CHAPTER I

Introduction to Integrable Systems

Integrable systems are special examples of differential systems defined by "Hamiltonian" vector fields[1] on a symplectic manifold.

In this chapter, I briefly review some useful concepts from symplectic geometry and give some examples of integrable systems. I refer readers to their preferred books on symplectic geometry (e.g., [**56, 15**] and others) and especially [**4**].

I.1. Symplectic Manifolds

I.1.a. Symplectic Vector Spaces. A symplectic vector space is a real vector space equipped with a *non-degenerate* skew-symmetric bilinear form. The prototypical (and, as we will see, essentially unique) example of a symplectic vector space is $\mathbb{R}^n \times \mathbb{R}^n$ equipped with the form

$$\omega((q,p),(q',p')) = p \cdot q' - p' \cdot q,$$

where, as usual, the dot $\cdot$ denotes the Euclidean dot product in $\mathbb{R}^n$.

In fact, as the following proposition asserts, this is the only example of a symplectic vector space.

PROPOSITION I.1.1. *Let ω be a nondegenerate, skew-symmetric bilinear form on a finite-dimensional vector space E. Then there is a basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ (called a symplectic basis) such that $\omega(e_i, f_j) = \delta_{i,j}$ and $\omega(e_i, e_j) = \omega(f_i, f_j) = 0$.*

PROOF. Since ω is nondegenerate, it is nonzero, and we can find two vectors e_1 and f_1 such that $\omega(e_1, f_1) = 1$. One verifies that ω, restricted to the (symplectic) orthogonal complement (with respect to ω) of the plane $\langle e_1, f_1 \rangle$, is a *non-degenerate* skew-symmetric form. Using the fact that a skew-symmetric bilinear form on a one-dimensional space is zero everywhere, induction on the dimension yields the desired result. □

In particular, the dimension of E is even and is the only isomorphism invariant of (E, ω). In a symplectic basis, the matrix of the bilinear form ω is

$$J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}.$$

[1]A vector field on a manifold *is* a differential equation, the differential equation whose solutions are its integral curves. See [**3**] and Footnote 1 in Appendix A.

EXAMPLE I.1.2. If $(e_1, \ldots, e_n)$ is an orthonormal basis for $\mathbb{R}^n$, then

$$((0, e_1), \ldots, (0, e_n), (e_1, 0), \ldots, (e_n, 0))$$

is a symplectic basis for $\mathbb{R}^n \times \mathbb{R}^n$.

Exercises I.1, I.2.

The Symplectic Group. This is the group of isometries of ω i.e., the linear isomorphisms g of E such that

$$\omega(gX, gY) = \omega(X, Y) \quad \text{for all } X, Y \in E.$$

In a symplectic basis, where $\omega(X, Y)$ is written as ${}^tXJY$, the symplectic group consists of matrices g such that

$${}^tgJg = J.$$

This is an algebraic subgroup of $\mathrm{GL}(2n; \mathbb{R})$, since the equations defining it are polynomial equations in the entries of g; we denote this group by $\mathrm{Sp}(2n; \mathbb{R})$. Its Lie algebra, the tangent space at the identity, consists of matrices satisfying

$${}^tAJ + JA = 0,$$

a linear equation obtained by differentiating ${}^tgJg - J = 0$ at the identity.
Exercises I.5, I.25.

Isotropic and Lagrangian Subspaces. We say that a subspace F of a symplectic space E is *isotropic* if it is contained in its symplectic orthogonal complement ($F \subset F^\circ$) of F. We say that F° is *co-isotropic*.

Since the form ω is nondegenerate, the equality

$$\dim F + \dim F^\circ = \dim E$$

holds for any subspace F. If we let $\dim E = 2n$, then $\dim F \leqslant n$ for any isotropic subspace F. There are in fact isotropic subspaces of dimension exactly n, for example the subspace generated by the "half" $(e_1, \ldots, e_n)$ of the symplectic basis. These subspaces are called *Lagrangian subspaces*.
Exercises I.3, I.4, I.6.

