infty} \frac{\theta_{e_n, e_n}}{n},$$

or, in matrix form,

$$æ := \operatorname{diag}\,(1, \frac{1}{2}, \frac{1}{3}, \dots).$$

Then æ is a strictly positive element in $\mathcal{K}(H_A)$. Indeed, on one hand, by Proposition 2.2.1, we have $æ \in \mathcal{K}(H_A)$. On the other hand, if $\rho : \mathcal{K}(H_A) \to \mathbf{C}$ is a state such that $\rho(æ) = 0$, then $\rho(\theta_{e_n,e_n}) = 0$ for any n, since all $\theta_{e_n,e_n} \ge 0$. Then, for any $x = (x_1, x_2, \dots) \in H_A$, one has

$$\rho(\theta_{e_n,x}\theta_{x,e_n}) = \rho\left(\left(\sum_{j=1}^{\infty} \theta_{e_n, e_j x_j} \right) \left(\sum_{j=1}^{\infty} \theta_{e_n, e_j x_j} \right)^* \right)$$
$$= \rho\left(\sum_{j=1}^{\infty} \theta_{e_n, e_j x_j} \theta^*_{e_n, e_j x_j} \right) = \rho\left(\sum_{j=1}^{\infty} \theta_{e_n, e_j x_j} \theta_{e_j x_j, e_n} \right)$$
$$= \rho\left(\sum_{j=1}^{\infty} \theta_{e_n \cdot \langle e_j x_j, e_j x_j\rangle, e_n} \right) \le \|x\|^2 \sum_{j=1}^{\infty} \rho(\theta_{e_n,e_n}) = 0,$$

where the last inequality follows from

$$\langle \theta_{e_n \cdot \langle e_j x_j, e_j x_j\rangle, e_n}(z), z\rangle = \langle e_n \cdot \langle e_j x_j, e_j x_j\rangle \langle e_n, z\rangle, z\rangle$$
$$= \langle z, e_n\rangle x_j^* x_j \langle e_n, z\rangle \le \|x\|^2 \langle \theta_{e_n,e_n} z, z\rangle.$$

Thus, for any $x, y, z \in \mathcal{M}$,

$$\theta_{x,y}(z) = x \cdot \langle y, z\rangle = \theta_{x,e_n}\theta_{e_n,y}(z)$$

and

$$|\rho(\theta_{x,y})| = |\rho(\theta_{x,e_n}\theta_{e_n,y})| \le \rho^{1/2}(\theta_{x,e_n}\theta_{e_n,x})\, \rho^{1/2}(\theta_{e_n,y}\theta_{y,e_n}) = 0,$$

due to the vanishing of the second factor. So ρ vanishes on a dense subset, hence, everywhere on $\mathcal{K}(H_A)$. We have shown that æ is a strictly positive element of $\mathcal{K}(H_A)$. Then $æ_n := æ^{1/n}$ is a countable approximate unit in $\mathcal{K}(H_A)$ and $\pi æ_n \iota$ is a countable approximate unit in $\mathcal{K}(\mathcal{M})$. Indeed, if $k \in \mathcal{K}(\mathcal{M})$, $\pi\iota k = k$, then $\iota k \pi \in \mathcal{K}(H_A)$ and

$$\|k - \pi æ_n \iota\| = \|\pi(\iota k \pi - æ_n)\iota\| = \|\iota k \pi - æ_n\| \longrightarrow 0 \qquad (n \longrightarrow \infty).$$

□

2.3. Complementable submodules and projections in Hilbert C^*-modules

Recall that a closed submodule $\mathcal{N}$ in a Hilbert C^*-module $\mathcal{M}$ is called orthogonally complementable if $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}^\perp$. As we have already seen, a closed submodule in a Hilbert C^*-module is not necessarily orthogonally complementable.

Definition 2.3.1. A closed submodule $\mathcal{N}$ in a Hilbert C^*-module $\mathcal{M}$ is called (*topologically*) *complementable* if there exists a closed submodule $\mathcal{L}$ in $\mathcal{M}$ such that $\mathcal{N} + \mathcal{L} = \mathcal{M}$, $\mathcal{N} \cap \mathcal{L} = \{0\}$.

The following example shows that there exist topologically complementable submodules that are not orthogonally complementable.

Example 2.3.2. Let $J \subset A$ be a closed ideal such that the equality $Ja = 0$, $a \in A$, implies that $a = 0$. Put $\mathcal{M} := A \oplus J$,

$$\mathcal{N} := \{(b, b) : b \in J\}.$$

Then

$$\mathcal{N}^\perp = \{(c, -c) : c \in J\}.$$

Therefore $\mathcal{N} \oplus \mathcal{N}^\perp = J \oplus J \neq \mathcal{M}$. However, the submodule $\mathcal{L} = \{(a, 0) : a \in A\} \subset \mathcal{M}$ is a topological complement to $\mathcal{N}$ in $\mathcal{M}$.

Recall that we denote the nonorthogonal direct sum of Hilbert C^*-modules by $\mathcal{N} \widetilde{\oplus} \mathcal{L}$. A decomposition into a direct sum $\mathcal{M} = \mathcal{N} \widetilde{\oplus} \mathcal{L}$ allows us to define the projection P onto $\mathcal{N}$ along $\mathcal{L}$. The operator P is A-linear and, by the closed graph theorem, is bounded, hence $P \in \mathrm{End}_A(\mathcal{M})$. However, as is clear from Example 2.3.2, the projection P can be nonadjointable. But if $\mathcal{M} = \mathcal{N} \oplus \mathcal{L}$, then the corresponding projection is selfadjoint, $P \in \mathrm{End}^*_A(\mathcal{M})$. Since it is more convenient to work with orthogonal decompositions, we would like to describe situations where such a decomposition exists.

Theorem 2.3.3 ([**90**]). *Let $\mathcal{M}$, $\mathcal{N}$ be Hilbert A-modules and $T \in \mathrm{Hom}^*_A(\mathcal{M}, \mathcal{N})$ an operator with closed image. Then*

(i) $\mathrm{Ker}\, T$ *is an orthogonally complementable submodule in* $\mathcal{M}$,
(ii) $\mathrm{Im}\, T$ *is an orthogonally complementable submodule in* $\mathcal{N}$.

Proof. (i) Let $\mathrm{Im}\, T = \mathcal{N}_0$ and let $T_0 : \mathcal{M} \longrightarrow \mathcal{N}_0$ be an operator such that its action coincides with the action of T. By the open mapping theorem, the image of the unit ball $T_0(B_1(\mathcal{M}))$ contains some ball of radius $\delta > 0$ in $\mathcal{N}_0$. Therefore for each $y \in \mathcal{N}_0$, one can find some $x \in \mathcal{M}$ such that $T_0 x = y$ and $\|x\| \leq \delta^{-1} \|y\|$. One has

$$\|T_0^* y\|^2 = \|\langle y, T_0 T_0^* y\rangle\| \leq \|y\| \cdot \|T_0 T_0^* y\|,$$

and hence,

$$\|y\|^2 = \|\langle T_0 x, y\rangle\| = \|\langle x, T_0^* y\rangle\| \le \|x\| \cdot \|T_0^* y\| \le \delta^{-1} \|y\| \cdot \|y\|^{1/2} \|T_0 T_0^* y\|^{1/2}.$$

We obtain that for any $y \in \mathcal{N}_0$,

$$\|y\| \le \delta^{-2} \|T_0 T_0^* y\|.$$

Let us show that the spectrum of the operator $T_0 T_0^*$ does not contain the origin. Suppose the opposite, i.e., that $0 \in \mathrm{Sp}(T_0 T_0^*)$. Let f be a continuous function on $\mathbf{R}$ such that

$$f(0) = 1 = \|f\|, \qquad f(t) = 0 \text{ if } |t| \ge \frac{1}{2}\delta^{-2}.$$

Using functional calculus in the C^*-algebra $\mathrm{End}_A^*(\mathcal{M})$, we define the operator $S \in \mathrm{End}_A^*(\mathcal{M})$ by the formula $S = f(T_0 T_0^*)$. Then $\|S\| = 1$ and $\|T_0 T_0^* S\| \le \frac{1}{2}\delta^{-2}$. We can choose an element $x \in \mathcal{M}$ such that $\|x\| = 1$, $\|Sx\| > \frac{1}{2}$. Then the inequality

$$\|T_0 T_0^* Sx\| \le \frac{1}{2}\delta^{-2} < \delta^{-2} \|Sx\|$$

contradicts the assumption (with $y = Sx$). So $0 \notin \mathrm{Sp}(T_0 T_0^*)$. Therefore the operator $T_0 T_0^*$ is invertible and, in particular, surjective. For any $z \in \mathcal{M}$ one can find an element $w \in \mathcal{N}_0$ such that $T_0 z = T_0 T_0^* w$. Then $z - T_0^* w \in \mathrm{Ker}\, T$ and

$$z = (z - T_0^* w) + T_0^* w \in \mathrm{Ker}\, T + \mathrm{Im}\, T_0^*.$$

Since the module $\mathrm{Im}\, T_0^*$ is obviously orthogonal to $\mathrm{Ker}\, T$, it is a complement of $\mathrm{Ker}\, T$. This completes the proof of (i).

(ii) Since $\mathcal{M} = \mathrm{Ker}\, T \oplus \mathrm{Im}\, T_0^*$, the submodule $\mathrm{Im}\, T_0^*$ is closed. Note that $\mathrm{Im}\, T_0^* = \mathrm{Im}\, T^*$, so one can apply the proof of (i) to the case of the operator T^* instead of T and it gives the orthogonal decomposition $\mathcal{N} = \mathrm{Ker}\, T^* \oplus \mathrm{Im}\, T$. □

Corollary 2.3.4. *If $P \in \mathrm{End}_A^*(\mathcal{M})$ is an idempotent, then its image $\mathrm{Im}\, P$ is an orthogonally complementable submodule in $\mathcal{M}$.*

Corollary 2.3.5. *Let $\mathcal{M}, \mathcal{N}$ be Hilbert A-modules and let $F : \mathcal{M} \to \mathcal{N}$ be a topologically injective (i.e., $\|Fx\| \ge \delta \|x\|$ for some $\delta > 0$ and for all $x \in M$) adjointable A-homomorphism. Then $F(\mathcal{M}) \oplus F(\mathcal{M})^{\perp} = \mathcal{N}$.*

Corollary 2.3.6. *Let $\mathcal{M}$ be a Hilbert A-module and let $J : \mathcal{M} \to \mathcal{M}$ be a selfadjoint topologically injective A-homomorphism. Then J is an isomorphism.*

Lemma 2.3.7 ([**90**]). *Let $\mathcal{M}$ be a finitely generated Hilbert submodule in a Hilbert module $\mathcal{N}$ over a unital C^*-algebra. Then $\mathcal{M}$ is an orthogonal direct summand in $\mathcal{N}$.*

Proof. Let $x_1, \dots, x_n \in \mathcal{M}$ be a finite set of generators. Define the operator $F : L_n(A) \longrightarrow \mathcal{N}$ by the formula $F(e_i) = x_i$, where $e_i \in L_n(A)$, $i = 1, \dots, n$, is the standard basis. It is easy to see that F is adjointable with the adjoint $F^* : \mathcal{N} \longrightarrow L_n(A)$ acting by the formula $F^*(x) = (\langle x_1, x\rangle, \dots, \langle x_n, x\rangle)$, where $x \in \mathcal{N}$. By Theorem 2.3.3 the module $\mathrm{Im}\, F = \mathcal{M}$ is an orthogonal direct summand. □

Lemma 2.3.8. *Let A be a unital C^*-algebra, $H_A = \mathcal{M} \widetilde{\oplus} \mathcal{N}$, $p : H_A \to \mathcal{M}$ a projection and $\mathcal{N}$ a projective module. Then $H_A = \mathcal{M} \oplus \mathcal{M}^{\perp}$ if and only if p is adjointable.*

PROOF. If p^* exists, then $(1-p)^* = 1-p^*$ exists as well. Therefore, by Theorem 2.3.3, $\operatorname{Ker}(1-p) = \mathcal{M}$ is the image of a selfadjoint projection.

To prove the converse, let us verify first that $H_A = \mathcal{N}^\perp + \mathcal{M}^\perp$. By the Kasparov stabilization theorem, one can assume, without loss of generality, that $\mathcal{N} = \operatorname{span}_A\langle e_1, \dots, e_n\rangle$, $\mathcal{N}^\perp = \operatorname{span}_A\langle e_{n+1}, e_{n+2}, \dots\rangle$. Let g_i be the image of e_i under the projection $\mathcal{N}$ onto $\mathcal{M}^\perp$:

$$e_1 = f_1 + g_1, \dots, e_n = f_n + g_n, \quad f_i \in \mathcal{M},\ g_i \in \mathcal{M}^\perp.$$

Since the projection induces an isomorphism of A-modules $\mathcal{N} \cong \mathcal{M}^\perp$, the elements $g_1, \dots, g_n$ are free generators and $\langle g_k, g_k\rangle > 0_A$ (i.e., the spectrum of this positive operator is separated from 0 and hence it is invertible). So, if

$$f_k = \sum_{i=1}^{\infty} f_k^i e_i, \quad \text{then} \quad e_k - f_k^k e_k = \sum_{i \neq k} f_k^i e_i + g_k.$$

On the other hand,

$$1 = \langle e_k, e_k\rangle = \langle f_k, f_k\rangle + \langle g_k, g_k\rangle, \quad 1 - (f_k^k)(f_k^k)^* \geq \langle g_k, g_k\rangle > 0,$$

i.e., the spectrum of the element $1 - f_k^k$ is separated from the origin, hence this element is invertible in A,

$$e_k = \frac{1}{1 - f_k^k}\left(\sum_{i \neq k} f_k^i e_i + g_k\right) \in \mathcal{N}^\perp + \mathcal{M}^\perp \quad (k = 1, \dots, n).$$

Thus $\mathcal{N}^\perp + \mathcal{M}^\perp = H_A$. Let $x \in \mathcal{N}^\perp \cap \mathcal{M}^\perp$. Since any element $y \in H_A = \mathcal{M}\widetilde{\oplus}\mathcal{N}$ has the form $y = m+n$, one has $\langle x, y\rangle = \langle x, m\rangle + \langle x, n\rangle = 0$; in particular, $\langle x, x\rangle = 0$, thus $x = 0$. Therefore $H_A = \mathcal{N}^\perp \widetilde{\oplus} \mathcal{M}^\perp$. Consider the map $q = \begin{cases} 1 & \text{on } \mathcal{N}^\perp \\ 0 & \text{on } \mathcal{M}^\perp \end{cases}$, which is a bounded projection since $H_A = \mathcal{N}^\perp \widetilde{\oplus} \mathcal{M}^\perp$. Let $x + y \in \mathcal{M}\widetilde{\oplus}\mathcal{N}$, $x_1 + y_1 \in \mathcal{N}^\perp \widetilde{\oplus} \mathcal{M}^\perp$. Then

$$\langle p(x+y), x_1 + y_1\rangle = \langle x, x_1 + y_1\rangle = \langle x, x_1\rangle,$$
$$\langle x + y, q(x_1 + y_1)\rangle = \langle x + y, x_1\rangle = \langle x, x_1\rangle.$$

Therefore p^* exists and equals q. □

2.4. Full Hilbert C^*-modules

Let $\mathcal{M}$ be a Hilbert A-module. Recall that it is full if the closure $\langle \mathcal{M}, \mathcal{M}\rangle \subset A$ of the linear span of all $\langle x, x\rangle$, $x \in \mathcal{M}$, coincides with A. One can always consider any Hilbert module as a full Hilbert module over the C^*-algebra $\langle \mathcal{M}, \mathcal{M}\rangle$. The following statement shows that full modules are easier to handle.

THEOREM 2.4.1 ([**71, 36**]). *Let A be a σ-unital C^*-algebra and let $\mathcal{M}$ be a full Hilbert A-module. Then:*

- (i) *There exists a Hilbert A-module $\mathcal{N}$ such that $l_2(\mathcal{M}) \cong H_A \oplus \mathcal{N}$. If A is unital, then there exist a number n and a Hilbert A-module $\mathcal{N}'$ such that $\mathcal{M} \oplus \dots \oplus \mathcal{M} = \mathcal{M}^n \cong A \oplus \mathcal{N}'$.*
- (ii) *If the module $\mathcal{M}$ is countably generated, then $l_2(\mathcal{M}) \cong H_A$.*

PROOF. Consider the set

$$S = \left\{ c \in A : \|c\| \le 1, c = \sum_{i=1}^{k} \langle x_i, x_i \rangle, k \in \mathbf{N}, x_i \in \mathcal{M} \right\}.$$

To prove the theorem, two following lemmas are required.

