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Mirror Symmetry

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Introduction

The present book consists of a set of notes from a course given by me at the Institut Henri Poincaré in the context of the algebraic geometry semester at the Émile Borel Center during the spring of 1995.

The goal of the course was to present recent results connected with mirror symmetry. As this topic is far from being perfectly understood mathematically, these recent results have developed in a number of different directions, from the profound study conducted by Batyrev on families of hypersurfaces with trivial canonical bundle on toric Fano varieties and the combinatorial construction of the mirror family to the discovery of the "quantum product" in the cohomology of a symplectic manifold. This last topic goes far beyond the scope of mirror symmetry, but it was motivated by the desire to give a mathematical definition of such objects as the Gromov–Witten potential, which is one of its essential ingredients, and to prove the main property of that potential, namely that it satisfies the "WDVV" equation.

This burgeoning situation is reflected in the division of the book into autonomous chapters, which, although they are connected by common themes (variation of Hodge structure, Calabi–Yau manifolds and their rational curves, and naturally, mirror symmetry) do not necessarily involve any very intimate logical connection.

In this introduction I propose nevertheless to give a synthesis of the subject intended to orient the book around the subject of mirror symmetry and also to show that the mathematical papers to which it has given rise, which are described in this book, interesting though they are intrinsically, are far from providing a justification as tangible as that proposed by physicists in the language of field theory (and unfortunately on the basis of quantum formalism and Feynman integrals, which seem incapable of being rigorously justified).

A *Calabi–Yau manifold* is a compact complex Kähler manifold X having trivial canonical bundle, that is, possessing a holomorphic form η that never vanishes, belonging to $H^0(X, \overset{n}{\wedge}\Omega_X^n)$, where Ω_X is the holomorphic cotangent bundle of X and where $n = \dim X$.

Throughout the following we assume that

$$H^2(\mathcal{O}_X) = \{0\},$$

even though some interesting studies have been conducted in the case of K3 surfaces (that is, simply connected Calabi–Yau twofolds), for which this assumption does not hold. Under this condition the Kähler cone of X is open in $H^2(X, \mathbb{R})$, and it is possible to introduce the "complexified Kähler cone" of this manifold, which is one of the aspects of the moduli space used by physicists. This cone is the open set

$$K(X) \subset H^2(X, \mathbb{C})/2i\pi H^2(X, \mathbb{Z})$$

defined by the condition

$$\omega \in K(X) \Longleftrightarrow \operatorname{Re}\omega \text{ is a Kähler class.}$$

Mirror symmetry consists essentially of the existence of a mirror family $\{X'\}$ of Calabi–Yau manifolds of the same dimension such that $K(X)$ uniformizes (that is, is a covering of) the moduli space Def X of deformations of the complex structure of X' and $K(X')$ uniformizes that of X. The exact nature of this covering is not completely understood, but it should be provided by a partial "marking" of the cohomology of X' (resp. X).

The elliptic curves ($n = 1$) provide the simplest example of this phenomenon: the complexified Kähler cone $K(E)$ (which is independent of the complex structure of E) can be canonically identified, through integration over E, with the set

$$\{\lambda \in \mathbb{C}/2i\pi\mathbb{Z};\ \operatorname{Re}\lambda > 0\}.$$

On the other hand, the set of marked complex structures on E can be identified by the period map with the set

$$\mathcal{H} = \{\tau \in \mathbb{C} \big|\ \operatorname{Im}\tau > 0\}$$

and the translations by $u \in \mathbb{C}$ on this set correspond simply to the action of upper-triangular integral matrices with diagonal equal to the identity on the set of markings of an elliptic curve E (that is to say, in this case, the symplectic isomorphisms $H^1(E,\mathbb{Z}) \cong \mathbb{Z}^2$ endowed with the standard symplectic form). The map

$$K(E) \longrightarrow \mathcal{H}/\mathbb{Z},$$
$$\lambda \mapsto \tau = -\frac{\lambda}{2i\pi} \ \operatorname{Mod}\mathbb{Z}$$

thus gives a uniformization of the kind predicted by mirror symmetry.

In higher dimensions a new phenomenon appears: the family $\{X'\}$ is generally different from the family $\{X\}$; for the underlying topological manifolds are different simply because their Betti numbers are different. However, the Hodge numbers of X', that is, the numbers

$$h^{p,q}(X') := \dim H^{p,q}(X') =: \dim H^q(X', \Omega^p_{X'}),$$

can be derived from the Hodge numbers of X as follows.

At a point $\omega \in K(X)$ having image X' the uniformization

$$K(X) \longrightarrow \text{ moduli space of } X'$$

induces an isomorphism

$$H^1(\Omega_X) \cong H^1(T_{X'})$$

on the level of the tangent spaces, where we have used the natural identifications

$$T_{K(X),\omega} \cong H^2(X,\mathbb{C}) \cong H^1(\Omega_X),$$

the second of which is a consequence of the assumption $H^2(\mathcal{O}_X) = \{0\}$.

