Preface

Perturbation theory is a natural methodology that can be phrased as follows: when the governing rule of the system in question is slightly changed by ε, an approximate solution is obtained by summing up the direct effect of the change on the object, the secondary effect produced by the (main part of) the direct effect, and all higher degree (or, at least sufficiently high degree) effects. Because of its naturality, it is no exaggeration to say that perturbation theory is constantly used in every field of the exact sciences, not only in mathematics. But nature sometimes—or we should rather say 'always'—enshrines in it seemingly complicated but actually sublimely beautiful structures, which seems so complicated that human beings cannot perceive it to be 'natural' at first. Perturbation theory is also one example of such subtleties that nature presents; most of the perturbation problems we encounter are the so-called singular perturbations whose characteristic property is that the aspect of the problem for $\varepsilon = 0$ is substantially different from that for $\varepsilon \neq 0$. As the analytic counterpart of this singular character, perturbation series almost surely diverge in singular perturbations. Probably because of this 'seeming complexity', pure mathematicians seem to have been little interested in singular perturbation theory, and we think it is not without reason. Actually, in our opinion, 'algebraic analysis of singular perturbation theory' or 'exact WKB analysis' is a field of mathematics whose core meaning can be grasped only after human beings master 'microlocal analysis'. (We hope the reader will feel so in Chapter 2, Section 2.3.)

The principal aim of this book is to sketch the recent results of our group; we have tried to describe the route to the goal, rather than the goal itself, as we understand it. Hence we confine our discussion to the case where the 'governing rule of the system' is given through a differential equation. In spite of such a restriction, we still hope this tiny monograph may trigger further development of singular perturbation theory, recalling the substantial effect on constructive field theory of the work of Bender-Wu [**14**] that discusses the eigenvalue problems in quantum mechanics, not quantum field theory.

It was Professor Mikio Sato who launched the authors' interest in the (algebraic analysis of) singular perturbation theory. The elder author (T.K.) sincerely thanks Professor Sato for having led him to the field that was seemingly alien to his own subject at that time, just at the time when one may often be tempted to choose one's subject in the 'natural' extension of one's past achievements. The younger author (Y.T.) thanks Professor Sato for having let him know of such a fruitful field at the early stage of his life as a mathematician. Both authors are truly indebted to Professor Sato for his exceptionally appropriate advice. In a word, this book (except for Chapter 4) is a report of the seminar conducted by mentor Sato, where we discussed in depth Bender-Wu [**14**], Pham [**51**], Voros [**65**], etc. The authors express their heartiest thanks to Professor Takashi Aoki for their stimulating discussions with him, from which they benefited much at every stage of their research. They express their heartiest thanks also to Professor Michio Jimbo, who gave them leads to their discussion in Chapter 4. We are thankful to the editors Kenji Ueno, Kazuhiko Aomoto, and Michio Jimbo, who gave us the opportunity to write this book, and also to Professor Orlando Neto, who arranged the Summer School supported by EU (July, 1996, Lisbon) for us to give a series of lectures on the material of this book. Without these opportunities, we could not have determined to write a book on such a rapidly changing and progressing subject as exact WKB analysis. Last but not least, we sincerely thank Professor Kazuo Murota for his careful reading of the manuscript and for providing us with invaluable advice to improve this monograph.

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Takahiro Kawai
Yoshitsugu Takei