Preface

Linear differential equations are looked upon from two points of view.

The first point of view is based on understanding a linear differential equation as a linear mapping between two infinite-dimensional vector spaces. Let us consider the situation when the linear mapping gives an "almost" linear isomorphism. Here, "almost" means "ignoring finite-dimensional vector subspaces." In other words, the finite-dimensional subspaces, which are ignored, are responsible for the possibility that the two infinite-dimensional vector spaces are not canonically isomorphic. This situation yields a linear mapping between certain finite-dimensional vector subspaces. The object obtained by using this finite-dimensional approximation is called the index.

Another point of view is based on the description of a linear differential equation using local coordinates. Then the linear differential equation can be formally regarded as a section of a certain fiber bundle. The object obtained using this approach is the homotopy class of the section and is called (the homotopy class of) the principal symbol.

Each object is invariant under continuous deformation, with appropriate requirements, of the linear differential equation. In this sense, both of them are topological objects.

The two points of view are related to each other. The condition that the principal symbol is almost invertible is a sufficient condition for the linear differential operator to be an almost linear isomorphism (at least on closed manifolds). The former is called ellipticity for partial differential equations, and the latter is called Fredholm property, which is a notion in functional analysis. Ellipticity implies Fredholm property.

In order to compute the index following its definition, we must solve the linear differential equation to obtain a finite-dimensional

approximation. Among various approaches to the index, the most basic one is given by the integer

$$\text{ind } P = \dim \text{Ker } P - \dim \text{Coker } P,$$

where we denote the linear mapping by P.

The Atiyah-Singer index theorem describes how to determine the index of an elliptic differential operator by its principal symbol using only topological tools.

As a slogan, the index theorem states the coincidence of the analytical index, which is the original definition, and the topological index, which is defined using topological tools.

The index theorem as a coincidence of two numbers originated in the work of Atiyah and Singer. Its applications to differential topology have also been explored since then, but this area of research is not active at this moment. In recent years, the index theorem has become rather a daily tool used in global analysis involving non-linear differential equations. A few decades ago, the index theorem itself was the focal point of the formation of global analysis.

Is research on the index theorem "done with"?

The author believes that this is not the case. One piece of evidence is that there are still active research fields surrounding the index theorem. Outside of such obviously related places, it seems to the author that the study of each subject in the background of the index theorem is still in progress. It is not yet clear what future theory the index theorem uses as its first step, but surely it will be something interesting. The contours of this future theory can be vaguely seen when we review the development of the index theorem.

At this moment, it is important to understand the index theorem not as a mountain chain, behind grand theories, but to digest it as a natural element of the scenery when we follow the fundamental properties of differential equations.

I would like to present such understanding of the index theorem by giving proofs that are as elementary as possible. This is what I always had in mind when writing this book. I would be happy if readers would give opinions and criticism.

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