More generally, as follows from the "construction" of mirror symmetry by physicists, for every p and q we should have isomorphisms

$$H^q(\Omega^p_X) \cong H^q(\textstyle\bigwedge^p T_{X'}).$$

Finally, the choice of a holomorphic n-form $\eta \in H^0(K_X)$ (where the form η is unique up to a constant multiple) determines isomorphisms given by the inner product

$$H^q(\overset{p}{\wedge} T_{X'}) \cong H^q(\Omega^{n-p}_{X'}),$$

which, when composed with the preceding isomorphisms, provide non-canonical isomorphisms

$$H^q(\Omega^p_X) \cong H^q(\Omega^{n-p}_{X'})$$

and hence a series of equalities:

$$h^{p,q}(X) = h^{n-p,q}(X').$$

We recall finally that according to Hodge theory the following direct-sum decomposition holds for each k:

$$H^k(X) = \bigoplus_{p+q=k} H^{p,q}(X).$$

This decomposition yields the relation between the Hodge numbers and the Betti numbers of X:

$$b_k(X) = \sum_{p+q=k} h^{p,q}(X).$$

When $n = 3$, comparison of the Hodge numbers of X and X' yields a comparison of the Betti numbers. Indeed:

- the assumption $H^2(\mathcal{O}) = \{0\}$ implies $b_2 = h^{1,1}$;
- on the other hand we have $h^{3,0} = 1$ and $h^{2,1} = h^{1,2}$ (since $H^1(\Omega^2_X)$ and $H^2(\Omega_X)$ are duals of each other); hence $b_3 = 2 + 2h^{2,1}$.

Finally, the relation $H^2(\mathcal{O}) = \{0\}$ is equivalent by Serre duality to the relation

$$H^1(K_X) = H^1(\mathcal{O}_X) = \{0\}$$

and hence to the relation $b_1(X) = 0$, so that we have

$$\begin{cases} b_1(X') = 0, \\ b_2(X) = \frac{1}{2}\bigl(b_3(X) - 2\bigr) = b_4(X'), \\ b_3(X') = 2 + 2b_2(X). \end{cases}$$

The constructions of mirror families available at present reduce essentially to projective algebraic geometry, the most general one being that of Batyrev, which involves partial desingularizations of hypersurfaces with trivial canonical bundle on toric Fano varieties. The latter are compactifications of $(\mathbb{C}^*)^{n+1}$ with ample anti-canonical bundle to which the natural action of $(\mathbb{C}^*)^{n+1}$ on itself can be extended.

Batyrev has shown that these manifolds are in one-to-one correspondence with convex polyhedra with integer vertices in $\mathbb{R}^{n+1}$ having 0 as their only interior lattice point, and such that the dual polyhedron also has integer vertices (the so-called *reflexive property*). The mirror family is then obtained by partial desingularization—unfortunately the desingularization procedure is generally not unique—of the family of hypersurfaces with trivial canonical bundle on the toric Fano variety associated with the dual polyhedron.

The best-known example of such a construction is that of the physicists Candelas, de la Ossa, Green, and Parkes in the outstanding paper [**43**], which is admirably expounded in the language of mathematicians by Morrison [**53**]. One considers the family of Calabi–Yau threefolds given by smooth hypersurfaces of degree 5 in $\mathbb{P}^4$; such a hypersurface has the following Hodge numbers:

$$h^{1,1} = 1, \quad h^{2,1} = 101.$$

The mirror family must therefore have the Hodge numbers

$$h^{1,1} = 101, \quad h^{2,1} = 1,$$

and the number of parameters for the deformations of the complex structure on this family must be 1.

This family is the following: consider the quintic polynomials (depending on a complex parameter $\lambda \in \mathbb{C}$) of the form

$$F_\lambda = \sum_{i=0}^{4} X_i^5 + \lambda X_0 \cdots X_4.$$

Each polynomial F_λ is invariant under the group

$$G = (\mathbb{Z}/5\mathbb{Z})^5/\mathrm{diag},$$

which acts on $\mathbb{P}^4$ by multiplying coordinates by a fifth root of unity.

The subgroup $H \subset G$ defined by the condition

$$(\alpha_0, \ldots, \alpha_4) \in H \Leftrightarrow \sum_i \alpha_i = 0 \quad \text{in} \quad \mathbb{Z}/5\mathbb{Z}$$

acts on $X_\lambda := \operatorname{div} F_\lambda$, and the action induced on $H^{3,0}(X_\lambda)$ is trivial.

It can thus be shown that the quotient X_λ/H admits a natural desingularization which is a Calabi–Yau desingularization. The family $\{\widetilde{X_\lambda/H}\}$ of dimension 1 is the required mirror family.

Morrison has expounded the calculus introduced in [**43**] from the point of view of Hodge theory; the essential ingredients are the following. A canonical coordinate at infinity is sought on a curve with coordinate λ (actually λ^5). In general such natural coordinates exist on the moduli space of a Calabi–Yau manifold (or rather the Kuranishi family, which is smooth) and depend on the choice of a partial marking of the cohomology. The point is that the monodromy around infinity provides such a marking naturally.

