

Contents

| | |
|---|----|
| Preface | xi |
| Scott Sheffield and Thomas Spencer | |
| Introduction | 1 |
| David C. Brydges | |
| Lectures on the Renormalisation Group | 7 |
| Acknowledgment | 9 |
| Lecture 1. Scaling limits and Gaussian measures | 11 |
| 1.1. Introduction | 11 |
| 1.2. Theoretical physics | 13 |
| 1.3. Some results | 15 |
| 1.4. Gaussian measures on \mathbb{R}^Λ | 15 |
| 1.5. Example: One dimension | 18 |
| 1.6. Local and global functions | 19 |
| 1.7. Example: Particles on a lattice | 21 |
| 1.8. The importance of the partition function | 22 |
| 1.9. Appendix. Green's functions | 24 |
| Lecture 2. Renormalisation group in hierarchical models | 27 |
| 2.1. Massless Gaussian measure | 27 |
| 2.2. Finite range decompositions | 27 |
| 2.3. Motivation | 29 |
| 2.4. The renormalisation group and hierarchical models | 30 |
| 2.5. Hierarchical models | 31 |
| 2.6. The formal infinite volume limit and trivial fixed point | 34 |
| 2.7. Analysis | 35 |
| 2.8. Expanding directions, relevant operators | 35 |
| 2.9. Trivial fixed point | 36 |
| 2.10. Appendix. Stable manifold theorem | 37 |
| Lecture 3. Example: the hierarchical Coulomb gas | 41 |
| 3.1. Example: Hierarchical Coulomb gas | 41 |
| 3.2. Finite volume | 46 |
| 3.3. Fractional charge observable and confinement | 47 |
| 3.4. Appendix. Notes on the rigorous renormalisation group | 49 |
| Lecture 4. Renormalisation group for Euclidean models | 53 |
| 4.1. Euclidean lattice and the dipole model | 53 |
| 4.2. The initial I_0, K_0 | 54 |
| 4.3. The basic scaling mechanism | 55 |
| 4.4. Coordinates (I_j, K_j) | 55 |
| 4.5. Euclidean replacement for Lemma 2.14 | 57 |

| | |
|--|-----|
| Lecture 5. Coordinates and action of renormalisation group | 61 |
| 5.1. Euclidean replacement for Lemma 2.14 continued | 61 |
| 5.2. Formulas for \tilde{I}, J . | 64 |
| Lecture 6. Smoothness of (RG) | 67 |
| 6.1. Choice of spaces and smoothness of (RG) | 67 |
| 6.2. Norms | 71 |
| 6.3. Open problems | 78 |
| 6.4. Appendix. Geometry and counting lemmas | 78 |
| 6.5. Appendix. Proof of Theorem 6.14 | 82 |
| Bibliography | 91 |
| | |
| Alice Guionnet | |
| Statistical Mechanics and Random Matrices | 95 |
| Introduction | 97 |
| 1. Motivations | 98 |
| 2. The different scales; typical results | 104 |
| Lecture 1. Wigner matrices and moments estimates | 109 |
| 1. Wigner's theorem | 109 |
| 2. Words in several independent Wigner matrices | 118 |
| 3. Estimates on the largest eigenvalue of Wigner matrices | 120 |
| Lecture 2. Gaussian Wigner matrices and Fredholm determinants | 123 |
| 1. Joint law of the eigenvalues | 123 |
| 2. Joint law of the eigenvalues and determinantal law | 124 |
| 3. Determinantal structure and Fredholm determinants | 126 |
| 4. Fredholm determinant and asymptotics | 127 |
| Lecture 3. Wigner matrices and concentration inequalities | 131 |
| 1. Concentration inequalities and logarithmic Sobolev inequalities | 131 |
| 2. Smoothness and convexity of the eigenvalues of a matrix and of traces of matrices | 135 |
| 3. Concentration inequalities for random matrices | 139 |
| 4. Brascamp-Lieb inequalities; applications to random matrices | 141 |
| Lecture 4. Matrix models | 147 |
| 1. Combinatorics of maps and non-commutative polynomials | 149 |
| 2. Formal expansion of matrix integrals | 153 |
| 3. First order expansion for the free energy | 160 |
| 4. Discussion | 167 |
| Lecture 5. Random matrices and dynamics | 171 |
| 1. Free Brownian motions and related stochastic differential calculus | 172 |
| 2. Consequences | 179 |
| 3. Discussion | 182 |
| Bibliography | 185 |

| | |
|----------------------------------|------------|
| Richard Kenyon | |
| Lectures on Dimers | 191 |
| 1. Overview | 193 |
| 1.1. Dimer definitions | 193 |
| 1.2. Uniform random tilings | 194 |
| 1.3. Limit shapes | 196 |
| 1.4. Facets | 198 |
| 1.5. Measures | 199 |
| 1.6. Other random surface models | 200 |
| 2. The height function | 200 |
| 2.1. Graph homology | 200 |
| 2.2. Heights | 201 |
| 3. Kasteleyn theory | 202 |
| 3.1. The Boltzmann measure | 203 |
| 3.2. Gauge equivalence | 203 |
| 3.3. Kasteleyn weighting | 203 |
| 3.4. Kasteleyn matrix | 204 |
| 3.5. Local statistics | 205 |
| 4. Partition function | 206 |
| 4.1. Rectangle | 206 |
| 4.2. Torus | 207 |
| 4.3. Partition function | 209 |
| 4.4. Height change distribution | 209 |
| 5. Gibbs measures | 209 |
| 5.1. Definition | 209 |
| 5.2. Periodic graphs | 210 |
| 5.3. Ergodic Gibbs measures | 210 |
| 5.4. Constructing EGMs | 211 |
| 5.5. Magnetic field | 211 |
| 6. Uniform honeycomb dimers | 212 |
| 6.1. Inverse Kasteleyn matrix | 213 |
| 6.2. Decay of correlations | 213 |
| 6.3. Height fluctuations | 214 |
| 7. Legendre duality | 215 |
| 8. Boundary conditions | 217 |
| 9. Burgers equation | 218 |
| 9.1. Volume constraint | 220 |
| 9.2. Frozen boundary | 220 |
| 9.3. General solution | 220 |
| 10. Amoebas and Harnack curves | 222 |
| 10.1. The amoeba of P | 222 |
| 10.2. Phases of EGMs | 223 |