I.1.b. Definition of Symplectic Manifold. We now try to understand what the definition a symplectic *manifold* W ought to be. First of all, the tangent space T_xW at each x should be equipped with a skew-symmetric bilinear form ω_x such that all of these forms together define a a differential 2-form ω . We also want each ω_x to be non-degenerate. If the (necessarily even) dimension of W is $2n$, we can express this as[2]

$$\wedge^n \omega_x \neq 0 \in \bigwedge^{2n} T_x^\star W$$

for all x. In other words, the $2n$-form $\omega^{\wedge n}$ must be a volume form on W. In particular, there must be a volume form defined on W which implies that W must be an orientable manifold.

But this definition does not yet suffice. We also want the calculus arising from the form ω on W to be locally the same as the usual one on $\mathbb{R}^n \times \mathbb{R}^n$ arising from

[2]See Exercise I.2 if necessary.

the "constant" form (as described at the beginning of Section I.1.a). We write this form as a differential form as follows: let $X = (q, p)$ and $X' = (q', p')$ be vectors in $\mathbb{R}^n \times \mathbb{R}^n$, then we have

$$\omega(X, X') = p \cdot q' - p' \cdot q.$$

In other words,

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i = d\Big(\sum_{i=1}^{n} p_i dq_i\Big),$$

and, in particular, ω is an *exact* form. We could require a symplectic form to be exact. Unfortunately, however, this would rule out compact manifolds as symplectic manifolds. This is because of the following result.

PROPOSITION I.1.3. *No 2-form defined on a compact manifold W is both nondegenerate and exact.*

PROOF. We have noted that a form ω is nondegenerate if and only if $\omega^{\wedge n}$ is a volume form. Since W is compact, it follows from de Rham cohomology that no volume form on W is exact[3]. Now, if ω were exact, then we would have $\omega = d\alpha$, but then the volume form $\omega^{\wedge n}$ would be exact as well: $\omega^{\wedge n} = d(\alpha \wedge \omega^{\wedge(n-1)})$. □

In fact, if we want to compute *locally* as in $\mathbb{R}^n \times \mathbb{R}^n$, then what we need is a condition of local exactness, that is: ω is *closed*. This leads us to the good definition[4].

DEFINITION I.1.4. A *symplectic manifold* is a pair (W, ω) where W is a manifold and ω is a closed, nondegenerate 2-form. Such a form ω is said to be *symplectic*.

Of course, we will often abuse notation by simply calling W a symplectic manifold, without mentioning the form ω when there is no cause for confusion.

I.1.c. Examples of Symplectic Manifolds. The space $\mathbb{R}^n \times \mathbb{R}^n$ with the form $\sum dp_i \wedge dq_i$ is clearly a symplectic manifold.

Cotangent Bundles. Let V be a manifold, and consider the total space of its cotangent bundle with the projection

$$\pi : T^\star V \longrightarrow V.$$

On $T^\star V$, there is a *canonical* differential 1-form α, called the *Liouville form* (see Exercise I.7), defined by

$$\alpha_{(x,\varphi)}(X) = \varphi(T_{(x,\varphi)}\pi(X)),$$

where $x \in V$, $\varphi \in T^\star_x V$, and $X \in T_{(x,\varphi)}(T^\star V)$. It is easy to check that $\omega = d\alpha$ is nondegenerate (and even easier to check that it is closed!). If $(q_1, \ldots, q_n)$ are local coordinates on V and $(p_1, \ldots, p_n)$ are the "dual" coordinates, then $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ is a system of local coordinates such that $\alpha = \sum p_i dq_i$ and $\omega = \sum dp_i \wedge dq_i$. We often say that ω is the "canonical symplectic form" on the

[3]For a basic knowledge of de Rham cohomology, see, for example, [**54**] or Volume I, Chapter VII or [**75**].

[4]See also Footnote 20.

cotangent bundle[5]. The example above of $\mathbb{R}^n \times \mathbb{R}^n$ can be considered as the special case where $V = \mathbb{R}^n$.

The Space of Lines. The space $\mathcal{D}_n$ of oriented (affine) lines in $\mathbb{R}^{n+1}$ is a symplectic manifold of dimension $2n$. It can be identified with

$$\left\{(p,u) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|p\|^2 = 1 \text{ and } p \cdot u = 0\right\}$$

(see Figure 1), which is symplectic as a submanifold of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ or as the (co)tangent bundle of the sphere S^n.