The logarithm t of the coordinate q thereby produced is then assumed to coincide through the mirror map with the natural coordinate existing on the complexified Kähler cone of the original family. The second point is that the same partial marking of the cohomology makes it possible likewise to trivialize the bundle $\mathcal{H}^{3,0}$ of rank 1, whose fiber at the point λ is the vector space $H^{3,0}(\widetilde{X_\lambda/H})$. Thus we have a function of q given by the value of the "Yukawa couplings" (that is, a cubic form on the tangent space to the family), normalized by the trivializing section of the bundle $\mathcal{H}^{3,0}$ over the field of logarithmic vectors $q\partial/\partial q$. The power series expansion is explicitly calculable and can be derived immediately from that of certain solutions of the Picard–Fuchs equation from the family $\{X_\lambda\}$.

The extraordinary mathematical novelty of this article thus lies in the identification of this series $\psi(q)$, where $q = e^t$, with the series

$$5 + \sum_{d>0} N(d) \frac{e^{td}}{1 - e^{td}}$$

where $N(d)$ is the number of immersed rational curves of degree d on a general quintic of $\mathbb{P}^4$. (This number should be finite according to a conjecture of Clemens.)

At the time these notes were written the prediction thereby obtained for the $N(d)$ had been verified for $d \leq 4$, which is already quite remarkable given the astronomical allure of these numbers. We note on the other hand that, despite the progress made by Kontsevich [**51**] on the problem of evaluating the $N(d)$, there were not at the time any methods making it possible to calculate these numbers except one by one.

A major achievement in the story of mirror symmetry is the work [**G**], where Givental proves, among other things, the correctness of these predictions.

The identification of these two series is a consequence of the "physical" construction of mirror symmetry. Assigning a complex structure on X and a complexified Kähler parameter $\omega = \alpha + i\beta$ determines via Yau's theorem a Kähler–Einstein metric of Kähler class α, while to β there corresponds a class of 2-forms that are closed modulo exact forms and forms that are an integral multiple of $2i\pi$ on every cycle of X of dimension 2. These assignments make it possible to construct an "$N = 2$-supersymmetric" σ-model given by an action $S(\phi, \psi)$ where ϕ is a mapping of a Riemann surface Σ onto X, and ψ is a section of $\phi^*(T_x) \otimes S$, S being the spinor bundle corresponding to the choice of a Spin structure on Σ.

The dependence of this action relative to the choice of a form $\tilde{\beta}$ representing β hardly affects the theory, since $\tilde{\beta}$ contributes to S only in the term $\int_\Sigma \phi^*(\tilde{\beta})$, so that a different choice $\widetilde{\beta'}$ modifies the action by a "boundary term" and a term that assumes as values multiples of $2i\pi$ on surfaces without boundary. The boundary terms do not contribute to the Euler–Lagrange equations describing the critical points of S, and on the other hand only the exponential of the action appears, for example, in the calculation of correlation functions.

The action $S(\phi, \psi)$ is invariant (modulo a boundary term) by an infinite-dimensional Lie superalgebra of infinitesimal transformations, the "Virasoro $N = 2$-superalgebra," whose odd part, made up of "supersymmetric transformations" is better understood if one represents $S(\phi, \psi)$ as an expansion in components of an action $S(\Phi)$ associated with superdifferentiable maps Φ of a Riemann super-surface Σ on X. The even part of this Lie superalgebra is made up of two copies of the Virasoro algebra and reflects the conformal invariance of the action $S(\phi, \psi)$. The $N = 2$-supersymmetry of this action is a consequence of the fact that the metric is a Kähler metric.

The physicists wish to represent by quantification certain functionals on the space of classical solutions of the Euler equations, called "observables", on a Hilbert space. The construction of such a representation would be equivalent to assigning the "correlation functions" of the theory, that is to say, the Feynman integrals

$$\langle \mathcal{O}_1(p_1) \cdots \mathcal{O}_r(p_r) \rangle = \int_{\Sigma, \phi, \psi} \prod_i \mathcal{O}_i\big((\phi, \psi)(p_i)\big) e^{-S(\phi, \psi)}\, d\Sigma\, d\phi\, d\psi,$$

where p_i are fixed points of Σ and the $\mathcal{O}_i$ are differential forms on X, the integral over Σ denoting the integral over the complex structures of Σ up to isomorphism.

Physicists have arguments suggesting that the invariance of the action $S(\phi, \psi)$ with respect to $N = 2$-supersymmetry may be preserved at the quantum stage (that is, a central extension of the superalgebra is represented on the Hilbert space $\mathcal{H}$ in such a way that its action on observables coincides with the operator bracket on $\mathcal{H}$) precisely when the metric on X is a Kähler–Einstein metric.