LEMMA 2.4.2 ([**16**]). *For any $a \in A$, $a \ge 0$ and any $\varepsilon > 0$, there exists $c \in S$ such that $\|(1-c)a\| < \varepsilon$.*

PROOF. Since the module $\mathcal{M}$ is full, one can find a finite set of elements $y_i \in \mathcal{M}$ such that

$$\left\| a - \sum_{i=1}^{k} \langle y_i, y_i \rangle \right\| < \varepsilon/2. \tag{2.5}$$

Put

$$x_i = y_i(\varepsilon^2 + \sum_{j=1}^{k} \langle y_j, y_j \rangle)^{-1/2}, \quad i = 1, \ldots, k;$$
$$c = \sum_{i=1}^{k} \langle x_i, x_i \rangle, \qquad b = \sum_{i=1}^{k} \langle y_i, y_i \rangle.$$

Then

$$\|c\| = \left\| (\varepsilon^2 + b)^{-1/2} b (\varepsilon^2 + b)^{-1/2} \right\| \le 1,$$

hence $c \in S$. Let $f(t) := \varepsilon^4 t^2 (\varepsilon^2 + t)^{-2}$. Applying this function to the element b we obtain the estimate

$$\begin{aligned} \|f(b)\| &= \left\| \varepsilon^4 b^2 (\varepsilon^2 + b)^{-2} \right\| = \left\| \varepsilon^2 (\varepsilon^2 + b)^{-1} b^2 \varepsilon^2 (\varepsilon^2 + b)^{-1} \right\| \\ &= \left\| (1-c) b^2 (1-c) \right\| \le \varepsilon^2/4. \end{aligned}$$

Therefore $\|(1-c)b\| \le \varepsilon/2$ and, together with the estimate (2.5), this proves the lemma. ☐

LEMMA 2.4.3 ([**16**]). *There exists a sequence (x_i), $x_i \in \mathcal{M}$, such that the sequence of partial sums of the series $\sum_{i=1}^{k} \langle x_i, x_i \rangle$ is a countable approximate unit of the algebra A. If A is unital, then there exist a finite number k and elements $x_1, \ldots, x_k \in \mathcal{M}$ such that $\sum_{i=1}^{k} \langle x_i, x_i \rangle = 1$.*

PROOF. Consider first the case of a unital C^*-algebra. Then, by Lemma 2.4.2, one can find an element $c \in S$ such that $\|1 - c\| < 1/2$. Therefore the element c is invertible and $c = \sum_{j=1}^{k} \langle y_j, y_j \rangle$ for some k and some $y_j \in \mathcal{M}$. Put $x_j = y_j \cdot c^{-1/2}$; then $\sum_{j=1}^{k} \langle x_j, x_j \rangle = 1$.

In the case of a nonunital C^*-algebra let $h \in A$ be a strictly positive element. By induction we shall construct a sequence (c_j) in S such that

$$\sum_{j=1}^{k} c_j \le 1; \qquad \left\| \left(1 - \sum_{j=1}^{k} c_j \right) h \right\| < \frac{1}{2^k}.$$

By Lemma 2.4.2, we can find an element $c_1 \in S$ such that $\|(1 - c_1)h\| < \frac{1}{2}$. Under the assumption that the elements $c_1, \ldots, c_k$ are already found, by Lemma 2.4.2 we can find an element $d \in S$ such that

$$\left\| (1-d)\Big(1-\sum_{j=1}^{k} c_j\Big)^{1/2} h \right\| < \frac{1}{2^{k+1}} \tag{2.6}$$

and put

$$c_{k+1} = \Big(1-\sum_{j=1}^{k} c_j\Big)^{1/2} d \Big(1-\sum_{j=1}^{k} c_j\Big)^{1/2}.$$

Since $\left\| \Big(1-\sum_{j=1}^{k} c_j\Big)^{1/2} \right\| \le 1$ and $d \in S$, we have $c_{k+1} \in S$. Since $\|d\| \le 1$ and $c_{k+1} \le 1-\sum_{j=1}^{k} c_j$, we have $\sum_{j=1}^{k+1} c_j \le 1$. Finally, inequality (2.6) implies that

$$\begin{aligned} \left\| \Big(1-\sum_{j=1}^{k+1} c_j\Big) h \right\| &= \left\| \Big(1-\sum_{j=1}^{k} c_j\Big)^{1/2} (1-d) \Big(1-\sum_{j=1}^{k} c_j\Big)^{1/2} h \right\| \\ &\le \left\| \Big(1-\sum_{j=1}^{k} c_j\Big)^{1/2} \right\| \left\| (1-d) \Big(1-\sum_{j=1}^{k} c_j\Big)^{1/2} h \right\| < \frac{1}{2^{k+1}}, \end{aligned}$$

thus completing the induction step. So we obtain that

$$\left\| \Big(1-\sum_{j=1}^{k} c_j\Big) h \right\| \longrightarrow 0$$

as $k \to \infty$. Since the strictly positive element h generates the whole C^*-algebra A [**104**], the lemma is proved. □

Let us continue the proof of the theorem. By Lemma 2.4.3, we can choose a sequence (x_i) in the module $\mathcal{M}$ such that the partial sums of $\sum \langle x_i, x_i \rangle$ form an approximate unit in A. Define the map $T : A \longrightarrow l_2(\mathcal{M})$ by the formula

$$T(a) = (x_1 a, \dots, x_k a, \dots), \qquad a \in A. \tag{2.7}$$

Since the series $\langle Ta, Ta \rangle = \sum_{i=1}^{\infty} a^* \langle x_i, x_i \rangle a = a^* a$ is convergent in A, one has $T(a) \in l_2(\mathcal{M})$. Moreover, the adjoint operator is well defined, $T^* : l_2(\mathcal{M}) \longrightarrow A$, $T^*(y_i) = \sum_{i=1}^{\infty} \langle x_i, y_i \rangle \in A$ for $(y_i) \in l_2(\mathcal{M})$. Since T^*T acts identically on A, the operator T is an isometry and

$$l_2(\mathcal{M}) = \operatorname{Im} T \oplus \operatorname{Ker} T^* \cong A \oplus \mathcal{N},$$

where $\mathcal{N} = \operatorname{Ker} T^*$. This completes the proof of statement (i) of the theorem for the case of a nonunital C^*-algebra. In the case of a unital C^*-algebra the previous reasoning can be applied literally if we replace the module $l_2(\mathcal{M})$ by $\mathcal{M}^k$ and replace the infinite sequence in (2.7) by a finite one.

We pass to the proof of statement (ii). Renumber the sequences in the module $l_2(\mathcal{M})$ with the help of a bijection $\mathbf{N} \longrightarrow \mathbf{N} \times \mathbf{N}$. Then elements of the module $l_2(\mathcal{M})$ can be viewed as sequences (m_{ij}), $m_{ij} \in \mathcal{M}$, and, for each $i \in \mathbf{N}$, the set of sequences (m_{ij}), $j \in \mathbf{N}$, is isomorphic to the module $l_2(\mathcal{M})$. Such a renumbering defines an isomorphism $l_2(\mathcal{M}) \cong l_2(l_2(\mathcal{M}))$. Taking into account the isomorphism $l_2(\mathcal{M}) \cong A \oplus \mathcal{N}$, we conclude that

$$l_2(\mathcal{M}) \cong l_2(l_2(\mathcal{M})) \cong l_2(A \oplus \mathcal{N}) = l_2(A) \oplus l_2(\mathcal{N}).$$

Note that the Hilbert module $l_2(\mathcal{N})$ is countably generated, hence

$$l_2(\mathcal{M}) \cong l_2(A) \oplus l_2(\mathcal{N}) \cong l_2(A)$$

by the Kasparov stabilization theorem. □

2.5. Dual modules. Self-duality

For a Hilbert A-module $\mathcal{M}$ let us denote by $\mathcal{M}'$ the set of all bounded A-linear maps from $\mathcal{M}$ to A. The structure of a vector space over the field $\mathbf{C}$ is introduced by the formula $(\lambda \cdot f)(x) := \overline{\lambda} f(x)$, where $\lambda \in \mathbf{C}$, $f \in \mathcal{M}'$, $x \in \mathcal{M}$. This definition may seem artificial, however it is convenient since it allows us to define a linear inclusion of the module $\mathcal{M}$ into $\mathcal{M}'$ (there is also the alternate approach: to define $\mathcal{M}'$ as the set of all *anti*-linear maps from $\mathcal{M}$ into A). The formula

$$(f \cdot a)(x) = a^* f(x),$$

where $a \in A$, introduces the structure of a right A-module on $\mathcal{M}'$. This module is complete with respect to the norm $\|f\| = \sup\{\|f(x)\| : \|x\| \leq 1\}$. Such modules are called *dual* (*Banach*) modules. The elements of the module $\mathcal{M}'$ are called *functionals* on the Hilbert module $\mathcal{M}$. Note that there is an obvious isometric inclusion $\mathcal{M} \subset \mathcal{M}'$, which is defined by the formula $x \mapsto \langle x, \cdot \rangle = \widehat{x}$. Sometimes we shall write $\langle f, x \rangle$ instead of $f(x)$.

DEFINITION 2.5.1. A Hilbert module $\mathcal{M}$ is called *self-dual* if $\mathcal{M} = \mathcal{M}'$.

The condition of self-duality is very strong. Below we shall see that there are quite a few self-dual modules: any Hilbert module over a C^*-algebra A is self-dual iff A is finite dimensional. Self-dual Hilbert C^*-modules behave quite like Hilbert spaces. In the same way as in the case of Hilbert spaces, the following statements can be easily checked.

PROPOSITION 2.5.2 ([**100**]). *Let $\mathcal{M}$ be a self-dual Hilbert A-module, $\mathcal{N}$ an arbitrary Hilbert A-module and $T : \mathcal{M} \longrightarrow \mathcal{N}$ a bounded operator, $T \in \mathrm{Hom}_A(\mathcal{M}, \mathcal{N})$. Then there exists an operator $T^* : \mathcal{N} \longrightarrow \mathcal{M}$ such that the equality $\langle x, T^* y \rangle = \langle Tx, y \rangle$ holds for all $x \in \mathcal{M}$, $y \in \mathcal{N}$.*

COROLLARY 2.5.3. *Let $\mathcal{M}$ be a self-dual Hilbert A-module. Then* $\mathrm{End}_A(\mathcal{M}) = \mathrm{End}^*_A(\mathcal{M})$.

PROPOSITION 2.5.4. *Let $\mathcal{M}$ be a self-dual Hilbert A-module and let $\mathcal{M} \subset \mathcal{N}$. Then $\mathcal{N} = \mathcal{M} \oplus \mathcal{M}^{\perp}$.*

PROOF. Since $\mathcal{M}$ is self-dual, $i : \mathcal{M} \to \mathcal{N}$ is an isometric adjointable inclusion. Therefore $\mathcal{M} = i\mathcal{M}$ has an orthogonal complement by Corollary 2.3.5. □

If A is a unital C^*-algebra, then the Hilbert module $L_n(A)$ is obviously self-dual. For an arbitrary module this is not true; moreover, the Banach module $\mathcal{M}'$ may not admit the structure of a Hilbert C^*-module at all. A description of the dual module for the standard Hilbert module H_A is given by the following statement.

PROPOSITION 2.5.5. *Consider the set of sequences $f = (f_i)$, $f_i \in A$, $i \in \mathbf{N}$, such that the norms of partial sums $\left\| \sum_{i=1}^N f_i^* f_i \right\|$ are uniformly bounded. If A is a* *unital C^*-algebra, then this set coincides with H'_A, the action of f on elements of the module H_A is defined by the formula*

$$f(x) = \sum_{i=1}^{\infty} f_i^* x_i, \tag{2.8}$$

where $x = (x_i) \in H_A$, and the norm of f is given by the equality

$$\|f\|^2 = \sup_N \left\| \sum_{i=1}^{N} f_i^* f_i \right\|. \tag{2.9}$$

PROOF. Let $f \in H'_A$ and let e_i be the standard basis in H_A. Put $f_i = (f(e_i))^*$. We shall show that the sequence (f_i) determines a functional f in a unique way. Let us assume that there exists a functional $g \neq f$, $g(e_i) = f(e_i)$. Choose $x \in H_A$ such that $\|f(x) - g(x)\| = C \neq 0$. Denote by $x^{(N)}$ the image of x under the projection onto the submodule $L_N(A) \subset H_A$, $x^{(N)} = \sum_{i=1}^N e_i x_i = (x_1, \ldots, x_N, 0, \ldots)$. Let us find a number N such that

$$\left\| x - x^{(N)} \right\| = \left\| \sum_{i=N+1}^{\infty} x^* x \right\|^{1/2} < \frac{C}{2(\|f\| + \|g\|)}.$$

Since $f(x^{(N)}) = g(x^{(N)})$, we have $\left\|f(x - x^{(N)}) - g(x - x^{(N)})\right\| = C$. But, on the other hand, we have

$$\begin{aligned} \left\| f(x - x^{(N)}) - g(x - x^{(N)}) \right\| &\leq (\|f\| + \|g\|) \left\| x - x^{(N)} \right\| \\ &< (\|f\| + \|g\|) \frac{C}{2(\|f\| + \|g\|)} \leq C/2. \end{aligned}$$

The obtained contradiction shows that $f = g$. The Cauchy–Bunyakovskii inequality

$$\left\| \sum_{i=1}^{N} f_i^* x_i \right\|^2 \leq \left\| \sum_{i=1}^{N} f_i^* f_i \right\| \left\| \sum_{i=1}^{N} x_i^* x_i \right\| \tag{2.10}$$

shows that

$$\|f\|^2 \leq \sup_N \left\| \sum_{i=1}^{N} f_i^* f_i \right\|. \tag{2.11}$$

Note that if we take $x_i = f_i / \left\| \sum_{i=1}^N f_i^* f_i \right\|^{1/2}$, then equality is obtained in (2.10). Put $f^{(N)} = (f_1, \ldots, f_N, 0, \ldots)$, $f^{(N)} \in L_N(A)' \cong L_N(A)$. It is clear that

$$\|f\| \geq \left\| f^{(N)} \right\|. \tag{2.12}$$

But $\left\|f^{(N)}\right\|^2 = \left\| \sum_{i=1}^N f_i^* f_i \right\|$, hence (2.11) and (2.12) imply (2.9). Since for any $\varepsilon > 0$ one can find a number N such that the estimate

$$\left\| \sum_{i=N}^{N+n} f_i^* x_i \right\| \leq \left\| \sum_{i=N}^{N+n} f_i^* f_i \right\| \cdot \left\| \sum_{i=N}^{N+n} x_i^* x_i \right\| \leq \|f\|^2 \left\| \sum_{i=N}^{N+n} x_i^* x_i \right\| < \|f\|^2 \varepsilon$$

holds for all $n > 0$, the convergence of the series (2.8) follows. $\square$

Note that for the functional $f = (\varphi_i)$ from Example 2.1.2, the partial sums $\sum_{i=1}^{N} \varphi_i^* \varphi_i$ are uniformly bounded, however the corresponding series is not convergent.

Let us describe an interesting example of a dual module.

EXAMPLE 2.5.6 ([**44**]). Let $A = \mathcal{B}(H)$ be the algebra of all bounded operators on a separable Hilbert space H. Consider pairwise orthogonal projections $p_i \in A$, $i \in \mathbf{N}$, such that the series $\sum_i p_i$ converges w^*-weakly to $1 \in A$, and each projection p_i is equivalent to 1. We can consider $H = \bigoplus_i H_i$ as an orthogonal sum of Hilbert spaces isomorphic to H, $u_i : H \to H_i$ being isometries, so that

$$p_i = u_i u_i^*, \qquad 1 = u_i^* u_i.$$

As was shown above (see Proposition 2.5.5),

$$l_2(A)' = \left\{ \{a_i\} \middle| a_i \in A, i \in \mathbf{N}, \left\{ \sup_{N \in \mathbf{N}} \left\| \sum_{i=1}^{N} a_i a_i^* \right\| \right\} < \infty \right\}$$

is a C^*-Hilbert module over A with respect to the inner product

$$\langle \{a_i\}, \{b_i\} \rangle := w^* \text{-lim} \sum_i a_i b_i^*.$$

The maps

$$S : A \to l_2(A)', \qquad S : a \mapsto a \cdot \{u_i\},$$
$$S^{-1} : l_2(A)' \to A, \qquad S^{-1} : \{a_i\} \mapsto w^* \text{-lim} \sum_i a_i u_i^*$$

define an isometric isomorphism of A and $l_2(A)'$.