Exercises I.7, I.9, I.13.

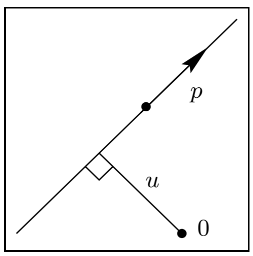

FIGURE 1. The space of lines

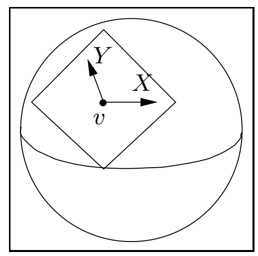

FIGURE 2. The symplectic form on the sphere

Surfaces. Any differential 2-form on a surface W is closed. Moreover, in dimension 2, saying that a 2-form is nondegenerate is equivalent to saying that it is nowhere zero. On a surface, the notion of a symplectic form thus coincides with that of a volume form. Consequently all orientable surfaces can be considered to be symplectic manifolds.

EXAMPLE I.1.5. We consider the unit sphere S^2 in $\mathbb{R}^3$. Its tangent space at a point v is the plane orthogonal to the unit vector v. Set

$$\omega_v(X,Y) = v \cdot (X \times Y)$$

(see Figure 2). This is a nondegenerate (straightforward check), skew-symmetric, bilinear form, and thus a symplectic form.

Exercise I.8.

It can be shown (this is the famous Theorem of Darboux; see Section I.5) that all symplectic manifolds are locally isomorphic to $\mathbb{R}^n \times \mathbb{R}^n$ with the form $\sum dp_i \wedge dq_i$.

Complex Symplectic Manifolds. In Chapter III, we will need *complex* symplectic manifolds. A *complex symplectic* manifold is:

- a complex analytic manifold W (the change of coordinate maps are analytic)

[5]This is another example of an exact symplectic form—on a noncompact manifold, of course.

- equipped with a nondegenerate, closed, complex 2-form ω (on each tangent space, ω is a nondegenerate complex bilinear form) that is *holomorphic*, i.e., we have in complex analytic local coordinates that

$$\omega = \sum f_{i,j} dz_i \wedge dz_j,$$

where the $f_{i,j}$ are holomorphic functions.

I.1.d. Hamiltonian Vector Fields. If $H : W \to \mathbb{R}$ is a function, the symplectic form allows us to associate to H a vector field X_H, which is a sort of gradient called the *Hamiltonian vector field* (or sometimes called the "symplectic gradient"). This is the vector field determined by the relation

$$\omega_x(Y, X_H(x)) = (dH)_x(Y) \quad \text{for all } Y \in T_xW$$

(or, equivalently, by $i_{X_H}\omega = -dH$).

Note that the vector field X_H vanishes at x if and only if x is a critical point of the function H

$$X_H(x) = 0 \Longleftrightarrow (dH)_x = 0.$$

In particular, the singularities (or zeros) of a Hamiltonian vector field are the critical points of a function.

Note also that the function H is constant on the trajectories, or integral curves, of the field X_H: since ω_x is skew-symmetric, we have $(dH)(X_H) = 0$ or $X_H \cdot H = 0$. Exercise I.11.

I.1.e. The Poisson Bracket. Let f and g be two differentiable functions on W. We define their "Poisson bracket" by the formula

$$\{f, g\} = X_f \cdot g = dg(X_f).$$

It is easy to check that

$$\{f, g\} = dg(X_f) = \omega(X_f, X_g) = -\omega(X_g, X_f) = -df(X_g) = -X_g \cdot f = -\{g, f\}.$$