This suggests associating with (X, ω) a representation of the Virasoro $N = 2$-superalgebra having "central charge" $c = 3n$. According to Segal [**36**], one may regard this representation as the infinitesimal version of a "theory of $N = 2$-superconformal fields" associated with (X, ω). According to Gepner, this correspondence should be bijective (provided one considers only integer "$U(1)$ charge" representations). Mirror symmetry would be the correction that needs to be added to that statement and would correspond to the following phenomenon: the $N = 2$-superalgebra admits four series of generators

$$G_r^+, \quad G_r^-, \quad \overline{G}_r^+, \quad \overline{G}_r^-, \quad r \in \mathbb{Z} + \frac{1}{2},$$

whose geometric significance depends on the interpretation of the superalgebra in terms of supersymmetric transformations: it turns out that one can construct an involution on the superalgebra acting on the odd generators as follows:

$$G_r^+ \mapsto G_r^-, \quad G_r^- \mapsto G_r^+,$$

$$\overline{G}_r^+ \mapsto \overline{G}_r^+, \quad \overline{G}_r^- \mapsto \overline{G}_r^-,$$

compatible with the Lie superbracket.

This involution has no geometric interpretation: the idea is that the representation obtained by composing the initial representation with this involution is the representation associated with the mirror (X', ω'), or again that one has the same representation with a different labeling of the generators, which must be interpreted geometrically by passing to the mirror. This yields formally the comparison of the Dolbeault cohomologies of X and X'. Indeed, following Witten [**38**], one can identify

$$\bigoplus_{p,q \geq 0} H^q(X, \overset{p}{\wedge} T_X)$$

with the subspace of the Hilbert space $\mathcal{H}$ formed of "chiral-chiral" states, that is to say, those annihilated by the generators

$$G_r^+, \quad \overline{G}_r^+, \quad \big(r \geq -\frac{1}{2}\big) \quad \text{and} \quad G_r^-, \quad \overline{G}_r^- \quad \big(r \geq \frac{1}{2}\big),$$

the bigrading (p, q) on the second space being furnished by the coupling of the eigenvalues of the operators J_0 and $\overline{J}_0$, where for $m \in \mathbb{Z}$ the operators J_m and $\overline{J}_m$ form a series of even generators of the superalgebra, called the *current* $U(1)$, on which the involution acts by

$$J_m \mapsto -J_m, \quad \overline{J}_m \mapsto \overline{J}_m.$$

Similarly,

$$\bigoplus_{p,q \geq 0} H^q(X, \overset{p}{\wedge} \Omega_X)$$

can be identified with the "antichiral-chiral" states, that is, annihilated by

$$G_r^-, \quad \overline{G}_r^+, \quad \big(r \geq -\frac{1}{2}\big) \quad \text{and} \quad G_r^+, \quad \overline{G}_r^- \quad \big(r \geq \frac{1}{2}\big),$$

the bigrading $(-p, q)$ on the second space being furnished by the coupling of the eigenvalues of the operators J_0, $\overline{J}_0$. By definition the chiral-chiral states of a theory are the antichiral-chiral states of the theory obtained by composition with the involution that interchanges G^+ and G^-, and the bigrading undergoes simply the interchange $(p,q) \mapsto (-p,q)$, corresponding to the change of sign of J_0. The preceding thus provides a series of isomorphisms

$$H^q(\Omega_X^p) \cong H^q(\overset{p}{\wedge} T_{X'})$$

for the mirror (X', ω') of (X, ω).

Finally the assumptions of the theory of conformal fields (and more particularly the state/operator field correspondence) make it possible to construct a graded product on the space of chiral-chiral (resp. antichiral-chiral) states and in particular since this space is of rank 1 in bidegree (n,n) (resp. $(-n,n)$), a homogeneous form of degree n on its component of bidegree $(1,1)$ (resp. $(-1,1)$, which is isomorphic to $H^1(T_X)$ (resp. $H^1(\Omega_X)$).

The interpretation of these forms (physical Yukawa couplings) in terms of correlation functions of the σ-model determined by (X,ω), and the asymptotic expansion of Feynman integrals then made it possible for Witten [**38**] to describe the couplings obtained over $H^1(\Omega_X)$ and $H^1(T_X)$ respectively, at least for the case $n = 3$.

The former, denoted Y^ω is independent of the complex structure of X and depends instead on the parameter ω; in contrast, the latter, denoted Y^η, depends only on the complex structure of X and the choice of the holomorphic form $\eta \in H^{3,0}(X)$, whose square reflects the choice of the isomorphism $H^3(\overset{3}{\wedge} T_X) \cong \mathbb{C}$. Witten gives the following descriptions:

- the form Y^η can be identified with the composition

$$S^3 H^1(T_X) \longrightarrow H^3(\overset{3}{\wedge} T_X) \overset{\eta^2}{\cong} \mathbb{C};$$

- the form Y^ω is given by the formula

$$Y^\omega(\gamma) = \int_X \gamma^3 + \sum_{0 \neq A \in H_2(X,\mathbb{Z})} N(A) \exp\Big(-\int_A \omega\Big)\Big(\int_A \gamma\Big)^3$$

where $N(A)$ is a rational number that is an adequate substitute for the number of rational curves of class A in X, and is obtained by an integration over the set of rational curves of class A when the latter is not discrete.