Let φ be a positive linear functional on A. If $\mathcal{M}$ is a Hilbert A-module and if $N_\varphi = \{x \in \mathcal{M} : \varphi(\langle x, x \rangle) = 0\}$ is a linear subspace of $\mathcal{M}$, then $\mathcal{M}/N_\varphi$ is a pre-Hilbert space with the inner product $(\cdot,\cdot)_\varphi$ given by the formula

$$(x + N_\varphi, y + N_\varphi)_\varphi = \varphi(\langle x, y \rangle), \quad x, y \in \mathcal{M}.$$

We denote the norm defined by this scalar-valued inner product by $\|\cdot\|_\varphi$ and the Hilbert space obtained by completion of $\mathcal{M}/N_\varphi$ with respect to this norm by H_φ. Let $f \in \mathcal{M}'$. By Theorem 2.1.4 and Corollary 2.1.6, we have

$$f(x)^* f(x) \leq \|f\|^2 \langle x, x \rangle$$

for all $x \in \mathcal{M}$. Therefore, if $x \in N_\varphi$, then

$$\varphi(f(x)^* f(x)) = 0 = \varphi(f(x)).$$

Hence, the map

$$x + N_\varphi \longmapsto \varphi(f(x)) \tag{2.13}$$

defines a linear functional on $\mathcal{M}/N_\varphi$. Since

$$\begin{aligned} |\varphi(f(x))| &\leq \|\varphi\|^{1/2} \varphi(f(x)^* f(x))^{1/2} \leq \|\varphi\|^{1/2} \|f\| \varphi(\langle x, x \rangle)^{1/2} \\ &= \|\varphi\|^{1/2} \|f\| \|x + N_\varphi\|_\varphi, \end{aligned}$$

the functional (2.13) is bounded. Then there exists a unique vector $f_\varphi \in H_\varphi$ such that $\|f_\varphi\|_\varphi \leq \|f\| \|\varphi\|^{1/2}$ and $(f_\varphi, x + N_\varphi)_\varphi = \varphi(f(x))$ for all $x \in \mathcal{M}$. For $x \in \mathcal{M}$, we shall denote by $\widehat{x}$ the functional $\langle x, \cdot \rangle \in \mathcal{M}'$. Note that $\widehat{y}_\varphi = y + N_\varphi$ for all $y \in \mathcal{M}$.

Let ψ be another positive linear functional on A such that $\psi \le \varphi$. Then $N_\varphi \subset N_\psi$ and the natural map $x + N_\varphi \longmapsto x + N_\psi$ can be extended to the map

$$V_{\varphi,\psi} : H_\varphi \longrightarrow H_\psi, \qquad \|V_{\varphi,\psi}\| \le 1.$$

It is easy to see that $V_{\varphi,\psi}(\widehat{x}_\varphi) = \widehat{x}_\psi$. It turns out that the same holds for all elements of $\mathcal{M}'$.

PROPOSITION 2.5.7. *Let $\mathcal{M}$ be a Hilbert A-module and let φ and ψ be positive linear functionals on A with $\psi \le \varphi$. Then $V_{\varphi,\psi}(f_\varphi) = f_\psi$ for any functional $f \in \mathcal{M}'$.*

PROOF. Let $f \in \mathcal{M}'$. Since the quotient space $\mathcal{M}/N_\varphi$ is dense in H_φ, one can choose a sequence $\{y_n + N_\varphi\}$ of elements in $\mathcal{M}/N_\varphi$ such that $\|y_n + N_\varphi - f_\varphi\|_\varphi \to 0$. Then

$$V_{\varphi,\psi}(f_\varphi) = \lim_n V_{\varphi,\psi}(y_n + N_\varphi) = \lim_n (y_n + N_\psi).$$

To prove the statement, it is sufficient to show that $\psi(\langle y_n, x\rangle) \to \psi(f(x))$ for all $x \in \mathcal{M}$. But

$$\begin{aligned} &|\psi(\langle y_n,x\rangle - f(x))|^2 \\ &\qquad \le \|\psi\| \cdot \psi(\langle y_n, x\rangle\langle x, y_n\rangle - \langle y_n, x\rangle f(x)^* - f(x)\langle x, y_n\rangle + f(x)f(x)^*) \\ &\qquad \le \|\varphi\| \cdot \varphi(\langle y_n, x\rangle\langle x, y_n\rangle - \langle y_n, x\rangle f(x)^* - f(x)\langle x, y_n\rangle + f(x)f(x)^*) \end{aligned}$$

for each $n \in \mathbf{N}$. Since

$$\varphi(\langle y_n, x\rangle f(x)^*) = \varphi(\langle y_n, x \cdot (f(x))^*\rangle) \to \varphi(f(x \cdot (f(x))^*)) = \varphi(f(x)f(x)^*),$$

it is sufficient to show that $\varphi(\langle y_n, x\rangle\langle x, y_n\rangle - f(x)\langle x, y_n\rangle) \to 0$. Note that

$$\begin{aligned} \varphi(\langle y_n, x\rangle\langle x, y_n\rangle - f(x)\langle x, y_n\rangle) &= \varphi(\langle y_n, x \cdot \langle x, y_n\rangle\rangle - f(x \cdot \langle x, y_n\rangle)) \\ &= (y_n + N_\varphi - f_\varphi, x \cdot \langle x, y_n\rangle + N_\varphi)_\varphi, \end{aligned}$$

and the sequence $\{x \cdot \langle x, y_n\rangle + N_\varphi\}$ is bounded with respect to the norm $\|\cdot\|_\varphi$. Indeed,

$$\begin{aligned} \|x \cdot \langle x, y_n\rangle + N_\varphi\|_\varphi^2 &= \varphi(\langle x \cdot \langle x, y_n\rangle\rangle, \langle x \cdot \langle x, y_n\rangle\rangle)) \\ &= \varphi(\langle y_n, x\rangle\langle x, x\rangle\langle x, y_n\rangle) \\ &\le \|x\|^2 \cdot \varphi(\langle y_n, x\rangle\langle x, y_n\rangle) \\ &\le \|x\|^2 \cdot \varphi(\|x\|^2 \cdot \langle y_n, y_n\rangle) \\ &= \|x\|^4 \cdot \|y_n + N_\varphi\|^2, \end{aligned}$$

and the sequence $\{y_n + N_\varphi\}$ is bounded. Since $\|y_n + N_\varphi - f_\varphi\|_\varphi \to 0$, the statement is proved. □

It turns out that the property of a Hilbert C^*-module over a unital C^*-algebra being self-dual is very close to the property of being finitely generated and projective. We complete this section with a complete description [**131**] of that interplay in the commutative case.

DEFINITION 2.5.8. A C^*-algebra A is called *module-infinite* (MI) if each countably generated Hilbert A-module is projective and finitely generated if and only if it is self-dual.

Recall that a projective finitely generated module over a unital C^*-algebra is always self-dual.

DEFINITION 2.5.9. A commutative unital C^* algebra $A = C(Y)$ is said to be DI (*divisibly infinite*) if for any infinite sequence u_i of elements of norm $1 \geq \|u_i\| \geq C > 0$ in A there exist a subsequence $i(k)$ and elements $0 \leq b_k \in A$ of norm 1 such that

(i) the supports of b_k in Y are pairwise disjoint;

(ii) for each k there exist points y_k, y'_k such that $b_k(y_k) = 1$, $y'_k \notin \bigcup_j \operatorname{supp} b_j$, $|u_{i(k)}(y'_k)| \geq \delta$, $|u_{i(k)}(y_k)| \geq \delta$, and the sequences $\{y_k\}$ and $\{y'_k\}$ have a common accumulation point. In particular, $\sum_k b_k^s$ is a discontinuous function for any integer $s \geq 1$.

THEOREM 2.5.10. *If a commutative unital* C^**-algebra* A *is* DI*, then it is* MI.

PROOF. We have to prove that any countably generated self-dual Hilbert A-module $\mathcal{M}$ is finitely generated and projective. By the Kasparov stabilization theorem 1.4.2, one has $\mathcal{M} \oplus l_2(A) \cong l_2(A)$, where $l_2(A)$ is the standard Hilbert C^*-module (see p. 6). Denote by $p : l_2(A) \to \mathcal{M} \subset l_2(A)$ the corresponding orthogonal projection and let $p_j : l_2(A) \to E_j \cong A^j$ be the orthogonal projection onto the first j standard summands of $l_2(A)$. Then either 1) $\|(1 - p_j)p\| \to 0$ as $j \to \infty$ or 2) this is not the case.

1) Let us show that in this case $\mathcal{M}$ is finitely generated and projective. One can argue as in [**92**]: for a sufficiently large j, the operator

$$J(x) = \begin{cases} p_j(x) & \text{if } x \in \mathcal{M}, \\ x & \text{if } x \in \mathcal{M}^\perp \cong l_2(A), \end{cases}$$

is close to the identity, hence it is an isomorphism. It maps $\mathcal{M}$ isomorphically onto a direct summand of E_j.

2) In this case (changing the standard decomposition, if necessary) we can find a sequence $j(k)$ such that for each k there exists an element $z_k \in \mathcal{M}$ of norm 1 such that its $j(k)$-th entry $z_k^{j(k)}$ (with respect to the standard decomposition) has norm greater than some fixed $C > 0$. Let us choose functions b_k as in Definition 2.5.9 (for simplicity we assume that we do not need to pass to a subsequence for the second time). Then the formula $\beta(x) = \sum_k b_k x^{j_k}$ defines a functional on $l_2(A)$. It is evident that this functional is not adjointable on $l_2(A)$ since $b_k \not\to 0$ as $k \to \infty$. Let us show that there is no adjointable functional α on $l_2(A)$ with $\alpha|_{\mathcal{M}} = \beta|_{\mathcal{M}}$, so that $\beta|_{\mathcal{M}}$ is a nonadjointable functional on $\mathcal{M}$ and $\mathcal{M}$ is not a self-dual module. Indeed, suppose that there exists an element $a = (a_1, a_2, \dots) \in \mathcal{M} \subset l_2(A)$ such that $\alpha(x) := \sum_i a_i x^i = \beta(x)$ for any $x \in \mathcal{M}$. Then

$$\langle a, a \rangle \leq \sum_k b_k^2 \tag{2.14}$$

(the last element is a bounded measurable discontinuous function; see Definition 2.5.9 (ii)), since for any continuous positive function f_k equal to 1 on $\bigcup_{j \leq k} \operatorname{supp} b_j$ and equal to 0 on $\bigcup_{j > k} \operatorname{supp} b_j$ (the existence follows from normality of Y), one has $\beta f_k = \alpha f_k + (\beta - \alpha) f_k$. But βf_k is an element of $l_2(A)$ (or, more precisely, an adjointable functional $\beta f_k(x) = \sum_{j \leq k} b_k x^{j_k}$). Let $b f_k$ be the corresponding element. Then $b f_k = a f_k + (b - a) f_k$ is the decomposition corresponding to $l_2(A) = \mathcal{M} \oplus l_2(A)$. Indeed, $a \in \mathcal{M}$ and $\langle (b - a) f_k, x \rangle = 0$ for any $x \in M$, since

$\alpha(x)=\beta(x)$ for those x. The Pythagorean theorem for Hilbert C^*-modules shows that $\langle a,a\rangle f_k^2 \le \sum_{j\le k} b_j^2$. Taking all the f_k's, one obtains (2.14).

On the other hand,

$$\langle a,a\rangle(y_i) \ge \langle a,z_i\rangle\langle z_i,a\rangle(y_i) = \sum_{m,k} b_m z_i^{j(m)}\overline{z_i^{j(k)}} b_k(y_i) = b_i^2(y_i) z_i^{j(i)}(y_i) \ge \delta^2.$$

But (2.14) implies that $\langle a,a\rangle(y_i')=0$. Hence, $\langle a,a\rangle$ does not belong to A. This contradiction proves the theorem. □

Now we describe some consequences of this theorem.

THEOREM 2.5.11. *A commutative separable unital C^*-algebra A is* MI *if and only if its Gelfand spectrum Y has no isolated points.*

PROOF. If Y has an isolated point, then a separable Hilbert space arises as one of self-dual modules, hence, MI does not hold.

Now, suppose that Y has no isolated points, in particular it is infinite. Since Y is a compact Hausdorff separable space, the topology is generated by some metric ρ. For any given sequence u_i of elements with norm $\ge C$ we can find a sequence of different points $\widetilde{y}_i$ such that $|u_i(\widetilde{y}_i)| > 2C/3$.

Since Y is compact, one can pass to a convergent subsequence $y_k=\widetilde{y}_{i(k)}$. For a convergent sequence, by induction, we can choose ε_k such that the corresponding ε_k-neighborhoods U_k of y_k are pairwise disjoint. Then we can choose $y_k' \ne y_k$ inside these neighborhoods so close to y_k that $|u_{i(k)}| \ge C/2$. Finally one can choose functions b_k such that $b_k(y_k)=1$ and $\operatorname{supp} b_k \subset U_k \setminus y_k'$. It remains to take $\delta = C/2$. □

COROLLARY 2.5.12. *Suppose that X is a compact connected separable Hausdorff G-space. Then $C_G(X)$ is* MI *if and only if X/G has at least two separated points.*

PROOF. The Gelfand spectrum Y of $C_G(X)$ is a quotient space of X/G with respect to the equivalence of nonseparated points. Since X is connected, Y is connected too. So, by Theorem 2.5.11, $C_G(X)$ is MI if and only if Y is not reduced to one point. □

2.6. Banach-compact operators

DEFINITION 2.6.1. Let $\mathcal{M},\mathcal{N}$ be Hilbert A-modules and let $\mathcal{M}'$ be the dual module for $\mathcal{M}$. Consider the closure $\mathcal{B}K(\mathcal{M},\mathcal{N})$ of the linear span of all operators

$$\theta_{y,f}(x) = y\cdot f(x),$$

$x\in\mathcal{M}$, $y\in\mathcal{N}$, $f\in\mathcal{M}'$, in the Banach space $\operatorname{Hom}_A(\mathcal{M},\mathcal{N})$. We call the elements of the set $\mathcal{B}K(\mathcal{M},\mathcal{N})$ *Banach-compact operators.*

In the case $\mathcal{N}=\mathcal{M}$ the set $\mathcal{B}K(\mathcal{M},\mathcal{N})$ is equipped with the natural structure of a Banach algebra. If $T\in\operatorname{End}_A(\mathcal{M})$ is a (not necessarily adjointable) operator, then the equalities

$$\theta_{y,f}Tx = y\cdot f(Tx) = \theta_{y,f\circ T}(x), \qquad T\theta_{y,f}(x) = Ty\cdot f(x) = \theta_{Ty,f}(x)$$

show that $\mathcal{B}K(\mathcal{M})$ is a two-sided ideal in the Banach algebra $\operatorname{End}_A(\mathcal{M})$.

In the case of the standard Hilbert module over a unital C^*-algebra we give one more (geometric) description of compact and Banach-compact operators. Let

$S \subset H_A$ be a bounded set. We call it *A-pre-compact* if for each $\varepsilon > 0$ there exists a free finitely generated A-module $\mathcal{N} \cong L_n(A)$, $\mathcal{N} \subset H_A$, such that $\operatorname{dist}(S, \mathcal{N}) < \varepsilon$.

PROPOSITION 2.6.2. *Let* $T \in \operatorname{End}_A(H_A)$ (*resp.* $T \in \operatorname{End}^*_A(H_A)$). *Then the following conditions are equivalent:*

(i) $T \in \mathcal{B}K(H_A)$ (*resp.*, $T \in \mathcal{K}(H_A)$);
(ii) *the image* $T(B_1(H_A))$ *of the unit ball* $B_1(H_A)$ *is A-pre-compact.*

PROOF. If statement (i) holds, it is sufficient to prove that one can find an approximating module $\mathcal{N} \cong L_n(A)$ for a finite set of elements in H_A and this can be easily done by the Dupré – Fillmore method, as in the proof of Theorem 1.4.5. So suppose that (ii) is fulfilled. Then for any $\varepsilon > 0$, one can find elements $b_1, \dots, b_k \in H_A$, generating the module $\mathcal{N} \subset H_A$, such that $\langle b_i, b_j \rangle = \delta_{ij}$ and $\operatorname{dist}(T(B_1(H_A)), \mathcal{N}) < \varepsilon$, where $B_1(H_A)$ is the unit ball of H_A. Denote by $P_{\mathcal{N}}$ the projection onto $\mathcal{N}$ and consider the operator $P_{\mathcal{N}}T$. It can be decomposed as

$$P_{\mathcal{N}}Tx = b_1\langle f_1, x\rangle + \cdots + b_n\langle f_n, x\rangle, \tag{2.15}$$

where $f_i \in H'_A$. Since $x \in B_1(H_A)$, one can find an element $b \in \mathcal{N}$ such that $\|Tx - b\| < \varepsilon$. Therefore

$$\begin{aligned} \|Tx - P_{\mathcal{N}}Tx\| &= \|Tx - b + b - P_{\mathcal{N}}Tx\| \\ &= \|Tx - b\| + \|P_{\mathcal{N}}(b - Tx)\| \le \varepsilon + \|P_{\mathcal{N}}\|\,\varepsilon = 2\varepsilon, \end{aligned} \tag{2.16}$$

hence $\|T - P_{\mathcal{N}}T\| \le 2\varepsilon$ and T lies in the norm closure of the set of operators of the form (2.15). If T is adjointable, then $P_{\mathcal{N}}T$ is adjointable as well, hence $f_i \in H_A$ and T is compact. □

2.7. C^*-Fredholm operators and index. Mishchenko's approach

The material of this section is taken mainly from [**92**]. Recall first the definition of K-groups.