This sequence of equalities shows that the Poisson bracket is skew-symmetric in f and g. Also by definition, it is a derivation (in each of its variables), that is, it satisfies the Leibniz identity

$$\{f, gh\} = \{f, g\} h + g \{f, h\}.$$

The Poisson Bracket and Hamiltonian Vector Fields. From the general formula

$$\mathcal{L}_X i_Y - i_Y \mathcal{L}_X = i_{[X,Y]}$$

and the Cartan formula

$$\mathcal{L}_X = d i_X + i_X d,$$

we see that

$$\begin{aligned} i_{[X_f, X_g]}\omega &= \mathcal{L}_{X_f} i_{X_g}\omega - i_{X_g}\mathcal{L}_{X_f}\omega \\ &= d i_{X_f} i_{X_g}\omega + i_{X_f} d i_{X_g}\omega - i_{X_g} d i_{X_f}\omega - i_{X_g} i_{X_f} d\omega \\ &= d i_{X_f} i_{X_g}\omega = d(\omega(X_g, X_f)) = d\{f, g\}, \end{aligned}$$

that is,

$$[X_f, X_g] = X_{\{f,g\}}.$$

Thus, we also have that

$$[X_f, X_g] \cdot h = \{\{f, g\}, h\}.$$

We will deduce from this that the Poisson bracket also satisfies the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

and thus it defines a Lie algebra structure on $\mathcal{C}^\infty(W)$. Moreover the map

$$\begin{aligned} \mathcal{C}^\infty(W) &\longrightarrow \mathcal{X}(W) \\ f &\longmapsto X_f \end{aligned}$$

is a morphism from the Lie algebras $\mathcal{C}^\infty(W)$ (equipped with the Poisson bracket) into the Lie algebra of vector fields (equipped with the Lie bracket of vector fields).

Proof of the Jacobi Identity. We apply the definition of the bracket of vector fields,

$$[X_f, X_g] \cdot h = X_f \cdot (X_g \cdot h) - X_g \cdot (X_f \cdot h),$$

and that of the Poisson bracket together with the equality shown above to obtain

$$\begin{aligned} \{\{f, g\}, h\} &= [X_f, X_g] \cdot h \\ &= X_f \cdot (X_g \cdot h) - X_g \cdot (X_f \cdot h) \\ &= X_f \cdot \{g, h\} - X_g \cdot \{f, h\} \\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\}. \end{aligned}$$

Since the Poisson bracket is skewsymmetric, this is equivalent to the Jacobi identity. □

Remark I.1.6. More generally, a *Poisson bracket* on a manifold V is a Lie algebra structure, denoted by $\{\ ,\ \}$, on the vector space $\mathcal{C}^\infty(V)$ of functions on V that satisfies the Leibniz identity. There will be some non-symplectic examples in Section IV.4.a.

Exercises I.10, I.15, I.16, I.17.

I.2. Hamiltonian Systems, Examples

A Hamiltonian system is simply a differential system associated to a Hamiltonian vector field. It is a differential equation

$$\dot{x}(t) = X_H(x(t))$$

for some function H on a symplectic manifold W. For example, if H is a function on $\mathbb{R}^n \times \mathbb{R}^n$, then the associated Hamiltonian system is the differential system[6]

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q}, \end{cases}$$

where, in abbreviated notation, $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$.

[6]See Exercise I.10 for an expression in terms of coordinates for the field X_H.

Examples of Hamiltonian systems that I present in this book include pendulums, solids (tops), geodesic flows (which get special treatment in Section I.4), and some other academic examples.

I.2.a. The Harmonic Oscillator. A harmonic oscillator is a differential system describing, for example, the motion of a spring (without damping) or an RLC electrical circuit without resistor ($R = 0$). The corresponding differential equation is $\ddot{q} = -aq$, for some real $a > 0$.

In this case, the symplectic manifold is $\mathbb{R} \times \mathbb{R}$, and we assume a to be 1 (see Exercise I.12 for the general case). The Hamiltonian is then

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2,$$

and the differential system is simply

$$\dot{q} = p, \quad \dot{p} = -q.$$

This is of course equivalent to the differential equation $\ddot{q} = -q$, and its solutions are given by $q = A\cos(t - t_0)$, $p = -A\sin(t - t_0)$. The supports of the trajectories are the level sets of H, i.e., the circles centered at the origin.

We can construct Hamiltonian systems of higher dimension by coupling several harmonic oscillators, that is considering Hamiltonians on $\mathbb{R}^{2n}$ of the form

$$H(q_1, \ldots, q_n, p_1, \ldots, p_n) = \frac{1}{2}\sum p_i^2 + \sum a_i q_i^2.$$

Exercise I.12.