Just as the mirror couples (X,ω) and (X',ω') have by definition the same associated conformal theories, their Yukawa couplings will be identified via the isomorphisms

$$H^1(T_X) \cong H^1(\Omega_{X'}), \quad H^1(T_{X'}) \cong H^1(\Omega_X)$$

by an adequate choice of η and η'.

It was in this way—and assuming that the natural choice of η' mentioned above is the correct one—that Candelas, de la Ossa, Green, and Parkes derived the identity of the two series and thus the value of the $N(d)$. Of course, the procedure assumes that the form of the mirror map has been determined a priori. That is done by the canonical coordinates and is the subtlest point of [**43**].

This theory gives a natural appearance to mirror symmetry that no mathematical approach has yet been able to equal.

Nevertheless, one can hardly consider it satisfactory since it rests on the hypothetical construction of the correspondence between Calabi–Yau manifolds endowed with one complexified Kähler parameter and $N = 2$-superconformal quantum field theories. It seems impossible, however, given the correctness of the predictions resulting from this approach, not to admit the existence of such a correspondence.

The problem that naturally arises mathematically, and which seems more important theoretically than mirror symmetry itself, is to give a mathematical realization of this correspondence.

Mathematical progress toward an understanding of the principle of mirror symmetry that goes beyond the construction and study of examples, resides essentially in the construction of analogous structures on the two moduli spaces $K(X)$ and $\operatorname{Def} X$ and in the formulation of mirror symmetry in terms of identification of these structures; the structures are stated and described in terms of variations of Hodge structure, which are the most subtle mathematical object associated with a deformation of complex structure.

The first analogy between these two moduli spaces is the following. We recall that $K(X)$ is open in $H^2(X, \mathbb{C})/2i\pi H^2(X, \mathbb{Z})$, and thus admits a natural flat structure, that is, the assignment of a local coordinate system, defined up to affine transformations. A first easy result is the existence of such a structure on $\operatorname{Def} X$ depending on a partial marking of the cohomology $H^n(X, \mathbb{Z})$, and given essentially by the periods of a generator of $H^{n,0}(X)$ on certain homology classes of X, which give coordinates on $\operatorname{Def} X$. The mirror map between $K(X)$ and $\operatorname{Def} X$ is thus practically determined by the condition of compatibility with these flat structures.

As mentioned above, for a Kähler manifold X there exists a direct-sum decomposition of each of the cohomology groups $H^k(X, \mathbb{C}) \cong H^k(X, \mathbb{C}) \otimes \mathbb{C}$:

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

provided by the Hodge theory and satisfying a certain number of conditions. For example, $H^{p,q}(X)$ is the complex conjugate of $H^{q,p}(X)$ and, for $k = n = \dim X$, $H^{p,q}(X)$ is orthogonal to $H^{p',q'}(X)$ relative to the intersection form $\langle\, ,\rangle$ of $H^n(X)$ for $(p', q') \neq (n-p, n-q)$. If the integer k is fixed, the local (or marked) period map of X associates with the Hodge decomposition of X_t on the fixed vector space $H^k(X, \mathbb{C})$ a deformation X_t of X accompanied by a $\mathcal{C}^\infty$ diffeomorphism of X_t with X inducing an isomorphism $H^k(X_t) \cong H^k(X)$.

It is pleasanter to consider the variation corresponding to the Hodge filtration

$$F^i H^k(X_t) = \bigoplus_{p \geq i} H^{p,k-p}(X_t),$$

which enjoys the following two remarkable properties:

- $F^i H^k(X_t)$ varies holomorphically as a function of t;
- the differential of the period map satisfies the Griffiths "transversality" condition

$$\frac{d}{dt}\big(F^i H^k(X_t)\big) \subset F^{i-1} H^k(X_t).$$

The period map is often one-to-one even for Calabi–Yau manifolds; it can be shown that it is immersive for $k = n$.

The problem is that, precisely because of the transversality condition, which is generally a non-trivial differential equation, the period map is almost never surjective, so that it can only rarely be used as it is to describe the moduli space of deformations of the complex structure of a manifold X.

In the case of Calabi–Yau manifolds, it would be very interesting to understand the connection between the period map and the construction of the conformal field theory proposed by the physicists. It is not clear that the latter must determine the former, but there is no doubt that a relation exists, since as was noted above, the spaces $H^{p,q}(X)$ are calculable from the point of view of the superconformal theory associated with X.