DEFINITION 2.7.1 ([**60**, § II.1]). Let M be an Abelian monoid. Consider the Cartesian product $M \times M$ and its quotient monoid with respect to the equivalence relation

$$(m, n) \sim (m', n') \;\Leftrightarrow\; \exists\, p, q :\; (m, n) + (p, p) = (m', n') + (q, q).$$

This quotient monoid is a group, which is denoted by $S(M)$ and is called the *symmetrization* of M. Consider now the additive category $\mathcal{P}(A)$ of projective modules over a unital C^*-algebra A and denote by $[\mathcal{M}]$ the isomorphism class of an object $\mathcal{M}$ from $\mathcal{P}(A)$. The set $\Phi(\mathcal{P}(A))$ of these classes has the structure of an Abelian monoid with respect to the operation $[\mathcal{M}] + [\mathcal{N}] = [\mathcal{M} \oplus \mathcal{N}]$. In this case the group $S(\Phi(\mathcal{P}(A)))$ is denoted by $K(A)$ or $K_0(A)$ and is called the *K-group* of A or the *Grothendieck group* of the category $\mathcal{P}(A)$. If A has no unit, then the natural map $A^+ \to \mathbf{C}$ induces a map of K-groups and $K_0(A)$ is defined by

$$K_0(A) := \operatorname{Ker}(K_0(A^+) \to K_0(\mathbf{C})).$$

The groups $K_n(A) := K_0(A \otimes C_0(\mathbf{R}^n))$ for integer $n > 0$ turn out to be 2-periodic in n and the definition can be extended to $n \in \mathbf{Z}$ by periodicity. For unital C^*-algebras one could use classes of stable homotopy of projections in A^n (i.e., allowing addition of direct summands) instead of classes of isomorphic projective modules.

More precisely, projections $p : A^n \to A^n$ and $q : A^m \to A^m$ are equivalent if there are m' and n' such that $n + n' = m + m' = s$ and the projections

$$A^s = A^n \oplus A^{n'} \overset{\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}}{\longrightarrow} A^n \oplus A^{n'} = A^s,$$

$$A^s = A^m \oplus A^{m'} \overset{\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}}{\longrightarrow} A^m \oplus A^{m'} = A^s$$

can be connected by a norm continuous homotopy in the set of all projections from $\mathrm{End}(A^s) = \mathrm{End}^*(A^s)$ (A is unital!). One can also consider the equivalence classes of projections in the algebraic sense. The details can be found in [**10, 60, 94, 134**].

Recall the following well-known statement.

Lemma 2.7.2 (cf. [**115**, Theorem 4.15]). *The set of epimorphisms is open in the space of bounded linear maps of a Banach space E to a Banach space G.*

Lemma 2.7.3 ([**92**, 1.4]). *Let $\mathcal{N}$ be a finitely generated Hilbert A-module over a unital C^*-algebra A and let $a_1, \dots, a_s$ be generators of $\mathcal{N}$. Then there exists a number $\varepsilon > 0$ such that if some elements $a'_1, \dots, a'_s \in \mathcal{N}$ satisfy*

$$\|a'_k - a_k\| < \varepsilon \qquad (k = 1, \dots, s),$$

then these elements also generate $\mathcal{N}$.

Proof. The map

$$f : L_s(A) \to \mathcal{N}, \qquad (0, \dots, 0, \underset{i}{1}, 0, \dots, 0) \mapsto a_i,$$

is an epimorphism. Hence, by Lemma 2.7.2, there exists $\varepsilon > 0$ such that if $\|g - f\| < s\varepsilon$, then g is an epimorphism. Let

$$g : L_s(A) \to \mathcal{N}, \qquad (0, \dots, 0, \underset{i}{1}, 0, \dots, 0) \mapsto a'_i.$$

Then for any $\alpha = (\alpha_1, \dots, \alpha_s) \in L_s(A)$ with norm $\|\alpha\| \le 1$, one has

$$\|(g - f)\alpha\| = \left\| \sum_{i=1}^{s} (a'_i - a_i)\alpha_i \right\| \le s\varepsilon,$$

hence g is an epimorphism and the elements $a'_1, \dots, a'_s$ generate $\mathcal{N}$. □

In this section we always assume that the algebra A is unital. Recall the definition of a Fredholm operator suggested by Mishchenko in [**92**].

Definition 2.7.4. A (bounded A-linear) operator $F : H_A \to H_A$ is called *A-Fredholm* if

(i) it is adjointable;
(ii) there exists a decomposition of the domain, $H_A = \mathcal{M}_1 \widetilde{\oplus} \mathcal{N}_1$, and the range, $H_A = \mathcal{M}_2 \widetilde{\oplus} \mathcal{N}_2$ (where $\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1, \mathcal{N}_2$ are closed A-modules and $\mathcal{N}_1, \mathcal{N}_2$ have a finite number of generators), such that F has the matrix form $F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$ with respect to these decompositions and $F_1 : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism.

THEOREM 2.7.5 ([**92**]). *Let* $H_A \cong \mathcal{M}\widetilde{\oplus}\mathcal{N}$, *where* $\mathcal{M}$ *and* $\mathcal{N}$ *are closed* A*-modules and* $\mathcal{N}$ *has a finite number of generators* $a_1, \dots, a_s$. *Then* $\mathcal{N}$ *is a projective* A*-module of finite type.*

PROOF. By Lemma 2.7.3, there exists some $\varepsilon > 0$ such that the estimate

$$\|a'_k - a_k\| < \varepsilon, \quad a'_k \in \mathcal{N}, \quad k = 1, \dots, s,$$

implies that $\{a'_k\}$ generate $\mathcal{N}$. Let $P : H_A \to \mathcal{N}$ be the projection along the summand $\mathcal{M}$ in the module H_A. Then P is bounded and A-linear. Therefore there exists some $\delta > 0$ such that $\|x\| < \delta$ implies $\|Px\| < \varepsilon$. Choose a number n_0 such that

$$\|a_k - \overline{a}_k\| < \delta, \qquad k = 1, \dots, s,$$

where $\overline{a}_k$ is the projection of a_k onto L_{n_0} along $L_{n_0}^{\perp}$. Write $\overline{a}_k$ with respect to the decomposition $H_A = \mathcal{M}\widetilde{\oplus}\mathcal{N}$ as the sum

$$\overline{a}_k = a'_k + a''_k, \quad a'_k \in \mathcal{N}, \quad a''_k \in \mathcal{M}.$$

Then $a_k - a'_k = P(a_k - \overline{a}_k)$. Therefore $\|a_k - a'_k\| < \varepsilon$ and $\{a'_k\}$ generate $\mathcal{N}$. Let $\overline{\mathcal{N}}$ be the module generated by $\{\overline{a}_k\}$. Then H_A coincides with the sum $\mathcal{M} + \overline{\mathcal{N}}$. Indeed, if $x \in H_A$, then

$$x = x_{\mathcal{M}} + \sum \lambda^k a'_k = (x_{\mathcal{M}} - \sum \lambda^k a''_k) + \sum \lambda^k \overline{a}_k.$$

Consider the operator Q projecting onto L_{n_0} along $L_{n_0}^{\perp}$. Then

$$\begin{aligned} Q(a_k) &= \overline{a}_k, & Q(\mathcal{N}) &= \overline{\mathcal{N}}, \\ P(\overline{a}_k) &= a'_k, & P(\overline{\mathcal{N}}) &= \mathcal{N}. \end{aligned}$$

Since a_k are close to a'_k, the composition of operators

$$\mathcal{N} \xrightarrow{Q} \overline{\mathcal{N}} \xrightarrow{P} \mathcal{N}$$

is an isomorphism. Therefore $Q : \mathcal{N} \to \overline{\mathcal{N}}$ and $P : \overline{\mathcal{N}} \to \mathcal{N}$ are isomorphisms. In particular, if $\sum_{k=1}^{s} \lambda_k a'_k = 0$ holds, then $\sum_{k=1}^{s} \lambda_k \overline{a}_k = 0$. Therefore $\mathcal{M} \cap \mathcal{N} = 0$, i.e., $H_A = \mathcal{M}\widetilde{\oplus}\overline{\mathcal{N}}$. It is clear that $\overline{\mathcal{N}}$ is a closed A-submodule in H_A and

$$L_{n_0} = (\mathcal{M} \cap L_{n_0})\widetilde{\oplus}\overline{\mathcal{N}}.$$

Indeed, $(\mathcal{M} \cap L_{n_0}) \cap \overline{\mathcal{N}} = 0$; on the other hand, if $x \in L_{n_0}$, then $x = x' + x''$, $x' \in \overline{\mathcal{N}}$, $x'' \in \mathcal{M}$. Since $\overline{\mathcal{N}} \subset L_{n_0}$, one has $x'' \in L_{n_0}$, i.e., $x'' \in \mathcal{M} \cap L_{n_0}$. Thus $\mathcal{N}$ is isomorphic to a direct summand in the free finitely generated A-module L_{n_0}. $\square$

THEOREM 2.7.6 ([**129**]). *In the decomposition mentioned in the definition of a Fredholm operator* (*see* 2.7.4), *one always can assume that the modules* $\mathcal{M}_0$ *and* $\mathcal{M}_1$ *are orthogonally complementable. More precisely, there exist decompositions for* F,

$$\begin{pmatrix} F_3 & 0 \\ 0 & F_4 \end{pmatrix} : H_A = V_0\widetilde{\oplus}W_0 \to V_1\widetilde{\oplus}W_1 = H_A,$$

such that $V_0^{\perp} \oplus V_0 = H_A$, $V_1^{\perp} \oplus V_1 = H_A$ *or* (*which is the same by Lemma* 2.3.8) *such that projections* $V_0\widetilde{\oplus}W_0 \to V_0$ *and* $V_1\widetilde{\oplus}W_1 \to V_1$ *are adjointable.*

Proof. Let $W_0 = \mathcal{N}_0$, $V_0 = W_0^\perp$. The orthogonal complement exists by Lemma 2.3.7. The restriction $F|_{W_0^\perp}$ is an isomorphism. Indeed, if $x_n \in W_0^\perp$, then let $x_n = x_1^n + x_2^n$, $x_1^n \in \mathcal{M}_0$, $x_2^n \in W_0$, $\|x_n\| = 1$. Let us assume that $\|Fx_n\| \to 0$; then $\|Fx_1^n + Fx_2^n\| \to 0$. Since $Fx_1^n \in \mathcal{M}_1$, $Fx_2^n \in \mathcal{N}_1$, $\mathcal{M}_1 \widetilde{\oplus} \mathcal{N}_1 = H_A$, this means that $\|Fx_1^n\| \to 0$ and $\|Fx_2^n\| \to 0$. The operator F_1 is an isomorphism, hence $\|x_1^n\| \to 0$. If $a_1, \dots, a_s$ are generators for $W_0 = \mathcal{N}_0$, then $0 = \langle x_n, a_j\rangle = \langle x_1^n, a_j\rangle + \langle x_2^n, a_j\rangle$,

$$\|\langle x_2^n, a_j\rangle\| = \|\langle x_1^n, a_j\rangle\| \le \|x_1^n\|\,\|a_j\| \to 0, \quad n \to \infty, \quad j = 1, \dots, s.$$

Since $x_2^n \in \mathcal{N}_0$, $x_2^n \to 0$ (as $n \to \infty$) and $x_n = x_1^n + x_2^n \to 0$. This contradiction with the condition $\|x_n\| = 1$ shows that $F|_{W_0^\perp}$ is an isomorphism. Let $V_1 = F(V_0)$. Since $W_0 = \mathcal{N}_0$, one can assume that $W_1 = \mathcal{N}_1$. Indeed, any $y \in H_A$ is of the form $y = m_1 + n_1 = F(m_0) + n_1$, where $m_1 \in \mathcal{M}_1$, $n_1 \in \mathcal{N}_1$, $m_0 \in \mathcal{M}_0$. Further, $m_0 = v_0 + n_0$, where $v_0 \in V_0$, $n_0 \in W_0 = \mathcal{N}_0$, and

$$y = F(v_0 + n_0) + n_1 = F(v_0) + (F(n_0) + n_1) \in V_1 + \mathcal{N}_1.$$

Thus $H_A = V_1 + W_1$.

Let $y \in V_1 \cap W_1 = V_1 \cap \mathcal{N}_1$, i.e., $n_1 = y = F(v_0)$, $n_1 \in \mathcal{N}_1$, $v_0 \in V_0$. Let us decompose v_0 into two summands, $v_0 = m_0 + n_0$, where $m_0 \in \mathcal{M}_0$, $n_0 \in \mathcal{N}_0$. Then

$$n_1 = F(m_0) + F(n_0),$$
$$F(m_0) = n_1 - F(n_0), \quad F(m_0) \in \mathcal{M}_1, \quad n_1 - F(n_0) \in \mathcal{N}_1,$$

whence we obtain that $F(m_0) = 0, \quad n_1 - F(n_0) = 0$. Since $F : \mathcal{M}_0 \cong \mathcal{M}_1$, $m_0 = 0$. Further, $v_0 \in V_0 = \mathcal{N}_0^\perp$, hence

$$0 = \langle v_0, n_0\rangle = \langle m_0 + n_0, n_0\rangle = \langle n_0, n_0\rangle, \quad n_0 = 0.$$

Thus $v_0 = m_0 + n_0 = 0$, $y = F(v_0) = 0$. Therefore $V_1 \cap W_1 = 0$ and $H_A = V_1 \widetilde{\oplus} W_1$.

By Corollary 2.3.5, the module V_1 has the orthogonal complement $V_1^\perp$, $V_1 \oplus V_1^\perp = H_A$ and the proof is complete. □

Remark 2.7.7. If we do not require the operator F to be adjointable, then there still exists a decomposition $F : \mathcal{N}_0^\perp \oplus \mathcal{N}_0 \to \mathcal{M}_1 \widetilde{\oplus} L_n$, where $L_n = \operatorname{span}_A(e_1, \dots, e_n)$, but $\mathcal{M}_1$ does not necessarily have an orthogonal complement. This result was obtained in [**54**].

Definition 2.7.8. Let the conditions of Definition 2.7.4 hold. By Theorem 2.7.5, $\mathcal{N}_1$ and $\mathcal{N}_2$ are projective A-modules and we can define the *index* of F by

$$\operatorname{index} F = [\mathcal{N}_1] - [\mathcal{N}_2] \in K(A).$$

In [**134**] this index is called Mindex. We suggest this notation be interpreted as emphasizing Mishchenko's role in introducing the index of C^*-Fredholm operators.

Theorem 2.7.9. *The index is well defined.*

Proof. It is necessary to check that the index does not depend on decompositions of the range and the domain in Definition 2.7.4. Let p_m be the projection onto L_m along $L_m^\perp$. Let F be an A-Fredholm operator and let $H_A = \mathcal{M}_1 \widetilde{\oplus} \mathcal{N}_1$ and

$H_A = \mathcal{M}_2 \widetilde{\oplus} \mathcal{N}_2$ be decompositions of the domain and of the range, respectively. Then

$$F = \begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix},$$

where $F_1 : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism. According to the proofs of Theorem 2.7.5 and Theorem 2.7.6, one can assume that

$$\mathcal{N}_1 \subset L_n, \qquad L_n = \mathcal{N}_1 \widetilde{\oplus} \mathcal{P}_1, \qquad \mathcal{M}_1 = \mathcal{P}_1 \oplus L_n^\perp,$$

where $\mathcal{P}_1$ is a projective finitely generated A-module. Let another decomposition of the domain and the range be given:

$$H_A = \mathcal{M}_1' \widetilde{\oplus} \mathcal{N}_1', \qquad H_A = \mathcal{M}_2' \widetilde{\oplus} \mathcal{N}_2'.$$

Then there exists $m \geq n$ such that

$$L_m = \mathcal{P}_1' \widetilde{\oplus} p_m(\mathcal{N}_1'), \qquad p_m(\mathcal{N}_1') \cong \mathcal{N}_1',$$

where $\mathcal{P}_1'$ is a projective finitely generated A-module. This is exactly the result of the proof of Theorem 2.7.5.