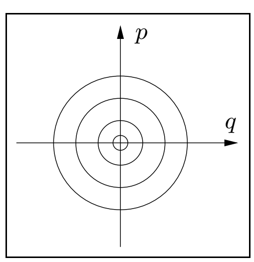

FIGURE 3. The harmonic oscillator

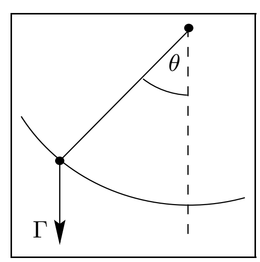

FIGURE 4. The simple pendulum

I.2.b. Rotations on the Sphere. As we saw in Example I.1.5, the unit sphere S^2 in the Euclidean space $\mathbb{R}^3$ can be equipped with a form ω defined by

$$\omega_v(X, Y) = v \cdot (X \times Y).$$

We consider the Hamiltonian $H(x, y, z) = z$. We call e_3 the "third" basis vector, so that $H(v) = v \cdot e_3$. The Hamiltonian vector field is the vector field X tangent to S^2 and satisfying

$$v \cdot (X \times Y) = -(dH)_v(Y) = -Y \cdot e_3$$

for all $Y \in T_v S^2 = v^\perp$. In other words, we have

$$Y \cdot (v \times X - e_3) = 0$$

for all $Y \in v^{\perp}$. This condition is thus equivalent to the fact that $v \times X - e_3$ is collinear to v, which implies $X_H(v) = v \times e_3$. The trajectories of X_H are the circular intersections of the sphere with horizontal planes.

I.2.c. The Simple Pendulum. The simple pendulum is the mechanical system consisting of a point mass attached to the end of a massless inelastic cord for which the other end is fixed, moving in a constant gravitational field (see Figure 4).

We assume here that the mass moves in a vertical plane. We denote by θ the angle between the cord and the vertical axis (direction of gravity). The differential equation describing the motion of the point mass is then $\ddot{\theta} = -\sin\theta$.

The phase space is $S^1 \times \mathbb{R}$. Using a real variable q whose reduction modulo 2π is θ and the dual variable $p = \dot{q}$, we see that this differential equation is equivalent to the system

$$\dot{q} = p, \qquad \dot{p} = -\sin q.$$

This is the Hamiltonian system associated to the function $H = \frac{1}{2}p^2 - \cos q = \frac{1}{2}p^2 - \cos\theta$, the total energy of the simple pendulum.

Exercise I.18.

We now consider some examples "with two degrees of freedom", i.e., on four-dimensional symplectic manifolds.

I.2.d. The Hénon–Heiles System. The symplectic manifold is $\mathbb{R}^2 \times \mathbb{R}^2$, and the Hamiltonian is the function

$$H = \frac{1}{2}(p_1^2 + p_2^2) - q_2^2(A + q_1) - \frac{\lambda}{3}q_1^3.$$

The vector field X_H is

$$X_H(q_1, q_2, p_1, p_2) = (p_1, p_2, q_2^2 + \lambda q_1^2, 2q_2(A + q_1)).$$

Let E be the (symplectic) subspace defined by $q_2 = p_2 = 0$. If $x = (q_1, 0, p_1, 0)$ is a point in E, we have

$$X_H(x) = (p_1, 0, \lambda q_1^2, 0).$$

Thus the Hamiltonian vector field X_H is *tangent* to the subspace E, and thus there are solutions to the Hamiltonian system which are contained in E. When $\lambda = 0$, these are the lines $(p_1 t + q_1^0, 0, p_1, 0)$. This "academic" example and its solutions will be utilized in Chapter III.

Exercise I.19.