Another, more precise, connection is the fact that the Yukawa couplings on the tangent space $H^1(T_X)$ to $\operatorname{Def} X$ calculated by the physicists as correlation functions have a very simple interpretation in terms of the variation of Hodge structure: the form η that normalizes them can be continued as a section of the bundle with fiber $H^{n,0}(X_t)$ at the point $t \in \operatorname{Def} X$; for n vector fields $u_1, \dots, u_n$ on $\operatorname{Def} X$ we therefore set

$$Y^\eta(u_1, \dots, u_n) = \big\langle \eta,_{u_1}\big(\cdots\big(_{u_n}(\eta)\big)\cdots\big)\big\rangle$$

where the derivatives $_{u_i}(\eta)$ are taken by regarding η as a function with values in the constant space $H^n(X, \mathbb{C})$. The polarization and transversality conditions imply easily that this does indeed define a symmetric homogeneous form of degree n on the tangent space to $\operatorname{Def} X$ at the point X. Griffiths has in addition shown that this form can be identified with the product

$$S^n H^1(T_X) \longrightarrow H^n(\overset{n}{\wedge} T_X) \overset{\eta^2}{\cong} \mathbb{C}$$

as the Yukawa coupling of the physicists. Thus we now have the following simple result, which will be stated only for the case $n = 3$, which is the one most often considered in the present book.

In the natural coordinates z_i on $\operatorname{Def} X$ derived from a partial marking of $H^3(X, \mathbb{Z})$ and for a natural choice of $\eta(z)$, a section of the bundle ($\mathcal{H}^{3,0}$ with fiber $H^{3,0}(X_z)$ at the point z) derived from the same marking, the form Y^η depends on a potential, that is, there exists a function $F(z)$ such that

$$\frac{\partial^3 F}{\partial z_i \partial z_j \partial z_k} = Y^\eta\Big(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\Big).$$

The most remarkable fact in favor of mirror symmetry is the existence of analogous entities on the complexified Kähler cone $K(X)$. The point is the use of Gromov–Witten invariants of the symplectic manifold underlying X to construct a potential whose derivatives will make it possible to construct a variation of complex Hodge structure parameterized by $K(X)$. The Gromov–Witten invariants [**76**], [**78**], [**83**] are multilinear forms on the cohomology $H^*(X)$ of a symplectic manifold (X, ω) depending on the choice of a class $A \in H_2(X, \mathbb{Z})$

$$\Phi_{A,m} : H^*(X, \mathbb{Q})^{\otimes m} \longrightarrow \mathbb{Q}.$$

In short, $\Phi_{A,m}(\beta_1 \otimes \cdots \otimes \beta_m)$ is obtained by integrating the class

$$\mathrm{pr}_1^* \beta_\wedge \cdots \wedge \mathrm{pr}_m^* \beta_m \in H^*(X^m)$$

over the image of the evaluation map

$$\begin{aligned} \mathrm{ev}_m : W_{A,J,\nu} \times (\mathbb{P}^1)^{(m-3)} &\to X^m \\ (\phi, x_1, \dots, x_{m-3}) &\mapsto \big(\phi(0), \phi(1), \phi(\infty), \phi(x_1), \dots, \phi(x_{m-3})\big). \end{aligned}$$

Here $W_{A,J,\nu}$ is the set of maps $\phi : \mathbb{P}^1 \to X$ such that $\phi_*\big([\mathbb{P}^1]\big) = A$ satisfying the inhomogeneous Cauchy–Riemann equation

$$\bar{\partial}_J \phi = (\mathrm{Id}, \phi)^* \nu$$

for a generic choice of pseudocomplex structure J on X compatible with the symplectic form ω and having parameter on $\mathbb{P}^1 \times X$ equal to

$$\nu \in \mathcal{C}^\infty\big(\mathrm{pr}_1^*(\Omega^{0,1}_{\mathbb{P}^1}) \otimes \mathrm{pr}_2^*(T^{1,0}_{X,J})\big).$$

The Gromov–Witten potential is then the function defined on (a conjecturally non-empty open set) $H^{\mathrm{even}}(X) = \bigoplus_i H^{2i}(X, \mathbb{C})$ by the series

$$\Psi_\omega(\gamma) = \sum_{\substack{A \in H_2(X,\mathbb{Z}) \\ m \geq 3}} \frac{1}{m!} \Phi_{A,m}(\gamma^{\otimes m}) \exp\Big(-\int_A \omega\Big).$$

Since X is a Calabi–Yau threefold, the potential Ψ_ω changes with ω only by translation, modulo a quadratic term in γ:

$$\Psi_{\omega'}(\gamma) = \Psi_\omega(\gamma - \omega' + \omega) + Q(\gamma), \quad \deg Q = 2.$$

It is this potential restricted to $H^2(X, \mathbb{C})$ that mirror symmetry identifies, in canonical coordinates and modulo a quadratic function, with the potential described above, from which the Yukawa couplings of the variation of Hodge structure of the mirror X' derive when X is a Calabi–Yau variety. Indeed, if we assume $n = 3$ for simplicity, this potential makes it possible to construct a variation of complex Hodge structure parameterized by $K(X)$ (assuming of convergence) as follows. The cubic derivatives of Ψ_ω make it possible to construct at each point $\gamma \in H^{\mathrm{even}}(X)$ where the series converges a product "$\bullet_\gamma$" on $H^{\mathrm{even}}(X)$ by the formula

$$\langle e_i \bullet_\gamma e_j, e_k \rangle = \frac{\partial^3 \Psi_\omega}{\partial t_i \partial t_j \partial t_k}$$

where e_i is a basis of $H^{\mathrm{even}}(X)$ and t_i are the linear coordinates corresponding to $H^{\mathrm{even}}(X)$.