Let us show that there exists $m \geq n$ such that if

$$L_m' = F(L_m) + \mathcal{N}_2, \qquad \text{and} \quad Q_m' : H_A \to H_A$$

is the projection on L_m' along $L_m'' = F(L_m^\perp)$,[1] then

$$L_m' = \mathcal{P}_2 \widetilde{\oplus} Q_m'(\mathcal{N}_2), \qquad Q_m'(\mathcal{N}_2) \cong \mathcal{N}_2,$$

where $\mathcal{P}_2$ is a projective finitely generated A-module, and

$$L_m' = \mathcal{P}_2' \widetilde{\oplus} Q_m'(\mathcal{N}_2'), \qquad Q_m'(\mathcal{N}_2') \cong \mathcal{N}_2',$$

where $\mathcal{P}_2'$ is a projective finitely generated A-module. Indeed, $H_A = L_m' \widetilde{\oplus} L_m''$. If $a_1, \dots, a_k$ are generators of the module $\mathcal{N}_2$, then

$$a_j = a_j' + a_j'', \quad a_j' \in L_m', \quad a_j'' \in L_m'', \quad j = 1, \dots, k.$$

For $m \to \infty$ we have $\|a_j''\| \to 0$ since $a_j'' = F(x_m^\perp)$, where x is arbitrary, $x_m^\perp$ is a projection of x onto $L_m^\perp$ and $\|x_m^\perp\| \to 0$ as $m \to \infty$. Then for m sufficiently large, we have

$$L_m' = (L_m' \cap \mathcal{M}_2) \widetilde{\oplus} Q_m'(\mathcal{N}_2), \qquad Q_m'(\mathcal{N}_2) \cong \mathcal{N}_2$$

(the proof repeats the proof of Theorem 2.7.5). Similarly

$$L_m' = (L_m' \cap \mathcal{M}_2') \widetilde{\oplus} Q_m'(\mathcal{N}_2'), \qquad Q_m'(\mathcal{N}_2') \cong \mathcal{N}_2'.$$

Since $m \geq n$, we have $L_m \cong \mathcal{N}_1 \widetilde{\oplus} \overline{\mathcal{P}}_1$, where $\overline{\mathcal{P}}_1$ is a finitely generated projective A-module. From the equalities

$$F(\overline{\mathcal{P}}_1) = F(L_m \cap \mathcal{M}_1) = \mathcal{P}_2, \qquad \overline{\mathcal{P}}_1 \subset \mathcal{M}_1,$$

we obtain that $F : \overline{\mathcal{P}}_1 \cong \mathcal{P}_2$ and it follows from relations $F(\mathcal{P}_1') = \mathcal{P}_2'$, $\mathcal{P}_1' \subset \mathcal{M}_1$ that $F : \mathcal{P}_1' \cong \mathcal{P}_2'$. Therefore we have the following equalities in $K(A)$:

$$\begin{aligned} [\mathcal{N}_1] + [\overline{\mathcal{P}}_1] &= [\mathcal{N}_1'] + [\mathcal{P}_1'] = [L_m], \quad [\overline{\mathcal{P}}_1] = [\mathcal{P}_2], \\ [\mathcal{N}_2] + [\mathcal{P}_2] &= [\mathcal{N}_2'] + [\mathcal{P}_2'] = [L_m'], \quad [\mathcal{P}_1'] = [\mathcal{P}_2']. \end{aligned}$$

Thus $[\mathcal{N}_1] - [\mathcal{N}_2] = [\mathcal{N}_1'] - [\mathcal{N}_2']$ and we have proved that the index is well defined. $\square$

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[1] This projection is well defined since $L_m^\perp \subset \mathcal{M}_1$ for $m \geq n$ and hence $F|_{L_m^\perp}$ is an isomorphism, whence it follows that $L_m'' \cong H_A$ is a closed A-module, $L_m' \cap L_m'' = 0$, $L_m' + L_m'' = H_A$. Therefore $H_A = L_m' \widetilde{\oplus} L_m''$ is a direct sum of closed A-modules and Q_m' is bounded and A-linear.

Lemma 2.7.10. *Let an operator $F : H_A \to H_A$ be A-Fredholm. Then there exists a number $\varepsilon > 0$ such that any adjointable operator D satisfying the condition $\|F - D\| < \varepsilon$ is an A-Fredholm operator and* $\operatorname{index} D = \operatorname{index} F$.

Proof. By the definition of Fredholmness

$$H_A = \mathcal{M}_1 \widetilde{\oplus} \mathcal{N}_1, \qquad H_A = \mathcal{M}_2 \widetilde{\oplus} \mathcal{N}_2, \qquad F = \begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix},$$

$F_1 : \mathcal{M}_1 \cong \mathcal{M}_2$. Then $\|F_1\| \le \|F\|$; moreover, if $D : H_A \to H_A$ is an arbitrary operator, then

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

and there exists a constant C such that $\|D_1\| \le C\|D\|$ (cf. [**92**]). Therefore, if D is an arbitrary operator satisfying the estimate $\|F - D\| < \varepsilon$, then $\|F_1 - D_1\| < C \cdot \varepsilon$. Since F_1 is an isomorphism, we can find $\delta > 0$ such that if $\|F_1 - D_1\| < \delta$ and D_1 is an operator, then D_1 is also an isomorphism. By taking $\varepsilon = \delta / C$ we obtain that for the operator

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

the element D_1 is an isomorphism. Then

$$U_2 D U_1 = \begin{pmatrix} D_1 & 0 \\ 0 & D_4 - D_3 D_1^{-1} D_2 \end{pmatrix},$$

where

$$U_2 = \begin{pmatrix} 1 & 0 \\ -D_3 D_1^{-1} & 1 \end{pmatrix} : (\mathcal{M}_2 \widetilde{\oplus} \mathcal{N}_2) \to (\mathcal{M}_2 \widetilde{\oplus} \mathcal{N}_2),$$

$$U_1 = \begin{pmatrix} 1 & -D_1^{-1} D_2 \\ 0 & 1 \end{pmatrix} : (\mathcal{M}_1 \widetilde{\oplus} \mathcal{N}_1) \to (\mathcal{M}_1 \widetilde{\oplus} \mathcal{N}_1)$$

are isomorphisms. Using U_1 and U_2 we obtain a new decomposition of the domain and the range into direct sums:

$$H_A = \mathcal{M}_1' \widetilde{\oplus} \mathcal{N}_1', \quad \mathcal{M}_1' = U_1(\mathcal{M}_1), \quad \mathcal{N}_1' = U_1(\mathcal{N}_1),$$

$$H_A = \mathcal{M}_2' \widetilde{\oplus} \mathcal{N}_2', \quad \mathcal{M}_2' = U_2^{-1}(\mathcal{M}_2), \quad \mathcal{N}_2' = U_2^{-1}(\mathcal{N}_2).$$

With respect to the new decomposition the matrix of D is equal to $U_2 D U_1$. Thus D is A-Fredholm with the index

$$[U_1(\mathcal{N}_1)] - [U_2^{-1}(\mathcal{N}_2)] = [\mathcal{N}_1] - [\mathcal{N}_2] = \operatorname{index} F.$$

□

Lemma 2.7.11. *Let F and D be A-Fredholm operators,*

$$F : H_A \to H_A, \qquad D : H_A \to H_A.$$

Then $DF : H_A \to H_A$ is an A-Fredholm operator and $\operatorname{index} DF = \operatorname{index} D + \operatorname{index} F$.

Proof. For F and D, consider decompositions from the definition,

$$H_A = \mathcal{M}_1 \widetilde{\oplus} \mathcal{N}_1 \xrightarrow{F} \mathcal{M}_2 \widetilde{\oplus} \mathcal{N}_2 \cong H_A,$$

$$H_A = \mathcal{M}_1' \widetilde{\oplus} \mathcal{N}_1' \xrightarrow{D} \mathcal{M}_2' \widetilde{\oplus} \mathcal{N}_2' \cong H_A,$$

where

$$F = \begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}, \qquad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_4 \end{pmatrix},$$

F_1 and D_1 are isomorphisms, and $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_1', \mathcal{N}_2'$ are projective finitely generated A-modules. As earlier, without loss of generality we can assume that

$$\mathcal{N}_2 \subset L_n, \qquad L_n = \mathcal{N}_2 \widetilde{\oplus} \mathcal{P}, \qquad \mathcal{M}_2 = L_n^{\perp} \oplus \mathcal{P}.$$

Moreover, since $F_1 : \mathcal{M}_1 \to \mathcal{M}_2$ is an isomorphism, one can change the decomposition into direct sums by setting

$$\overline{\mathcal{M}}_1 = F_1^{-1}(L_n^{\perp}), \quad \overline{\mathcal{N}}_1 = F_1^{-1}(\mathcal{P}) \widetilde{\oplus} \mathcal{N}_1, \quad \overline{\mathcal{M}}_2 = L_n^{\perp}, \quad \overline{\mathcal{N}}_2 = L_n.$$

Thus a number n can be chosen as large as necessary. Choose n in such a way that

$$L_n = \mathcal{P}' \widetilde{\oplus} p_n(\mathcal{N}_1'), \qquad \mathcal{P}' = \mathcal{M}_1' \cap L_n, \qquad p_n(\mathcal{N}_1') \cong \mathcal{N}_1',$$

where, as before, $p_n : H_A \to H_A$ is the projection onto L_n along $L_n^{\perp}$. Then

$$H_A = L_n^{\perp} \widetilde{\oplus} \mathcal{P}' \widetilde{\oplus} p_n(\mathcal{N}_1').$$

Put $\overline{\overline{\mathcal{M}}}_2 = L_n^{\perp}$, $\overline{\overline{\mathcal{N}}}_2 = \mathcal{P}' \widetilde{\oplus} \mathcal{N}_1'$. With respect to the new decomposition, $H_A = \overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \overline{\overline{\mathcal{N}}}_2$, the matrix of the operator F has the form

$$F = \begin{pmatrix} F_1 & F_2 \\ 0 & F_4 \end{pmatrix},$$

and F_1 is an isomorphism. Then

$$\begin{pmatrix} F_1 & F_2 \\ 0 & F_4 \end{pmatrix} \begin{pmatrix} 1 & -F_1^{-1}F_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}.$$

Denote by U the matrix $\begin{pmatrix} 1 & -F_1^{-1}F_2 \\ 0 & 1 \end{pmatrix}$ and put $\overline{\overline{\mathcal{M}}}_1 = U(\overline{\mathcal{M}}_1)$, $\overline{\overline{\mathcal{N}}}_1 = U(\overline{\mathcal{N}}_1)$. We obtained a new decomposition, $H_A = \overline{\overline{\mathcal{M}}}_1 \widetilde{\oplus} \overline{\overline{\mathcal{N}}}_1$, and the matrix F with respect to the decomposition

$$H_A = \overline{\overline{\mathcal{M}}}_1 \widetilde{\oplus} \overline{\overline{\mathcal{N}}}_1 \xrightarrow{F} \overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \overline{\overline{\mathcal{N}}}_2 = H_A$$

has the previous diagonal form. Consider the projection

$$T : H_A = \overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \mathcal{P}' \widetilde{\oplus} \mathcal{N}_1' \to \overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \mathcal{P}'.$$

Since $H_A \cong \mathcal{M}_2' \widetilde{\oplus} \mathcal{N}_1'$, the restriction $T|_{\mathcal{M}_2'} : \mathcal{M}_2' \to \overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \mathcal{P}'$ is an isomorphism. Consider the matrix D with respect to the decomposition

$$H_A = (\overline{\overline{\mathcal{M}}}_2 \widetilde{\oplus} \mathcal{P}') \widetilde{\oplus} \mathcal{N}_1' \xrightarrow{D} \mathcal{M}_2' \widetilde{\oplus} \mathcal{N}_2' = H_A.$$

This matrix has the form $D = \begin{pmatrix} D_1 & 0 \\ D_3 & D_4 \end{pmatrix}$, where D_1 is an isomorphism. Put

$$V\,D := \begin{pmatrix} 1 & 0 \\ -D_3 D_1^{-1} & 1 \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ D_3 & D_4 \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ 0 & D_4 \end{pmatrix}.$$

Therefore one can change the decomposition of the range,

$$H_A = \overline{\mathcal{M}}'_2 \widetilde{\oplus} \overline{\mathcal{N}}'_2, \qquad \overline{\mathcal{M}}'_2 = V(\mathcal{M}_2'), \qquad \overline{\mathcal{N}}'_2 = V(\mathcal{N}_2'),$$

in such a way that the matrix of the operator D with respect to the new decomposition

$$H_A = \overline{\mathcal{M}}'_2 \widetilde{\oplus} \overline{\mathcal{N}}'_2, \qquad \overline{\mathcal{M}}'_2 = V(\mathcal{M}_2'), \qquad \overline{\mathcal{N}}'_2 = V(\mathcal{N}_2'),$$

has diagonal form. Let us change the decomposition of the range once again:

$$\overline{\overline{\mathcal{M}}}'_2 = D(\overline{\overline{\mathcal{M}}}_2), \qquad \overline{\overline{\mathcal{N}}}_2 = D(\mathcal{P}')\widetilde{\oplus}\overline{\overline{\mathcal{N}}}'_2.$$

The matrix D with the respect to the latter decomposition

$$H_A = \overline{\overline{\mathcal{M}}}_2\widetilde{\oplus}\overline{\overline{\mathcal{N}}}_2 \xrightarrow{D} \overline{\overline{\mathcal{M}}}'_2\widetilde{\oplus}\overline{\overline{\mathcal{N}}}'_2 = H_A$$

has diagonal form. Then the composition DF with the respect to the decomposition

$$H_A = \overline{\overline{\mathcal{M}}}_1\widetilde{\oplus}\overline{\overline{\mathcal{N}}}_1 \to \overline{\overline{\mathcal{M}}}'_2\widetilde{\oplus}\overline{\overline{\mathcal{N}}}'_2 = H_A$$

has the form $DF = \begin{pmatrix} (DF)_1 & 0 \\ 0 & (DF)_4 \end{pmatrix}$, and $(DF)_1$ is an isomorphism. Taking into account the fact that $\operatorname{End}^* H_A$ is a C^*-algebra, we conclude that DF is an A-Fredholm operator and

$$\operatorname{index} F = [\overline{\overline{\mathcal{N}}}_1] - [\overline{\overline{\mathcal{N}}}_2], \quad \operatorname{index} D = [\overline{\overline{\mathcal{N}}}_2] - [\overline{\overline{\mathcal{N}}}'_2],$$

$$\operatorname{index} DF = [\overline{\overline{\mathcal{N}}}_1] - [\overline{\overline{\mathcal{N}}}'_2].$$

It follows that $\operatorname{index} DF = \operatorname{index} D + \operatorname{index} F$. □

LEMMA 2.7.12. *Let $K : H_A \to H_A$ be a compact operator. Then $1 + K$ is an A-Fredholm operator and* $\operatorname{index}(1 + K) = 0$.

PROOF. It is clear that $1 + K$ is adjointable. Choose a number n such that the inequality $\|K|_{L_n^\perp}\| < 1$ is fulfilled. With respect to the decomposition $H_A = L_n^\perp \oplus L_n$, we have the following matrix presentation:

$$K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}, \quad (1 + K) = \begin{pmatrix} 1 + K_1 & K_2 \\ K_3 & 1 + K_4 \end{pmatrix}.$$

By the estimate $\|K|_{L_n^\perp}\| < 1$, the operator $1 + K_1$ is invertible, hence, as before, there exist invertible operators U_1 and U_2 such that

$$U_2(1 + K)U_1 = \begin{pmatrix} 1 + K_1 & 0 \\ 0 & (1 + K_4) - K_3(1 + K_1)^{-1}K_2 \end{pmatrix}.$$

Then, with respect to the new decomposition, $H_A = \mathcal{M}_1\widetilde{\oplus}\mathcal{N}_1 \to \mathcal{M}_2\widetilde{\oplus}\mathcal{N}_2 = H_A$, where $\mathcal{M}_1 = U_1(L_n^\perp)$, $\mathcal{N}_1 = U_1(L_n)$, $\mathcal{M}_2 = U_2^{-1}(L_n^\perp)$, $\mathcal{N}_2 = U_2^{-1}(L_n)$, the operator $(1 + K)$ has the diagonal form and is thus an A-Fredholm operator with

$$\operatorname{index}(1 + K) = [U_1(L_n)] - [U_2^{-1}(L_n)] = 0.$$

□

LEMMA 2.7.13. *Consider an A-Fredholm operator $F : H_A \to H_A$ and let $K \in \mathcal{K}_A$. Then the operator $F + K$ is A-Fredholm and* $\operatorname{index}(F + K) = \operatorname{index} F$.