I.2.e. An Anharmonic Oscillator. The symplectic manifold is still $\mathbb{R}^4$ and the Hamiltonian is

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \lambda q_2^2 + (q_1^2 + q_2^2)^2,$$

with λ a nonzero constant. The Hamiltonian vector field is

$$X_H(q_1, q_2, p_1, p_2) = (p_1, p_2, -4(q_1^2 + q_2^2)q_1, -2\lambda q_2 - 4(q_1^2 + q_2^2)q_2).$$

As in Section I.2.d, the vector field X_H is tangent to the subspace E defined by $q_2 = p_2 = 0$. The solutions contained in this plane are supported by the curves

$$\frac{1}{2}p_1^2 + q_1^4 = h, \quad q_2 = p_2 = 0.$$

I.2.f. The Spherical Pendulum. In the system called the "simple pendulum", the mass moves in a plane. Now we allow it to move in space, namely on the surface of a sphere centered at the fixed end of the cord. By a "spherical pendulum" we mean the system defined on the unit sphere in $\mathbb{R}^3$ by the Hamiltonian

$$H = \frac{1}{2} \|p\|^2 - \Gamma \cdot q,$$

where p and q are vectors in $\mathbb{R}^3$ and Γ is the vertical gravitational field.

The phase space is

$$TS^2 = \left\{(q, p) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|q\|^2 = 1 \text{ and } q \cdot p = 0\right\}$$

(which, of course, is the tangent bundle of the sphere S^2), equipped with the restriction of the symplectic form on $\mathbb{R}^3 \times \mathbb{R}^3$, which continues to be a symplectic form (see Exercise I.13 if necessary).

The Hamiltonian vector field is the unique vector field (Y, X) tangent to TS^2 satisfying

$$(dH)_{(q,p)}(\eta, \xi) = \omega((\eta, \xi), (Y, X)) = \xi \cdot Y - \eta \cdot X$$

for all vectors (η, ξ) tangent to TS^2. Now,

$$(dH)_{(q,p)}(\eta, \xi) = p \cdot \xi - \Gamma \cdot \eta,$$

and the fact that (Y, X) is tangent to TS^2 at (q, p) is expressed by the conditions $Y \cdot q = 0$ and $Y \cdot p + q \cdot X = 0$. The unique solution is $Y = p$ and $X = \Gamma - (q \cdot \Gamma + \|p\|^2)q$, and the Hamiltonian system is then

$$\begin{cases} \dot{q} = p \\ \dot{p} = \Gamma - (q \cdot \Gamma + \|p\|^2)q. \end{cases}$$

Exercises I.13, I.14.

I.2.g. The Rigid Body With A Fixed Point. Here we consider a rigid body of mass 1 (for mathematical convenience) with center of gravity G, fixed point O and in a constant gravitational field.

In writing the equations describing the motion, it is convenient to utilize a frame attached to the rigid body. The constant gravitational field is a vector Γ depending on time (in the moving frame). Also, we let M be the angular momentum It is related to the instantaneous rotation Ω of the solid by the relation $M = \mathcal{J}(\Omega)$, where $\mathcal{J}$ is a certain constant symmetric endomorphism called *the matrix of inertia* (see, for example, [**8**]).

Then the total energy of the solid is

$$H(\Gamma, M) = \underbrace{\tfrac{1}{2} M \cdot \Omega}_{\text{kinetic}} + \underbrace{\Gamma \cdot L}_{\text{potential}},$$

where L denotes the vector $\overrightarrow{GO}$.

By applying the laws of mechanics, one can show that the differential system describing the motion of the solid is

$$\begin{cases} \dot{\Gamma} = \Gamma \times \Omega \\ \dot{M} = M \times \Omega + \Gamma \times L. \end{cases}$$

Of course, the vector Γ is no longer constant; however, it remains a unit vector[7]. We also know that the quantity $\Gamma \cdot M$, the moment of the solid with respect to the vertical axis[8], is conserved. Thus the vectors Γ and M are constrained to remain on the submanifold

$$W_a = \left\{ (\Gamma, M) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|\Gamma\|^2 = 1 \text{ and } \Gamma \cdot M = a \right\}.$$

This submanifold is diffeomorphic to the tangent bundle of the sphere S^2 via the map

$$(\Gamma, M) \longmapsto (\Gamma, M - a\Gamma).$$

We can equip it with a symplectic form ω for which the differential system is the Hamiltonian system associated to the energy H. Be careful, ω is *not* the canonical symplectic structure for the tangent bundle of S^2. Instead, it is the form[9]

$$\omega_{(\Gamma,M)}((\xi, \eta), (\xi', \eta')) = (\xi \times M + \Gamma \times \eta) \cdot \xi' + (\Gamma \times \xi) \cdot \eta'.$$

Exercise I.20.