An essential property of the Gromov–Witten invariants, which holds for all symplectic manifolds and is trivially satisfied in the case of Calabi–Yau manifolds threefolds, is the associativity of the product "$\bullet_\gamma$". From this one deduces, following Dubrovin [**75**], that the connection ∇ defined on the (trivial) tangent bundle of the manifold $H^{\mathrm{even}}(X)$ by

$$\nabla_u(\sigma)(\gamma) = d_u(\sigma)(\gamma) + u \bullet_\gamma \sigma$$

(d being the trivial connection and u, σ tangent fields) is integrable.

Finally, in the case of Calabi–Yau threefolds, the product "$\bullet_\gamma$" is compatible with the grading $H^{\mathrm{even}}(X) = \bigoplus_i H^{2i}(X)$ for all γ. From this it can be deduced that the restriction of ∇ to $H^2(X, \mathbb{C})$ is an integrable connection on the trivial bundle with fiber $H^{\mathrm{even}}(X)$ satisfying the transversality condition

$$\nabla F^i H^{\mathrm{even}}(X) \subset F^{i-1} H^{\mathrm{even}}(X) \otimes \Omega_{H^2(X)}$$

for the filtration

$$F^3 H^{\mathrm{even}} = H^0(X), \qquad F^2 H^{\mathrm{even}} = H^0 \oplus H^2,$$

$$F^1 H^{\mathrm{even}} = H^0(X) \oplus H^2 \oplus H^4, \quad F^0 H^{\mathrm{even}} = H^{\mathrm{even}}(X),$$

which provides the stated complex variation of the Hodge structure.

At this stage one can formulate mirror symmetry by conjecturing that this variation of Hodge structure is that of the mirror family $\{X'\}$ via the mirror map

$$K(X) \longrightarrow \mathrm{Def}\, X'.$$

The problem is that there is no reason to believe that it has a geometric origin, that is, that it corresponds to the variation of Hodge structure of a family of manifolds parameterized by $K(X)$.

We note also that, even in the case described above of quintics of $\mathbb{P}^4$, the equality of the two potentials described above or, what amounts to the same thing, the two power series considered in [**43**], remains a conjecture.[1]

Nevertheless, Givental has given a justification of it (unfortunately not sufficient from the point of view of rigor, due to the use of infinite products, which are assumed to have a sense in the "equivariant Floer cohomology" but whose status is uncertain) very attractive theoretically and completely independent of that of the physicists. Givental notes first of all the existence of a $\mathcal{D}$-module structure on the S^1-equivariant cohomology of a symplectic manifold $\widetilde{M}$ under the following conditions: $\widetilde{M}$ is a covering with group $\mathbb{Z}$ of a symplectic manifold M endowed with a locally Hamiltonian action of S^1, on which this action becomes globally Hamiltonian, the Hamiltonian function H satisfying

$$q^*H = H + 1$$

for $q : \widetilde{M} \to \widetilde{M}$ generating the action of $\mathbb{Z}$. On $\widetilde{M}$ the symplectic form ω then extends to an equivariant form $\tilde{\omega}$ whose class satisfies

$$q^*\tilde{\omega} = \tilde{\omega} - h$$

where h is the standard (point) generator of $H^*_{S^1}$; if p is the operator of multiplication by the class of $\tilde{\omega}$ in $H^*_{S^1}(\widetilde{M})$, we thus have the relation

$$p \circ q^* - q^* \circ p = hq^*,$$

which, if we agree to treat h as a scalar, provides the required $\mathcal{D}$-module structure if we place p in correspondence with $h\partial/\partial t$ and q^* with multiplication by e^t.

Givental applies this construction to the case when M is the loop space of $\mathbb{P}^4$ and $\widetilde{M}$ its universal covering, the action of S^1 being given by the rotation of loops. In fact, he approximates this covering by the projective variety M_∞, the direct limit of the space of Laurent polynomials

$$M_k = \mathbb{P}\Big(\Big\{ \sum_{i=-k}^{k} \phi_i z^i,\ \phi_i \in \mathbb{C}^5 \Big\}\Big),$$

the action of $\mathbb{Z}$ on M_∞ being generated by multiplication by z. The S^1-equivariant cohomology of M_∞ has thus, as above, a $\mathcal{D}$-module structure.