PROOF. Consider decompositions of the module H_A into direct sums such that the matrix F has the diagonal form:

$$H_A = \mathcal{M}_1\widetilde{\oplus}\mathcal{N}_1 \xrightarrow{F} \mathcal{M}_2\widetilde{\oplus}\mathcal{N}_2 = H_A.$$

Without loss of generality we can assume that

$$L_n = \mathcal{N}_1\widetilde{\oplus}\mathcal{P}_1, \quad \mathcal{M}_1 = L_n^\perp\widetilde{\oplus}\mathcal{P}_1,$$

where $\mathcal{P}_1$ is a finitely generated closed A-module. Choose a number n sufficiently large to provide that $\|K|_{L_n^\perp}\| < \|F_1^{-1}\|^{-1}$. Consider the new decomposition of the module H_A:

$$\overline{\mathcal{M}}_1 = L_n^\perp, \quad \overline{\mathcal{N}}_1 = L_n, \quad \overline{\mathcal{M}}_2 = FL_n^\perp, \quad \overline{\mathcal{N}}_2 = F(\mathcal{P}_1)\widetilde{\oplus}\mathcal{N}_2.$$

Let $F = \begin{pmatrix} F_1 & 0 \\ 0 & F_4 \end{pmatrix}$ and $K = \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix}$ be matrices of F and K with respect to the decomposition $H_A = \overline{\mathcal{M}}_1\widetilde{\oplus}\overline{\mathcal{N}}_1 \to \overline{\mathcal{M}}_2\widetilde{\oplus}\overline{\mathcal{N}}_2 = H_A$. Then

$$F + K = \begin{pmatrix} F_1 + K_1 & K_2 \\ K_3 & F_4 + K_4 \end{pmatrix},$$

and the operator $F_1 + K_1$ is invertible. By repeating the construction of Lemma 2.7.10 (about operators close to a Fredholm operator), we obtain

$$\begin{aligned} \operatorname{index}(F + K) &= [\overline{\mathcal{N}}_1] - [\overline{\mathcal{N}}_2] = [L_n] - [F(\mathcal{P}_1) + \mathcal{N}_2] \\ &= [\mathcal{N}_1] + [\mathcal{P}_1] - [\mathcal{P}_1] - [\mathcal{N}_2] = \operatorname{index} F. \end{aligned}$$

□

THEOREM 2.7.14. *Let*

$$F : H_A \to H_A, \quad D : H_A \to H_A, \quad D' : H_A \to H_A$$

be adjointable operators such that

$$FD = \operatorname{Id}_{H_A} + K_1, \quad D'F = \operatorname{Id}_{H_A} + K_2, \quad K_1,\, K_2 \in \mathcal{K}(H_A).$$

Then F is an A-Fredholm operator.

PROOF. Consider a decomposition of H_A, for which the operator $FD = 1_{H_A} + K_1$ has the diagonal form (Lemma 2.7.12)

$$H_A = \mathcal{M}_1\widetilde{\oplus}\mathcal{N}_1 \xrightarrow{1+K_1} \mathcal{M}_2\widetilde{\oplus}\mathcal{N}_2 = H_A,$$

and assume that the decomposition of H_A satisfies the conditions of Theorem 2.7.6. Consider the projection

$$P : H_A = \mathcal{M}_2\widetilde{\oplus}\mathcal{N}_2 \to \mathcal{N}_2.$$

Viewed as an operator on H_A, it is a compact operator. The image of the operator $(1 - P)(1 + K_1) = (1 - P)FD$ is exactly equal to $\mathcal{M}_2$. It is easy to see that, up to an isomorphism,

$$(1 - P)FD = (1 - P)(1 + K_1) = 1 + (-P(1 + K_1)) + K_1 = 1 + \widetilde{K}_1,$$

$$D'(1 - P)F = D'F - D'PF = 1 + \widetilde{K}_2,$$

where $\widetilde{K}_1 \in \mathcal{K}_\mathcal{A}$, $\widetilde{K}_2 \in \mathcal{K}_\mathcal{A}$. By Lemma 2.7.13, one can assume without loss of generality that $F : H_A \to \mathcal{M}_2$ is an epimorphism. Otherwise, we will pass to the operator $(1 - P)F$. Consider now the decomposition for $1 + K_2$:

$$H_A = \overline{\mathcal{M}}_1\widetilde{\oplus}\overline{\mathcal{N}}_1 \xrightarrow{F} \mathcal{M}_2\widetilde{\oplus}\mathcal{N}_2 \xrightarrow{D'} \overline{\mathcal{M}}_2\widetilde{\oplus}\overline{\mathcal{N}}_2 = H_A.$$

The composition $D'F|_{\overline{\mathcal{M}}_1} : \overline{\mathcal{M}}_1 \to \overline{\mathcal{M}}_2$ is an isomorphism. Therefore, since $F : H_A \to \mathcal{M}_2$ is an epimorphism, $F : \overline{\mathcal{M}}_1\widetilde{\oplus}\overline{\mathcal{N}}_1 \to \mathcal{M}_2$ maps $\overline{\mathcal{M}}_1$ isomorphically into $\mathcal{M}_2$ and $\operatorname{Ker} F \subset \overline{\mathcal{N}}_1$, $\mathcal{M}_2 = F(\overline{\mathcal{M}}_1) + F(\overline{\mathcal{N}}_1)$. Let us show that $F(\overline{\mathcal{M}}_1) \cap F(\overline{\mathcal{N}}_1) = 0$. For this purpose, decompose F into a composition

$$\overline{\mathcal{M}}_1\widetilde{\oplus}\overline{\mathcal{N}}_1 \to (\overline{\mathcal{M}}_1\widetilde{\oplus}\overline{\mathcal{N}}_1)/\operatorname{Ker} F = \overline{\mathcal{M}}_1\widetilde{\oplus}(\overline{\mathcal{N}}_1/\operatorname{Ker} F) \xrightarrow{\widetilde{F}} \mathcal{M}_2,$$

where $\widetilde{F}$ is an isomorphism. Therefore

$$\mathcal{M}_2 = F(\overline{\mathcal{M}}_1)\widetilde{\oplus}\widetilde{F}(\overline{\mathcal{N}}_1/\operatorname{Ker} F) = F(\overline{\mathcal{M}}_1)\widetilde{\oplus}F(\overline{\mathcal{N}}_1).$$

Since the A-module $\overline{\mathcal{N}}_1$ is finitely generated, $F(\overline{\mathcal{N}}_1)$ is finitely generated too. We have obtained a decomposition

$$H_A = \overline{\mathcal{M}}_1\widetilde{\oplus}\overline{\mathcal{N}}_1 \to F(\overline{\mathcal{M}}_1)\widetilde{\oplus}[F(\overline{\mathcal{N}}_1)\widetilde{\oplus}\mathcal{N}_2] = H_A,$$

where $F|_{\overline{\mathcal{M}}_1} : \overline{\mathcal{M}}_1 \to F(\overline{\mathcal{M}}_1)$ is an isomorphism. □

LEMMA 2.7.15. *If adjointable operators D, D' and F are such that FD and $D'F$ are A-Fredholm operators, then F is an A-Fredholm operator.*

PROOF. By the definition of Fredholmness of FD and $D'F$, we can find adjointable operators T and T' such that

$$(FD)T = 1 + K,$$
$$T'(D'F) = 1 + K'.$$

By Theorem 2.7.14, the operator F is A-Fredholm. For T, for example, one can take an operator with the matrix $\begin{pmatrix} (F_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}$, where FD has the form $\begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$ in the sense of Definition 2.7.4. □

REMARK 2.7.16. In the case where A is a W^*-algebra, the properties of A-Fredholm operators are more similar to the properties of usual Fredholm operators. This problem will be addressed in Proposition 3.6.8.

REMARK 2.7.17. For applications to elliptic operators it is important to develop the theory for modules of the form $l_2(\mathcal{P})$. This can be done similarly (see [**121**]).

2.8. Representations of groups on Hilbert modules

In this section we assume that G always denotes a compact group. First of all, we prove an equivariant version of the Kasparov stabilization theorem. We follow here the original proof [**63**]. For closely related problems see also [**87**].

DEFINITION 2.8.1. For a C^*-algebra B, put

$$\mathcal{H}_B := \sum_{i=1}^{\infty}(V_i \otimes_{\mathbf{C}} B),$$

where $\{V_i\}$ is a countable set of finite-dimensional spaces, in which all irreducible unitary representations G are realized (up to isomorphism) and each representation is repeated an infinite number of times; the B-Hilbert completion of the infinite algebraic sum is implemented with respect to the norm given by the following B-inner product on summands:

$$(x_1 \otimes b_1, x_2 \otimes b_2) := \langle x_1, x_2\rangle_{V_i} \cdot b_1^* b_2, \qquad x_1,\ x_2 \in V_i.$$

DEFINITION 2.8.2. A C^*-algebra B with a continuous action of a group G is called a *G-C^*-algebra.*

DEFINITION 2.8.3. A Hilbert C^*-module $\mathcal{E}$ over a G-C^*-algebra B equipped with a unitary action of G is called a *G-B-module* if this group action agrees with the Hilbert module structure, i.e.,

$$g(xb) = g(x)g(b), \quad \langle g(x), g(y)\rangle = g(\langle x, y\rangle), \quad x,\ y \in \mathcal{E},\ \ b \in B,\ \ g \in G.$$

THEOREM 2.8.4 ([**63**]). *Let B be a G-C^*-algebra and let $\mathcal{E}$ be a countably generated Hilbert G-B-module. Then there exists an equivariant B-linear isomorphism*

$$\mathcal{E} \oplus \mathcal{H}_B \cong \mathcal{H}_B,$$

which respects the inner product.

PROOF. Denote by $\mathcal{E}^+$ the module $\mathcal{E}$ viewed as a B^+-module and, for a B^+-module $\mathcal{N}$, denote by $(\mathcal{N})_B$ the same module $\mathcal{N}$ viewed as a B-module. Let the action of G on B^+ be extended from that on B by the formula $g(1) = 1$. Suppose that we have proved the theorem for unital C^*-algebras, so that there is an equivariant B-linear isomorphism

$$\mathcal{E}^+ \oplus \mathcal{H}_{B^+} \cong \mathcal{H}_{B^+}.$$

Then

$$\mathcal{E} \oplus \mathcal{H}_B \cong ((\mathcal{E} \oplus \mathcal{H}_B)^+)B \cong (\mathcal{E}^+ \oplus \mathcal{H}_{B^+})B \cong (\mathcal{H}_{B^+})B = \mathcal{H}_B,$$

so we can restrict ourselves to the unital case.

Let $\{x_k\}$ be a countable system of generators for $\mathcal{E}$ and let $\{e_k\}$ be an orthonormal basis of $\mathcal{H}_B$ such that each e_k is of the form $e_k = v_k \otimes 1_B$, where $v_k \in V_{s(k)}$. In other words, if $\{w_k\}$ is a union of some orthonormal bases of all V_j, then $e_k = w_k \otimes 1_B$. Let $\{y_i\}$ be a system of elements in $\mathcal{E} \oplus \mathcal{H}_B$, in which each element of the form $x_k \oplus 0$ or $0 \oplus e_k$ is repeated an infinite number of times. We can assume that $y_1 = 0 \oplus e_1$ and put $W_1 = 0 \oplus V_1 \otimes B$. Suppose that, by induction, we have already constructed subspaces $W_1, \dots, W_n$ satisfying the following conditions:

(i) W_i is a $\mathbf{C}$-finite-dimensional G-invariant subspace in $\mathcal{E} \oplus \mathcal{H}_B$,

(ii) each W_i has a basis $(z_i^1, \dots, z_i^{K(i)}) = (f_1, \dots, f_p)$ such that

$$\langle z_i^j, z_i^j \rangle = 1_B, \qquad \langle z_i^j, z_r^s \rangle = 0 \text{ for } i \neq r \text{ or } j \neq s,$$

(iii) there exists $m = m(n)$ such that

$$W_1 + \cdots + W_n \subset \mathcal{E}_m := \mathcal{E} \oplus \left(\bigoplus_{i=1}^{m} V_i \otimes B \right),$$

and consequently $(W_1 + \cdots + W_n)B \subset \mathcal{E}_m$,

(iv) the distance between y_n and $(W_1 + \cdots + W_n)B$ does not exceed $1/n$.

Note that (i) and (ii) imply that the modules W_iB are pairwise orthogonal and G-invariant and $(W_1 + \cdots + W_n)B$ is also G-invariant. The last module is free, so, by Lemma 2.3.7, it has an orthogonal complement, which is G-invariant due to unitarity of the action.

Let us pass to the construction of W_{n+1}. Put

$$y'_{n+1} := \sum_{j=1}^{p} f_j \langle f_j, y_{n+1} \rangle, \qquad y''_{n+1} = y_{n+1} - y'_{n+1}.$$

Then

$$\langle w, y''_{n+1} \rangle = \langle \sum_{j=1}^{p} f_j b_j, y_{n+1} - \sum_{j=1}^{p} f_j \langle f_j, y_{n+1} \rangle \rangle$$
$$= \sum_{j=1}^{p} b_j^* [\langle f_j, y_{n+1} \rangle - \langle f_j, y_{n+1} \rangle] = 0$$

for any $w \in (W_1 + \cdots + W_n)B$. Since by the definition of the sequence y_j, the element y_{n+1} lies either in $\mathcal{E}$ or in some $V_i \otimes B$, we have $y''_{n+1} \in \mathcal{E}_{m'}$ for some $m' > m$. Consider the orthogonal complement $S_{n,m'}$ to $(W_1 + \cdots + W_n)B$ in $\mathcal{E}_{m'}$. It is an invariant module and $y''_{n+1} \in S_{n,m'}$. By the Mostow theorem on periodic vectors [**93**], the elements with **C**-finite-dimensional orbits are dense in $S_{n,m'}$. Hence one can find a vector $z \in S_{n,m'}$ such that $\|z - y''_{n+1}\| < \frac{1}{2n+2}$ and the linear span R of the orbit Gz of z is an invariant finite-dimensional subspace of $S_{n,m'}$. Since z is a totalizing vector, R is an irreducible G-module. Therefore there is $m'' > m'$ such that there exists an equivariant isomorphism $\Gamma : R \to V_{m''}$. Let $\{h_1, \ldots, h_k\}$ be an orthonormal basis of $V_{m''}$ and $r_i := \Gamma^{-1}(h_i)$, $i = 1, \ldots, k$. Then for the corresponding irreducible matrix representation $T : G \to U(k)$, we have

$$g(h_i) = \sum_{j=1}^{k} T_i^j(g) h_j, \qquad g(r_i) = \sum_{j=1}^{k} T_i^j(g) r_j, \qquad g \in G.$$

Since $R \subset \mathcal{E}_{m'}$ and $m'' > m'$, R is orthogonal to $V_{m''}$. More precisely, each element of R is orthogonal to $V_{m''} \otimes B$ in $\mathcal{E} \otimes \mathcal{H}_B$. Hence $\langle r_i, h_j \rangle = 0$ for any i and j. Let

$$z := \sum_{i=1}^{k} r_i \alpha_i, \quad \alpha_i \in \mathbf{C}, \qquad \alpha := \left(\sum_{i=1}^{k} |\alpha_i|^2 \right)^{1/2} + 1,$$

$$r'_i := r_i + (h_i \otimes 1_B) \cdot \frac{1}{(2n+2)\alpha}.$$

Then $\langle r'_i, r'_j \rangle = \langle r_i, r_j \rangle + \{(2n+2)\alpha\}^{-2} \delta_{ij}$ and the matrix $L := \|\langle r'_i, r'_j \rangle\|_{i,j=1}^k$ is positive and invertible in $M_k(\mathbf{C}) \subset M_k(B)$. Let $D = \|d_{ji}\| := L^{-1/2} \in M_k(B)$ and $r''_i := \sum_{j=1}^k r'_j d_{ji}$. Set W_{n+1} to be the **C**-linear span of vectors r''_i, $i = 1, \ldots, k$, or, equivalently, to the span of r'_i, since D has complex coefficients. Then $W_{m+1} \subset \mathcal{E}_{m''}$ and

$$\langle r''_i, r''_j \rangle = \sum_{p,q=1}^{k} \langle r'_p d_{pi}, r'_q d_{pj} \rangle = (D^* L D)_{ij} = \delta_{ij}.$$

Since all h_i and r_i are orthogonal to $W_1 + \cdots + W_n$, W_{n+1} is orthogonal to it as well. Further, put $F := L^{1/2}$, so that $r'_i := \sum_{j=1}^k r''_j F_{ji}$. Then

$$\begin{aligned}
g(r''_i) = g\left(\sum_{j=1}^{k} r'_j d_{ji} \right) &= \sum_{j=1}^{k} (g r'_j) d_{ji} = \sum_{j=1}^{k} \left(g r_j + (g h_j \otimes 1_B) \cdot \frac{1}{(2n+2)\alpha} \right) d_{ji} \\
&= \sum_{j=1}^{k} \left(\sum_{s=1}^{k} T_j^s(g) r_s + \left(\sum_{s=1}^{k} T_j^s(g) h_s \otimes 1_B \right) \cdot \frac{1}{(2n+2)\alpha} \right) d_{ji} \\
&= \sum_{j=1}^{k} \sum_{s=1}^{k} T_j^s(g) \left(r_s + (h_s \otimes 1_B) \cdot \frac{1}{(2n+2)\alpha} \right) d_{ji} = \sum_{j=1}^{k} \sum_{s=1}^{k} T_j^s(g) r'_s d_{ji} \\
&= \sum_{j=1}^{k} \sum_{s=1}^{k} T_j^s(g) \left(\sum_{t=1}^{k} r''_t F_{ts} \right) d_{ji} = \sum_{t=1}^{k} r''_t \left(\sum_{j=1}^{k} \sum_{s=1}^{k} T_j^s(g) F_{ts} d_{ji} \right) \in W_{n+1}.
\end{aligned}$$