I.3. Completely Integrable Systems

I.3.a. Definition of a Completely Integrable System. The Hamiltonian H is constant along integral curves of the Hamiltonian system it defines, which we can write as

$$X_H \cdot H = 0 \text{ or } dH(X_H) = 0.$$

We say that H is a *first integral* for the system. More generally, a function $f : W \to \mathbb{R}$ that is constant along the trajectories of a vector field X is called a *first integral*. For a Hamiltonian vector field X_H, the equality $X_H \cdot f = 0$ is equivalent to $\{f, H\} = 0$.

Informally, we say that a Hamiltonian system is completely integrable if there are "as many first integrals in involution as possible." We now explain what this means formally.

- Let $f_1, \ldots, f_k$ be first integrals of the system X_H. We say that they are "in involution" if $\{f_i, f_j\} = 0$ for all i and j. Each f_i is constant along the trajectories of the Hamiltonian vector field defined by the others.
- We could also talk of a subalgebra[10] $\mathcal{A}$ of $\mathcal{C}^\infty(W)$ containing H that is Abelian in the sense that

$$f, g \in \mathcal{A} \Longrightarrow \{f, g\} = 0.$$

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[7] If the units are suitably chosen.

[8] By definition, "vertical axis" is the direction of the gravitational field.

[9] The manifold W_a is an "adjoint orbit", which yields the form in question. See, for example, [**8**].

[10] See [**77**].

- Now, we turn to what is meant by "as many as possible." At each point x in W, the subspace of T_xW generated by the Hamiltonian vector fields of the functions in $\mathcal{A}$ is isotropic

$$\omega(X_f, X_g) = \pm\{f, g\} = 0,$$

  and so its dimension is at most $n = \frac{1}{2}\dim W$. Thus for each x in an dense open subset of W, we require that this subspace have maximal dimension n.
- Finally, note that the vectors X_{f_i} are independent at x if and only if the linear forms $(df_i)_x$ are independent.

DEFINITION I.3.1. The Hamiltonian vector field X_H on a symplectic manifold W of dimension $2n$ is *completely integrable* if it has n independent first integrals in involution.

EXAMPLE I.3.2. On $\mathbb{R}^n \times \mathbb{R}^n$ with the standard symplectic form, any Hamiltonian that depends only on the coordinates p_i,

$$H = H(p_1, \dots, p_n),$$

is completely integrable: the functions p_i are independent first integrals in involution.

Any Hamiltonian system on a surface is completely integrable. Similarly, a Hamiltonian system on a symplectic manifold of dimension 4 is integrable if and only if it has a "second first integral" that is functionally independent of the energy first integral H.

I.3.b. The Harmonic Oscillator and the Simple Pendulum. See Sections I.2.a and I.2.c for a description of these systems, which are Hamiltonian systems on manifolds of dimension 2 and thus (automatically) integrable.

I.3.c. The Hénon–Heiles System. There exists a second first integral for this system (introduced in Section I.2.d) for certain values of the parameters A and λ. In particular, this is the case for $\lambda = 6$.

Exercise I.21.

I.3.d. The Anharmonic Oscillator. It is easy to check that the function

$$K = \frac{1}{2}p_1^2 - \frac{1}{\lambda}(q_1p_2 - q_2p_1)^2 + q_1^2(q_1^2 + q_2^2)$$

is a first integral for the Hamiltonian system considered in Section I.2.e and that is independent of H.

I.3.e. The Spherical Pendulum. We retain the notation introduced in Section I.2.f. The system is invariant under rotations about the vertical axis, which allows us to find easily a second first integral, i.e., the moment about this axis

$$K(q, p) = (q \times p) \cdot \Gamma.$$

One can show directly that $\{H, K\} = 0$ (see Exercise I.22), but it is more enlightening to give a geometric argument. The Hamiltonian vector field associated to K is

$$X_K(q,p) = (-q \times \Gamma, -p \times \Gamma),$$

whose flow is that of rotations about the vertical axis,

$$\varphi_t(q,p) = (R_t q, R_t p)$$

(this is what it means for K to be the moment about this axis). This flow preserves the submanifold TS^2, since

$$\|R_t q\|^2 = \|q\|^2 = 1 \text{ and } R_t p \cdot R_t p = q \cdot p = 0.$$

Similarly, it fixes H because it leaves the vertical vector Γ invariant. Then, by differentiating the relation $H(\varphi_t(x)) = H(x)$ at $t = 0$, we see that the Poisson bracket of H and K is zero. Thus the spherical pendulum is a completely integrable system.