Givental then constructs a class that is presented as an infinite product in $H^*_{S^1}(M_\infty)$, but whose geometric interpretation in terms of counting rational curves on a general quintic of $\mathbb{P}^4$ is clear. It satisfies formally, for this $\mathcal{D}$-module structure, a differential equation that is none other than the Picard–Fuchs equation of the mirror family $\{\overline{X_\lambda/H}\}$ described above. (This differential equation has meromorphic functions of t as coefficients, and its solutions are precisely the periods of the type (3,0) form on which it depends, on locally constant homology cycles. As explained above, these periods are essential in the calculation of canonical coordinates and the normalization of Yukawa couplings on which the construction of the mirror map is based.)

[1]It is now proved in [**G**].

The text ends with a description of this construction, which is rather miraculous, but undoubtedly involves very intimately the phenomenon of mirroring with its central object, the symplectic geometry of the loop space, which is also the object studied by the physicists through string theory and the σ-model.

Organization of the text

The first chapter contains some results on algebraic or Kähler geometry concerning Calabi–Yau manifolds: the existence of Kähler–Einstein metrics, the smoothness of the Kuranishi family, and some specific properties of these manifolds from the point of view of Hodge theory.

Chapter 2 is devoted to a description of mirror symmetry as it emerges from "physics." On the one hand the ideas involved in conformal field theory appear to be of great scientific import; on the other hand, as explained in this introduction, there is not at present enough mathematical intuition on the mirror phenomenon to replace the theory of the physicists.

That is the motive for this chapter, whose inspiration is somewhat special, and which is not necessary for understanding the other chapters.

The other four chapters are devoted to the mathematical aspects of mirror symmetry mentioned in the introduction.

Chapter 3 explains, following Morrison in part, the outline of the calculation performed in [**43**], introducing the necessary notions of Hodge theory: Yukawa coupling, Picard–Fuchs equations, the monodromy theorem, Griffith-style description of cohomology and Hodge filtration of a hypersurface by residues.

Chapter 4 introduces toric geometry and is devoted to the Batyrev construction. Without entering into the most recent developments, we also explain, following Danilov and Khovanskii, the calculation of Hodge numbers of hypersurfaces of toric varieties, used by Batyrev to verify in part by means of these examples the predictions concerning the comparison of the Hodge numbers of a manifold and its mirror.

The last two chapters are connected by the introduction of Floer cohomology, which, though not necessary here, seems important given that it treats quantum cohomology from the point of view of symplectic geometry of the loop spaces.

Chapter 5 is devoted to quantum cohomology: here we describe Gromov–Witten invariants and their crucial property of "splitting," and we explain the formal consequences of this property, such as associativity of the quantic product and the partial differential equations known as the WDVV equation (Witten, Dijgraaf, Verlinde, Verlinde) satisfied by the Gromov–Witten potential.

Chapter 6 begins with an introduction to Floer cohomology, and a schematic description of the proof of the comparison theorem between Floer cohomology and the usual cohomology tensorized by the Novikov ring. This section has as its main purpose to justify the infinite products of Givental by explaining their natural interpretation (in homology) as "fundamental S^1-equivariant Floer homology classes." This chapter is also an introduction to S^1-equivariant cohomology, which is the only mathematical tool used in the Givental construction.

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Note added in Translation

Since the appearance of these notes the subject has developed in several directions.

– The subject of Gromov–Witten invariants has been thoroughly investigated: they have been constructed in a purely algebraic way ([**BM**], [**BF**]); formulas for their transformations under blowing up have been found [**LR**].

– Concerning the general understanding of mirror symmetry, a new and very promising avenue has been opened in [**SYZ**]. The authors propose there to construct the mirror by dualizing special Lagrangian torus fibrations on Calabi–Yau manifolds. Some related results have been obtained in [**H**] and [**G**]. Also, Baranniker and Kontsevich [**BK**] have found a beautiful way of fomulating mirror symmetry in dimensions larger than 3, consistent with the ideas of [**12**]; they construct the "thickened" moduli space for a Calabi–Yau manifold X (which had also been discovered by Ran) and show that it has a natural Frobenius structure (see Chapter 5), namely a flat structure, a flat (complex) metric, and a cubic form on the tangent space that depends on a potential and together with the metric defines an associative algebra structure on the tangent space. Mirror symmetry can then be formulated as the identification of this moduli space with $M^*(\widehat{X}, \mathbb{C})$, where $\widehat{X}$ is the mirror, together with its Frobenius structure given by the Gromov–Witten invariants.

– Finally, the most important progress in mirror symmetry, which is not covered in these notes, is the splendid proof by Givental of the mirror symmetry conjecture for Calabi–Yau complete intersection in toric varieties. While the heuristic and mysterious computation described here (Chapter 6) remains perhaps the most striking point, Givental succeeded [**G**] in making it into a proof, the main conceptual ingredient being the introduction of the equivariant Gromov–Witten invariants, a more realistic substitute for the "equivariant Floer cohomology of the loop space."