Thus W_{n+1} is G-invariant. Let us estimate the distance by taking $z' = \sum_{i=1}^k r'_i \alpha_i$, so that

$$\rho(z, W_{n+1}) \le \rho(z, z') = \left\| \sum_{i=1}^k (r_i - r'_i)\alpha_i \right\| = \left\| \sum_{i=1}^k (h_i \otimes 1_B) \cdot \frac{1}{(2n+2)\alpha} \alpha_i \right\|$$
$$= \frac{1}{(2n+2)\alpha} \cdot \left\| \sum_{i=1}^k h_i \alpha_i \right\| = \frac{1}{(2n+2)\alpha} \cdot \left(\sum_{i=1}^k |\alpha_i|^2 \right)^{1/2}$$
$$= \frac{1}{(2n+2)} \cdot \frac{\left(\sum_{i=1}^k |\alpha_i|^2 \right)^{1/2}}{\left(\sum_{i=1}^k |\alpha_i|^2 \right)^{1/2} + 1} < \frac{1}{(2n+2)}.$$

Therefore

$$\rho(y_{n+1}, (W_1 + \cdots + W_n)B) \le \rho(y''_{n+1}, z) + \rho(z, W_{n+1}B)$$
$$\le \frac{1}{2n+2} + \rho(z, z') \le \frac{1}{n+1} < \frac{1}{n}.$$

Thus, by induction, the **C**-subspaces W_i with properties (i)–(iv) are constructed for all i. From the explicit expression for r'_i we obtain that W_n is isomorphic to some V_i, i.e., is irreducible. Further, the B-Hilbert completion $\mathcal{M}$ (i.e., the closure in $\mathcal{E} \oplus \mathcal{H}_B$) of the algebraic orthogonal sum of modules $W_n B$ gives the whole $\mathcal{E} \oplus \mathcal{H}_B$. Indeed, by property (iv), the algebraic sum is dense in $\mathcal{E} \oplus \mathcal{H}_B$. So, $\mathcal{M} \cong \mathcal{E} \oplus \mathcal{H}_B$. It remains to prove that $\mathcal{M}$ is isomorphic to $\mathcal{H}_B$, i.e., that each irreducible representation is repeated infinitely many times among the modules $W_n B$. Suppose the opposite, then $\mathcal{M} \oplus \mathcal{H}_B \cong \mathcal{H}_B$, or

$$\mathcal{E} \oplus \mathcal{H}_B = \mathcal{E} \oplus \mathcal{H}_B \oplus \mathcal{H}_B \cong \mathcal{M} \oplus \mathcal{H}_B \cong \mathcal{H}_B.$$

□

Let us now prove the theorem on decomposition of representations [**91**]. Let $\mathcal{M}$ be a Hilbert B-module with a strongly continuous unitary representation G,

$$T : G \to U(\mathcal{M}) \subset \mathrm{End}^*_B(\mathcal{M}), \qquad g \mapsto T_g,$$

and suppose that the group acts trivially on B. Now let $\{V_s\}$ be the complete collection of pairwise nonequivalent unitary representations of G of dimensions d_s and let D^s_{pq} be their matrix elements, which are continuous functions on G. For an invariant normalized Haar measure dg on G, we define the operator

$$P^s_{pq} : \mathcal{M} \to \mathcal{M}, \qquad P^s_{pq}(x) := d_s \int_G \overline{D^s_{pq}(g)} T_g(x)\, dg. \tag{2.17}$$

Since a product of continuous complex-valued functions by a continuous module-valued function is integrable for a fixed $x \in \mathcal{M}$ and since the group is compact, the integral is well defined and equals some element of $\mathcal{M}$. We obtain a bounded operator. Indeed,

$$\|P^s_{pq} x\| \le d_s \int_G \left| \overline{D^s_{pq}(g)} \right| \|T_g(x)\|\, dg \le d_s \sup_{g \in G} \left| \overline{D^s_{pq}(g)} \right| \|x\|.$$

Therefore

$$\|P^s_{pq}\| \le d_s \sup_{g\in G} \left|\overline{D^s_{pq}(g)}\right|.$$

It is well known [**7**, I, §7.1, Theorem 5] that

$$\int_G D^s_{ij}(g)\overline{D^{s'}_{mn}(g)} = \begin{cases} 0, & s \neq s', \\ \frac{1}{d_s}\delta_{im}\delta_{jn}, & s = s'. \end{cases} \tag{2.18}$$

We need the following Peter–Weil theorem (see, e.g., [**7**, I, §7.2, Theorem 1]).

THEOREM 2.8.5. *The functions $\sqrt{d_s}D^s_{jk}(g)$ form a complete orthonormal system in $L^2(G)$.*

LEMMA 2.8.6. *The operators P^s_{pq} have the following properties:*

(i) *P^s_{pq} is adjointable and*

$$(P^s_{pq})^* = P^s_{qp}; \tag{2.19}$$

(ii) *the following equality holds:*

$$P^s_{pq}P^{s'}_{p'q'} = \delta^{ss'}\delta_{qp'}P^s_{pq'}. \tag{2.20}$$

(iii) *the following equalities hold:*

$$T_g P^s_{jm} = \sum_{i=1}^{d_s} D^s_{ij}(g)\, P^s_{im}, \tag{2.21}$$

$$P^s_{jm} T_g = \sum_{i=1}^{d_s} D^s_{mi}(g)\, P^s_{ji}. \tag{2.22}$$

PROOF. First of all, note that for unitary operators in $\mathcal{M}$, the mapping $F \mapsto F^*$ is continuous in the strong operator topology. In other words, for unitary operators, the strong continuity implies the $*$-strong one. Indeed,

$$\|(F'^* - F^*)x\| = \|(F'^{-1} - F^{-1})x\| = \|F'(F'^{-1} - F^{-1})Fz\| = \|Fz - F'z\| \to 0.$$

Therefore one can take T^*_g instead of T_g in (2.17), and then take it outside the integral. More precisely, the first equality in the chain

$$\begin{aligned}(P^s_{pq})^* &= d_s \int_G D^s_{pq}(g)T^*_g(x)\,dg = d_s \int_G D^s_{pq}(g^{-1})T_g(x)\,d(g^{-1}) \\ &= d_s \int_G \overline{D^s_{qp}(g)}T_g(x)\,dg = P^s_{qp}\end{aligned}$$

has to be verified at first at the level of integral sums and then one has to take the limit, which is possible due to the $*$-strong continuity. Remaining equalities in the chain above follow from the invariance of Haar measure and the relations $T^*_g = T^{-1}_g = T_{g^{-1}}$. Statement (i) is proved.

It follows from (2.17) that

$$P^s_{pq}P^{s'}_{p'q'} = d_s d_{s'} \int_G \int_G \overline{D^s_{pq}(g)}\,\overline{D^{s'}_{p'q'}(g')}T_g T_{g'}\,dg\,dg'.$$

Since $T_g T_{g'} = T_{gg'}$, we obtain, by taking $\widetilde{g} := gg'$, from

$$D^s_{pq}(g) = D^s_{pq}(\widetilde{g}\,g'^{-1}) = D^s_{pr}(\widetilde{g})D^s_{rq}(g'^{-1}) = D^s_{pr}(\widetilde{g})\overline{D^s_{qr}(g')}$$

and from relations (2.18) that

$$\begin{aligned} P^s_{pq} P^{s'}_{p'q'} &= d_s d_{s'} \int_G D^s_{qr}(g') \, \overline{D^{s'}_{p'q'}(g')} \, dg' \cdot \int_G \overline{D^s_{pr}(\widetilde{g})} T_{\widetilde{g}} \, d\widetilde{g} \\ &= d_{s'} \delta^{ss'} \frac{1}{d_{s'}} \delta_{qp'} \delta_{rq'} P^s_{pr} = \delta^{ss'} \delta_{qp'} P^s_{pq'}. \end{aligned}$$

To prove statement (iii) let us note that

$$\begin{aligned} T_g P^s_{jm}(x) &= d_s \int_G \overline{D^s_{jm}(h)} T_{gh}(x) \, dh = d_s \int_G \overline{D^s_{jm}(g^{-1}h)} T_h(x) \, dh \\ &= d_s \int_G \sum_{i=1}^{d_s} \overline{D^s_{ji}(g^{-1})} \, \overline{D^s_{im}(h)} T_h(x) \, dh \\ &= \sum_{i=1}^{d_s} \overline{D^s_{ji}(g^{-1})} \, d_s \int_G \sum_{i=1}^{d_s} \overline{D^s_{im}(h)} T_h(x) \, dh \\ &= \sum_{i=1}^{d_s} \overline{D^s_{ji}(g^{-1})} \, P^s_{im}(x) = \sum_{i=1}^{d_s} D^s_{ij}(g) \, P^s_{im}(x). \end{aligned}$$

The second equality of this statement can be proved similarly. □

Lemma 2.8.7. *The operators* $P^s_p := P^s_{pp}$ *are selfadjoint pairwise orthogonal projections.*

Proof. If we rewrite the statement of the lemma as

$$(P^s_p)^* = P^s_p, \qquad P^s_p P^{s'}_{p'} = \delta^{ss'} \delta_{pp'} P^s_p, \tag{2.23}$$

then the proof follows immediately from (2.19) and (2.20). □

Lemma 2.8.8. *Put*

$$P^s := \sum_{p=1}^{d_s} P^s_p = \sum_{p=1}^{d_s} P^s_{pp}.$$

The operators P^s *have the following properties:*

$$(P^s)^* = P^s, \tag{2.24}$$

$$P^s P^{s'} = \delta_{ss'} P^s, \tag{2.25}$$

$$T_g P^s = P^s T_g. \tag{2.26}$$

In other words, P^s *are selfadjoint invariant pairwise orthogonal projections in* $\mathcal{M}$.

Proof. By the definition of P^s, formulas (2.24) and (2.25) immediately follow from (2.23). To verify the third relation consider the character of the representation V_s,

$$\chi^s(g) := \sum_{p=1}^{d_s} D^s_{pp}(g),$$

which, being a trace, satisfies the relation $\chi^s(g) = \chi^s(hgh^{-1})$. One also has

$$\begin{aligned} P^s &= d_s \int_G \chi^s(g) T_g \, dg, \\ T_g P^s &= d_s T_g \int_G \chi^s(g') T_{g'} \, dg' = d_s \int_G \chi^s(g') T_{gg'g^{-1}} T_g \, dg' \\ &= d_s \int_G \chi^s(gg'g^{-1}) T_{gg'g^{-1}} \, dg' T_g = P^s T_g. \end{aligned}$$

□

LEMMA 2.8.9. *Define*

$$\mathcal{M}^s := P^s(\mathcal{M}), \qquad \mathcal{M}^\bullet := \bigoplus_{s=1}^{\infty} \mathcal{M}^s, \tag{2.27}$$

where the sum is supposed to be completed either as a Hilbert sum or as a closure in $\mathcal{M}$ of the algebraic sum (*which is the same*). *Then*

$$\mathcal{M}^\bullet = \mathcal{M}. \tag{2.28}$$

PROOF. Assume that a $\mathbf{C}$-linear functional f on $\mathcal{M}$ vanishes on $\mathcal{M}^\bullet$ and that $x \in \mathcal{M}$ is an arbitrary vector. Then, for any set of indices, we have $P^s_{ij}(x) \in \mathcal{M}^\bullet$, so that

$$0 = f(P^s_{ij}(x)) = d_s \int_G \overline{D^s_{ij}(g)} f(T_g(x)) \, dg.$$

Therefore, by the Peter–Weil Theorem 2.8.5, $f(T_g(x)) = 0$ holds almost everywhere and, by continuity, it vanishes everywhere. In particular, $f(T_e(x)) = f(x) = 0$. Hence, by the Hahn–Banach theorem, $\mathcal{M}^\bullet = \mathcal{M}$. □

THEOREM 2.8.10 ([**91**]). *Let $\mathcal{M}$ be a Hilbert B-module equipped with a strongly continuous unitary representation of G and let the group act trivially on B. Now let $\{V_s\}$ be the complete collection of pairwise nonequivalent unitary representations of G and let*

$$\mathcal{M}_s := \operatorname{Hom}_{G,\mathbf{C}}(V_s, \mathcal{M}) \subset \operatorname{Hom}_{\mathbf{C}}(V_s, \mathcal{M}) \cong V_s^* \otimes \mathcal{M}$$

be a Hilbert B-module with the inner B-valued product defined by the formula

$$\langle \varphi, \psi \rangle := \sum_{i,j=1}^{\dim V_s} \langle \varphi(h^s_i), \psi(h^s_j) \rangle_{\mathcal{M}}, \qquad h^s_1, \dots, h^s_{\dim V_s} \text{ orthonormal basis for } V_s.$$

Then, for the Hilbert sum, we have an equivariant B-linear isomorphism

$$\Gamma = \bigoplus_{s=1}^{\infty} \Gamma_s : \bigoplus_{s=1}^{\infty} V_s \otimes \mathcal{M}_s \cong \mathcal{M}, \qquad \Gamma_s : v \otimes \varphi \mapsto \varphi(v), \quad v \in V_s, \ \varphi \in \mathcal{M}_s,$$

and

$$\Gamma(V_s \otimes \mathcal{M}_s) = \mathcal{M}^s,$$

where $\mathcal{M}^s$ is defined by (2.27).

PROOF. To begin with, let us note that the Γ_s are algebraically injective. Indeed, let

$$0 = \Gamma_s \left(\sum_{j=1}^{d_s} h^s_j \alpha_j \otimes \varphi \right) = \varphi \left(\sum_{j=1}^{d_s} h^s_j \alpha_j \right).$$

Since, by the Schur lemma, φ is either an isomorphism or 0, the above equality can be true only if either $\sum_{j=1}^{d_s} h_j^s \alpha_j = 0$ or $\varphi = 0$. But then $\sum_{j=1}^{d_s} h_j^s \alpha_j \otimes \varphi = 0$.

By Lemma 2.8.9, it is sufficient to prove only that Γ_s maps $V_s \otimes \mathcal{M}_s$ bijectively onto $\mathcal{M}^s$.

Note that, by setting $\mathcal{M}_i^s := P_i^s(\mathcal{M}) = P_{ii}^s(\mathcal{M})$, we obtain, by relation (2.20), that the operators P_{ij}^s induce isomorphisms

$$P_{ij}^s : \mathcal{M}_j^s \to \mathcal{M}_i^s.$$

Thus $\mathcal{M}^s = \bigoplus_{j=1}^{d_s} \mathcal{M}_j^s$ is a sum of isomorphic modules.

Let $\{h_1^s, \dots, h_{d_s}^s\}$ be the orthonormal basis of V_s, with respect to which the matrix elements D_{ij}^s were defined. Define the homomorphism

$$\Phi^s : V_s \otimes [\mathcal{M}_1^s] \to \mathcal{M}^s, \qquad \Phi^s(h_j^s \otimes x) = P_{j1}^s(x), \tag{2.29}$$

where we have put $\mathcal{M}_1^s$ in square brackets to emphasize that there is no action of G on it. By the properties of the operators P_{j1}^s, the map Φ^s is an isomorphism. Since, by (2.21),

$$T_g \Phi^s(h_j^s \otimes x) = T_g P_{j1}^s(x) = \sum_{i=1}^{d_s} D_{ij}^s(g)\, P_{i1}^s(x)$$

and

$$\Phi^s(g(h_j^s) \otimes x) = \Phi^s\left(\sum_{i=1}^{d_s} D_{ij}^s(g) h_i^s \otimes x\right) = \sum_{i=1}^{d_s} D_{ij}^s(g) P_{i1}^s(x),$$

the map Φ^s is equivariant. Further, there is a map

$$\Psi^s : \mathcal{M}_1^s \to \mathcal{M}_s, \qquad \Psi^s(x)(v) := \Phi^s(v \otimes x).$$

Then

$$\Gamma_s \circ (\mathrm{Id}_{V_s} \otimes \Psi^s)(v \otimes x) = \Phi^s(v \otimes x).$$

Since we have an isomorphism on the right-hand side and since Γ_s is algebraically injective, Γ_s is an isomorphism (see Lemma 2.8.12), whence Ψ^s is an isomorphism. In particular, the image of Γ_s coincides with $\mathcal{M}^s$ and these images are orthogonal to each other. Hence Γ is topologically injective and its image coincides with $\mathcal{M}$. □

REMARK 2.8.11. Let the G-A-module $\mathcal{M}$ belong to the class $\mathcal{P}(A)$ of projective finitely generated modules. Then obviously $\mathcal{M}_s = \mathrm{Hom}_G(V_s, \mathcal{M}) \in \mathcal{P}(A)$. Let us show that only a finite number of summands do not vanish in the sum $\bigoplus_s$. Denote generators of $\mathcal{M}$ by $a_1, \dots, a_s$. Using the Mostow lemma [**93**], choose $\mathbf{C}$-periodic vectors $b_1, \dots, b_s$ so close to $a_1, \dots, a_s$ that they generate $\mathcal{M}$ as an A-module (see Lemma 2.7.3). By decomposing the linear span of the orbit Gb_j, which is a finite-dimensional G-$\mathbf{C}$-module, into irreducible modules, let us find a new system of generators $c_1, \dots, c_N$, now belonging to irreducible G-$\mathbf{C}$-modules. Then it is evident that the number of nonzero summands does not exceed N.

