Preface to the English Edition

After the Japanese edition was published in 2002, some progress has been made in the subject of §6 of Chapter 3, especially in finding the geometric meaning of elliptic cohomology, such as [**BDR04**] and [**ST04**].

As is stated in the preface to the Japanese edition, this field is still in embryo and it is not possible to give an overview of the subject. The authors, however, decided to take the opportunity of the English translation to update the references and expositions in order to provide a better view.

References in other sections are also updated. Typos and errors found in the process of translation are corrected. Some of the commutative diagrams are redrawn to make them easier to read.

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Authors

Preface

This is a book on generalized cohomology theories intended to be read by wide audiences who are not experts in algebraic topology.

Algebraic topology is a field in which topological problems are studied by algebraic invariants. Among such invariants, fundamental groups and homology groups of simplicial complexes are classical examples. Although the notion of homology was a bit obscure when it was discovered by H. Poincaré, the definition of homology groups for simplicial complexes was already well-established in the 1930s.

It used to be an important problem to prove that the homology groups are independent of chosen simplicial decompositions. Eilenberg introduced a new homology theory based on singular simplices to avoid this problem. Then Eilenberg and Steenrod proved that the singular homology theory coincides with simplicial homology theory by characterizing homology theory by seven axioms. It was also noticed that cohomology groups that are dual to homology groups could be characterized by similar axioms.

Atiyah and Hirzebruch defined topological K-theory by using vector bundles in the 1950s. The famous Bott periodicity extends K-theory to a system which satisfies the first six axioms out of the seven axioms of Eilenberg and Steenrod. This "incomplete cohomology theory" turns out to be very useful. It is used to give a very simple proof of the Hopf invariant one problem. Adams uses it to solve the vector field problem on spheres. The J-group, which is closely related to K-theory, is frequently used in surgery theory. One of the important steps to compute this J-group is the famous Adams conjecture, which is solved by Quillen. One of the tools used by Quillen is étale homotopy theory. The higher algebraic K-theory discovered by Quillen is also related to the J-group. These are still important subjects of study.

On the other hand, it was also discovered around the same period that the notion of cobordism, which was introduced by Thom and is very important in the study of the topology of manifolds, gives rise to a system that satisfies the first six axioms of Eilenberg-Steenrod.

Such systems have come to be known as generalized (co)homology theories. The study of generalized (co)homology theories leads to the notions of spectrum and infinite loop space and is now an important part of algebraic topology.

Considered as generalized homology theories, the coefficient rings of various cobordism theories are the cobordism groups of Thom. It is an important problem to determine these groups. Milnor makes use of the Adams spectral sequence, which is an important tool in stable homotopy theory, to determine the structure of the complex cobordism ring. On the other hand, Quillen discovered an important relationship between complex-oriented cohomology theories and formal group laws used in number theory. He also proved that complex cobordism is universal among these cohomology theories and that the complex cobordism ring is isomorphic to the Lazard ring.

This book begins with basic notions in homotopy theory, and then introduces the axioms of generalized (co)homology theory. In Chapter 3, basic properties of complex-oriented cohomology theories are explained. One of the most important facts is that the Chern classes of complex vector bundles can be defined for such cohomology theories.

The next chapter, Chapter 4, is devoted to K-theory. It includes the definition of K-theory, a proof of the Bott periodicity theorem, the definition of the Adams operation and its applications, the definition of the J-group, and then a sketch of a proof of the Adams conjecture.

Chapter 5 is an exposition of spectral sequences which is an important tool not only in algebraic topology but also in many areas of mathematics. A spectral sequence is a tool that approximates a complicated algebraic object by a sequence of chain complexes. In order to make use of a spectral sequence, it is important to know how well the final term approximates the target. This is called the convergence problem of spectral sequences and is not an easy problem. In Chapter 5, we first review a basic idea of spectral sequences and then study the general convergence problem following Boardman. In the rest of this chapter, general constructions of spectral sequences are given in a way which is applicable to other fields of mathematics.

Within this general framework, several examples which are frequently used in modern homotopy theory are given.

The main topics of the last chapter, Chapter 6, are complex cobordism and formal group laws. We first prove Quillen's result by using the Adams spectral sequence following Adams. Related to formal group laws, several important complex-oriented cohomology theories have been defined, e.g. Brown-Peterson cohomology, Morava K-theory, and elliptic cohomology. However, we give no more than definitions and basic properties for these topics. One of the reasons is that elliptic cohomology and related subjects are still under heavy development. They still lack a nice geometric description like K-theory or cobordism. It is very likely that current "state of the art theorems" in this area will become less important after a good geometric model is discovered.

In the appendices, we summarize three important tools in this book: simplicial technique, limits, and spectra.

There are other good textbooks on this subject, e.g. [**Ara75**] by S. Araki and [**Ada74**] by J.F. Adams, but these were written more than twenty years ago. More recent is Rudyak's book [**Rud98**], which treats cobordism and related topics in detail. The main idea of this book is to summarize important topics on generalized (co)homology theories in a concise form and to serve as a guide book for those who don't have advanced knowledge in algebraic topology. It should also be noted that this book is not intended to cover the latest topics in algebraic topology, like the homotopy theory of algebraic varieties, and so on.

Finally, the authors are grateful to the editors of this series, especially Kenji Ueno and Kenji Fukaya for giving them a chance to write this book. Thanks are also due to the editorial department of Iwanami Shoten who did amazing work to help the authors to make up for the delay of manuscripts.

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