| | |
|--|-----|
| 10.3. Harnack curves | 224 |
| 10.4. Example | 224 |
| 11. Fluctuations | 226 |
| 11.1. The Gaussian free field | 226 |
| 11.2. On the plane | 227 |
| 11.3. Gaussians and moments | 227 |
| 11.4. Height fluctuations on the plane | 227 |
| 12. Open problems | 229 |
| Bibliography | 229 |
| G. Lawler | |
| Schramm-Loewner Evolution (SLE) | 231 |
| Introduction | 233 |
| Lecture 1. Scaling limits of lattice models | 235 |
| 1. Self-avoiding walk (SAW) | 235 |
| 2. Loop-erased random walk | 239 |
| 3. Percolation | 241 |
| 4. Ising model | 242 |
| 5. Assumptions on limits | 243 |
| 6. Exercises for Lecture 1 | 243 |
| Lecture 2. Conformal mapping and Loewner equation | 245 |
| 1. Important results about conformal maps | 245 |
| 2. Half-plane capacity | 247 |
| 3. Loewner equation | 249 |
| 4. Maps generated by a curve | 251 |
| 5. A flow on conformal maps | 252 |
| 6. Doubly infinite time | 253 |
| 7. Distance to boundary | 254 |
| 8. Exercises for Lecture 2 | 255 |
| Lecture 3. Schramm-Loewner evolution (SLE) | 257 |
| 1. Definition | 257 |
| 2. Phases | 259 |
| 3. Dimension of the path | 260 |
| 4. Cardy's formula | 261 |
| 5. Conformal images of SLE | 262 |
| 6. Exercises for Lecture 3 | 264 |
| Lecture 4. SLE_κ in a simply connected domain D | 265 |
| 1. Drift and locality | 265 |
| 2. Girsanov | 266 |
| 3. The restriction martingale | 267 |
| 4. (Brownian) boundary bubbles | 268 |
| 5. Brownian loop measure | 270 |
| 6. The measure $\mu_D(z, w)$ for $\kappa \leq 4$ | 271 |
| 7. Exercises for Lecture 4 | 272 |

| | |
|--|-----|
| Lecture 5. Radial and two-sided radial SLE_κ | 275 |
| 1. Example: SAW II | 275 |
| 2. Radial SLE_κ | 277 |
| 3. Another definition | 279 |
| 4. Radial SLE_κ in a smaller domain | 280 |
| 5. Two-sided radial | 281 |
| 6. Exercises for Lecture 5 | 282 |
| Lecture 6. Intersection exponents | 283 |
| 1. One-sided | 283 |
| 2. Two-sided | 286 |
| 3. Nonintersecting SLE_κ | 287 |
| 4. Radial exponent and SAW III | 288 |
| 5. Exercises for Lecture 6 | 290 |
| Tables for reference | 291 |
| Bibliography | 293 |
| Wendelin Werner | |
| Lectures on Two-dimensional Critical Percolation | 297 |
| Overview | 299 |
| Lecture 1. Introduction and tightness | 301 |
| 1. 2D percolation | 301 |
| 2. Notations and prerequisites | 302 |
| 3. Russo-Seymour-Welsh | 304 |
| 4. First consequences | 306 |
| First exercise sheet | 309 |
| Lecture 2. The Cardy-Smirnov formula | 313 |
| 1. Preliminaries | 313 |
| 2. Smirnov's theorem | 315 |
| Lecture 3. Convergence to SLE(6) | 321 |
| 1. Our goal | 321 |
| 2. Hand-waving argument | 322 |
| 3. Tightness | 323 |
| 4. Loewner chains | 323 |
| 5. Capacity increases | 324 |
| 6. Side-remark concerning the proof of Cardy-Smirnov formula | 326 |
| 7. Identifying continuous martingales | 326 |
| 8. Recognizing SLE(6) | 327 |
| 9. Take-home message | 328 |
| Second exercise sheet | 331 |
| Lecture 4. SLE(6) computations | 333 |
| 1. Radial SLE | 333 |
| 2. Relation between radial and chordal SLE(6) | 334 |

| | |
|--|-----|
| 3. Relation to discrete radial exploration | 335 |
| 4. Exponent computations | 336 |
| Lecture 5. Arm exponents | 341 |
| 1. Some notations | 341 |
| 2. One-arm exponent | 341 |
| 3. Four-arms exponent | 343 |
| 4. Other exponents and bibliographical remarks | 349 |
| Lecture 6. Near-critical percolation | 351 |
| 1. Correlation length | 351 |
| 2. Outline of the proof | 352 |
| 3. A priori estimates | 353 |
| 4. Arm separation | 353 |
| 5. Using differential inequalities | 354 |
| 6. Concluding remarks | 358 |
| Bibliography | 359 |