LEMMA 2.8.12. *Let $F : L \to M$, $T : N \to L$ be continuous maps of Banach spaces, $S = FT$ an isomorphism and* $\mathrm{Ker}\, F = 0$. *Then F is an isomorphism.*

PROOF. Since S is an isomorphism and F is bounded, T is topologically injective and its image $T(N)$ is closed in L. Suppose it does not coincide with L. Choose a vector $0 \neq x \in L \setminus T(N)$. Then $0 \neq F(x) \notin FT(N)$. Indeed, let $F(x) = FT(y)$ for some $y \in N$. Since $z = Ty \in T(N)$, $z - x \neq 0$, while $F(z - x) = FT(y) - F(z) = 0$.

We get a contradiction to the condition $\operatorname{Ker} F = 0$. Hence, T is a topologically injective epimorphism, i.e., isomorphism, as well as $F = ST^{-1}$. □

Let us recall some facts about integrating operator-valued functions [**62**]. Let X be a compact space, A a C^*-algebra and $\varphi : C(X) \to A$ an involutive homomorphism of unital algebras. Let $F : X \to A$ be a continuous map and let, for each $x \in X$, the element $F(x)$ commute with the image of φ. In this case the integral

$$\int_X F(x)\, d\varphi \in A$$

can be defined as follows. Let $X = \bigcup_{i=1}^n U_i$ be an open covering and let $\sum_{i=1}^n \alpha_i(x) = 1$ be a subordinate partition of unity. Choose points $\xi_i \in U_i$ and form the integral sum

$$\sum (F, \{U_i\}, \{\alpha_i\}, \{\xi_i\}) = \sum_{i=1}^n F(\xi_i)\varphi(\alpha_i).$$

If the limit of such an integral sums exists, then it is called an integral.

If X is a Lie group G, it is natural here to use a Haar measure $\varphi : C(X) \to \mathbf{C}$, $\varphi(\alpha) = \int_G \alpha(g)\, dg$ and to define

$$\int_G Q(g)\, dg := \lim \sum_i Q(\xi_i) \int_G \alpha_i(g)\, dg$$

for a norm continuous map $Q : G \to \mathcal{B}(H)$, where the C^*-algebra A is viewed as a subalgebra in the algebra $\mathcal{B}(H)$ of bounded operators on a Hilbert space H. If $Q : G \to P^+(A) \subset \mathcal{B}(H)$, then, since

$$\int_G \alpha_i(g)\, dg \geq 0,$$

we obtain that

$$\sum_i Q(\xi_i) \cdot \int_G \alpha_i(g)\, dg \;\in\; P^+(A) \quad \text{and} \quad \int_G Q(g)\, dg \;\in\; P^+(A)$$

(the positive cone $P^+(A)$ is convex and closed). Hence we have proved the following lemma.

LEMMA 2.8.13. *Let $Q : G \to P^+(A)$ be a continuous function. Then, for the integral in the sense of* [**62**]*, the following inequality holds:*

$$\int_G Q(g)\, dg \geq 0.$$

THEOREM 2.8.14 ([**128**]). *Let* $\mathrm{GL} = \mathrm{GL}(A)$ *be the full general linear group, i.e., the group of invertible operators in* $\operatorname{End} l_2(A)$*, and suppose that for the group G, a representation $g \mapsto T_g$ $(g \in G, T_g \in \mathrm{GL})$ is given, and that the map*

$$G \times l_2(A) \to l_2(A), \quad (g, u) \mapsto T_g u$$

is continuous. Then there exists an A-valued inner product on $l_2(A)$ equivalent to the initial one (*i.e., generating an equivalent norm*) *and such that the representation $g \mapsto T_g$ is unitary with respect to this new product.*

PROOF. Let $\langle\, ,\, \rangle'$ be the initial inner product. For any $u, v \in l_2(A)$ there exists a continuous map $G \to A, \quad x \mapsto \langle T_x u, T_x v\rangle'$. Define the new product by the formula

$$\langle u, v\rangle = \int_G \langle T_x u, T_x v\rangle'\, dx,$$

where the integral can be considered in the sense of any of the two definitions in [**62**], since the map is norm continuous. It is easy to see that this new product gives a sesquilinear map $l_2(A) \times l_2(A) \to A$ satisfying properties (i)–(iv) of Definition 1.2.1 and that, by Lemma 2.8.13, $\langle u, u\rangle \geq 0$. Let us show that this map is continuous. Fix an arbitrary $u \in l_2(A)$. Then $x \mapsto T_x(u), \quad G \to l_2(A)$ is a continuous map defined on a compact space, thus the set $\{T_x(u) \mid x \in G\}$ is bounded. Therefore, by the uniform boundedness principle [**7**, v. 2],

$$\lim_{v \to 0} T_x(v) = 0 \tag{2.30}$$

uniformly in $x \in G$. If u is fixed, then

$$\|T_x(u)\| \leq M_u = \text{const}$$

and, by equality (2.30), one has

$$\begin{aligned} \|\langle u, v\rangle\| &= \| \int_G \langle T_x(u), T_x(v)\rangle'\, dx\| \\ &\leq M_u \cdot \text{vol}\, G \cdot \sup_{x \in G} \|T_x(v)\| \to 0 \quad (v \to 0). \end{aligned}$$

We have obtained the continuity at the point 0, hence on the whole space $l_2(A) \times l_2(A)$. For $T_x u = (a_1(x), a_2(x), \dots) \in l_2(A)$, the equality $\langle u, u\rangle = 0$ takes the form

$$\int_G \sum_{i=1}^{\infty} a_i(x) a_i^*(x)\, dx = 0.$$

Let A be viewed as a subalgebra of the algebra of bounded operators on a Hilbert space L with an inner product $(\, ,\,)_L$. For each $p \in L$ we have

$$\begin{aligned} 0 &= \left(\left(\int_G \sum_{i=1}^{\infty} a_i(x) a_i^*(x)\, dx \right) p, p \right)_L \\ &= \int_G \left(\sum_{i=1}^{\infty} a_i(x) a_i^*(x) p, p \right)_L dx = \int_G \left(\sum_{i=1}^{\infty} (a_i^*(x) p, a_i^*(x) p)_L \right) dx \end{aligned}$$

(cf. [**62**]). Therefore $a_i(x) = 0$ almost everywhere, hence, by continuity, $a_i(x) = 0$ for all x and $T_x u = 0$. In particular, $u = 0$.

Since each operator T_y is an automorphism, we obtain

$$\langle T_y u, T_y v\rangle = \int_G \langle T_{xy} u, T_{xy} v\rangle'\, dx = \int_G \langle T_z u, T_z v\rangle'\, dz = \langle u, v\rangle.$$

Now we show the equivalence of the two norms, which, in particular, implies the continuity of the representation. There is a number $N > 0$ such that $\|T_x\|' \leq N$

for any $x \in G$. Hence, by [**62**], we have

$$\|u\|^2 = \|\langle u,u\rangle\|_A = \|\int_G \langle T_x u, T_x u\rangle' \,dx\|_A$$
$$\leq \left(\sup_{x\in G} \|T_x u\|'\right)^2 \leq N^2(\|u\|')^2.$$

On the other hand, applying Theorem 2.1.4 and Lemma 2.8.13, we obtain that

$$\langle u,u\rangle' = \int_G \langle T_{g^{-1}}T_g u, T_{g^{-1}}T_g u\rangle' \,dg \leq \int_G \|T_{g^{-1}}\|^2 \langle T_g u, T_g u\rangle' \,dg$$
$$\leq \int_G N^2 \langle T_g u, T_g u\rangle' \,dg = N^2 \int_G \langle T_g u, T_g u\rangle' \,dg = N^2\langle u,u\rangle.$$

Then $(\|u\|')^2 = \|\langle u,u\rangle'\|_A \leq N^2\|\langle u,u\rangle\|_A = N^2\|u\|^2$. □

REMARK 2.8.15. Since $l_2(P)$ is a direct summand in $l_2(A)$, the previous theorem remains valid for $l_2(P)$ and for any other countably generated module $\mathcal{M}$.

DEFINITION 2.8.16. Denote by $\mathrm{GL}^* \subset \mathrm{GL}$ the *general linear group* formed by adjointable invertible operators.

REMARK 2.8.17. Before averaging we have had operators, which, in general, are nonadjointable, but after averaging we have obtained unitary operators out of them. In relation to this remark the following question arises: is it true that if a given operator represents an element of compact group, then it is adjointable? A negative answer to this question is contained in Example 2.3.2, since a decomposition into a direct (topological) sum defines a representation of the group $\mathbf{Z}/2\mathbf{Z}$.

COROLLARY 2.8.18 ([**129**]). *Let $\mathcal{M} = \mathcal{M}_1 \widetilde{\oplus} \mathcal{M}_2$ be a topological decomposition into a direct sum of closed Hilbert C^*-modules (not necessarily an orthogonal sum). Then there exists a new inner product on the module $\mathcal{M}$ equivalent to the initial one, with respect to which the given decomposition is orthogonal.*

PROOF. Define the operator $J : \mathcal{M} \longrightarrow \mathcal{M}$ by the formula

$$Jx = \begin{cases} x, & \text{if } x \in \mathcal{M}_1, \\ -x, & \text{if } x \in \mathcal{M}_2. \end{cases}$$

One can consider the operator J as a representation of the group $\mathbf{Z}/2\mathbf{Z}$ on the module $\mathcal{M}$ and, by Theorem 2.8.14, the inner product $\langle x,y\rangle_\beta = \langle x,y\rangle + \langle Jx,Jy\rangle$ is equivalent to the initial one. Orthogonality of $\mathcal{M}_1$ and $\mathcal{M}_2$ with respect to this inner product is evident. □

In Theorem 5.7.13 we shall show how the averaging Theorem 2.8.14 can be generalized from the case of a compact group to the case of an amenable group, but only for Hilbert W^*-modules.

2.9. Equivariant Fredholm operators

In this section we transfer the main definitions and statements related to A-Fredholm operators to the G-equivariant case, where G is a compact group. We will present only a list of results of [**128**]. The proofs (very similar to the nonequivariant case) and details can be found, e.g., in [**121**].

By the averaging Theorem 2.8.14 we can restrict ourselves to a unitary action of G on $l_2(P)$. Suppose the action of G on A is trivial and apply the Kasparov stabilization Theorem 2.8.4 to $l_2(P)$:

$$l_2(P) \oplus \mathcal{H}_A \cong \mathcal{H}_A, \tag{2.31}$$

where $\mathcal{H}_A = \sum_{i=1}^{\infty}(A \otimes_{\mathbf{C}} V_i)$ and $\{V_i\}$ is a countable collection of finite-dimensional vector spaces of all irreducible representations of G (up to their equivalence), and each representation is repeated countably many times. The isomorphism (2.31) is a G-isomorphism of Hilbert A-modules and the sum on the left-hand side of (2.31) is orthogonal. Put

$$R_m = \sum_{i=m+1}^{\infty}(A \otimes_{\mathbf{C}} V_i), \qquad R_m^{\perp} = \sum_{i=1}^{m}(A \otimes_{\mathbf{C}} V_i).$$

For any bounded G-A-operator $F : l_2(P_1) \to l_2(P_2)$ denote by $S(F) : \mathcal{H}_A \to \mathcal{H}_A$ the following (stabilized) operator:

$$\begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix} : \mathcal{H}_A \cong \mathcal{H}_A \oplus l_2(P_1) \to \mathcal{H}_A \oplus l_2(P_2) \cong \mathcal{H}_A.$$

Note that this stabilization is defined up to a G-A-isomorphism.

Theorem 2.9.1 ([**128**]). *Let $\mathcal{H}_A \cong \mathcal{M}\widetilde{\oplus}\mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ are closed G-A-modules, and let $\mathcal{N}$ have a finite system of generators $a_1, \dots, a_s$. Then $\mathcal{N}$ is a projective finitely generated G-A-module.*

Definition 2.9.2. A bounded G-A-operator

$$F : l_2(\mathcal{P}_1) \to l_2(\mathcal{P}_2)$$

is called *G-A-Fredholm* if

1) the operator F is adjointable;

2) there exist decompositions of the domain $\mathcal{H}_A = \mathcal{M}_1\widetilde{\oplus}\mathcal{N}_1$ and the range $\mathcal{H}_A = \mathcal{M}_2\widetilde{\oplus}\mathcal{N}_2$ of the operator $S(F)$ (where $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{N}_1$, $\mathcal{N}_2$ are closed G-A-modules and $\mathcal{N}_1$, $\mathcal{N}_2$ are finitely generated), such that the matrix of $S(F)$ with respect to these decompositions has the form $S(F) = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}$, where $F_1 : M_1 \to M_2$ is an isomorphism of G-A-modules. By the previous theorem, $\mathcal{N}_1$ and $\mathcal{N}_2$ are projective G-A-modules and we can define the index of F by

$$\operatorname{index} F = [\mathcal{N}_1] - [\mathcal{N}_2] \in K^G(A).$$

Remark 2.9.3. Since the action of G is unitary, an analog of 2.7.6 evidently holds.

Theorem 2.9.4 ([**128**]). *The index is well defined.*

Lemma 2.9.5 ([**128**]). *Let an operator $F : l_2(A) \to l_2(A)$ be G-A-Fredholm. Then there exists a number $\varepsilon > 0$ such that any bounded adjointable G-A-operator D satisfying the inequality $\|F - D\| < \varepsilon$ is a G-A-Fredholm operator and* $\operatorname{index} D = \operatorname{index} F$.

Lemma 2.9.6 ([**128**]). *Let F and D be G-A-Fredholm operators*:

$$F : l_2(\mathcal{P}_1) \to l_2(\mathcal{P}_2), \qquad D : l_2(\mathcal{P}_2) \to l_2(\mathcal{P}_3).$$

Then $DF : l_2(\mathcal{P}_1) \to l_2(\mathcal{P}_3)$ is a G-A-Fredholm operator and

$$\operatorname{index} DF = \operatorname{index} D + \operatorname{index} F.$$

LEMMA 2.9.7 ([**128**]). *Let $K : l_2(\mathcal{P}_1) \to l_2(\mathcal{P}_2)$ be a compact G-operator. Then $1 + K$ is a G-A-Fredholm operator and* $\operatorname{index}(1 + K) = 0$.

LEMMA 2.9.8 ([**128**]). *Suppose $F : l_2(\mathcal{P}_1) \to l_2(\mathcal{P}_2)$ is a G-A-Fredholm operator and an operator $K \in \mathcal{K}(l_2(\mathcal{P}_1), l_2(\mathcal{P}_2))$ is G-equivariant. Then the operator $F + K$ is G-A-Fredholm and* $\operatorname{index}(F + K) = \operatorname{index} F$.

LEMMA 2.9.9 ([**128**]). *Suppose $F : l_2(\mathcal{P}_1) \to l_2(\mathcal{P}_2)$ is a bounded adjointable G-A-operator, $D \in \operatorname{End}^* \mathcal{H}_A$ and $K \in \mathcal{K}(\mathcal{H}_A)$ are G-A-operators, and for D there exists a decomposition of $\mathcal{H}_A$, as in the definition of a G-A-Fredholm operator, and $D = S(F) + K$. Then F is a G-A-Fredholm operator.*

THEOREM 2.9.10 ([**128**]). *Suppose*

$$F : l_2(\mathcal{P}_1) \to l_2(\mathcal{P}_2), \qquad D : l_2(\mathcal{P}_2) \to l_2(\mathcal{P}_1), \qquad D' : l_2(\mathcal{P}_2) \to l_2(\mathcal{P}_1)$$

are bounded adjointable G-A-operators and

$$S(FD) = 1_{\mathcal{H}_A} + K_1, \quad S(D'F) = 1_{\mathcal{H}_A} + K_2, \quad K_1,\, K_2 \in \mathcal{K}(\mathcal{H}_A).$$

Then F is a G-A-Fredholm operator.

LEMMA 2.9.11 ([**128**]). *If, for bounded adjointable G-A-operators D, D' and F, the operators FD and $D'F$ are G-A-Fredholm operators, then F is a G-A-Fredholm operator too.*