Exercise I.22.

I.3.f. A Rigid Body: the Euler–Poinsot Case. This is the rigid body described in Section I.2.g when the fixed point is the center of gravity ($G = 0, L = 0$). It is easy to check that $K(M) = \|M\|^2$ is a first integral for this system and that is independent of the energy integral.

Remark I.3.3. In this case, the Hamiltonian system

$$\begin{cases} \dot{\Gamma} = \Gamma \times \Omega \\ \dot{M} = M \times \Omega \end{cases}$$

is reduced to the system $\dot{M} = M \times \Omega$ (with $M = \mathcal{J}(\Omega)$), which describes the motion of a "free" body (without gravity).

I.3.g. The Top. We continue to consider a rigid body, but this time one with an axis of rotation OG. This means that there is an orthonormal frame where the last vector is $L = \overrightarrow{GO}$ and the matrix of inertia is the diagonal matrix (ℓ, ℓ, m). We choose the units so that $\ell = 1$. Then the condition is that $M - \Omega$ is collinear to L.

This case, which Lagrange studied in 1788, is that of a "top". The tops one plays with do not really have a fixed point, but instead are constrained to having one end O of their axis of rotation in a given horizontal plane. For more about this system, see the classic work by Klein and Sommerfeld [**49**]. In this book, however, we consider "Lagrange's top" to be a reasonable approximation to the toy one usually plays with.

In this case, it is clear that the moment about the axis of rotation is a first integral.

Exercise I.20.

I.3.h. The Kowalevski Top. We continue to consider a rigid body with a fixed point but this time when the matrix of inertia $\mathcal{J}$ is the diagonal matrix $\mathcal{J} = (2, 2, 1)$ in a frame where the *first* vector is collinear to L. As in the case of Lagrange's top, there is an equatorial plane, but this time it contains the axis $\overrightarrow{OG}$. In the same basis, we write

$$M = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \Omega = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}, \quad L = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}.$$

Then the function

$$K = |(p + iq)^2 + (\gamma_1 + i\gamma_2)|^2$$

is a first integral called the "Kowalevski integral" (see [**53**]).

Remark I.3.4. These three special cases are the only cases of rigid bodies with a fixed point that are integrable systems. More precisely, in the other cases, there are no additional independent meromorphic integrals. (The polynomial analog is due to Husson [**43**] and the meromorphic case is due to Ziglin [**84, 85**]. This can also be established with the help of the Morales–Ramis theorem [**62**], as in Chapter III, see [**57**].)

I.3.i. Other Examples. There are many other interesting examples of integrable systems, notably among geodesic flows, which I will address in the next section. See, for example, the list on page 13 of [**8**]. Two of the most remarkable systems that will not appear in this book are the systems studied by Goldman [**35**][11] and Hitchin [**41**].

Exercise I.26.

I.4. Geodesic Flows

In this section we consider a Riemannian manifold V. Its metric defines a function $L : TV \to \mathbb{R}$, the "Lagrangian"

$$L(q, \xi) = \frac{1}{2}\langle \xi, \xi \rangle_q,$$

where q is a point in V and ξ is a vector in $T_q V$.

The geodesics of V are the extrema of the functional

$$c \longmapsto \int \|\dot{c}(t)\|^2 \, dt$$

and are given by the Euler–Lagrange equation. We thus let $p = \partial L/\partial \xi$ (which is the linear form $\langle \xi, \cdot \rangle_q$). The Euler–Lagrange equation can now be written as: $\dot{p} = \partial L/\partial q$. The Hamiltonian[12] is then the function on $T^\star V$ defined by

$$H(q, p) = p \cdot \xi - L(q, \xi) = \langle \xi, \xi \rangle_q - \frac{1}{2}\langle \xi, \xi \rangle_q = \frac{1}{2}\langle \xi, \xi \rangle_q$$

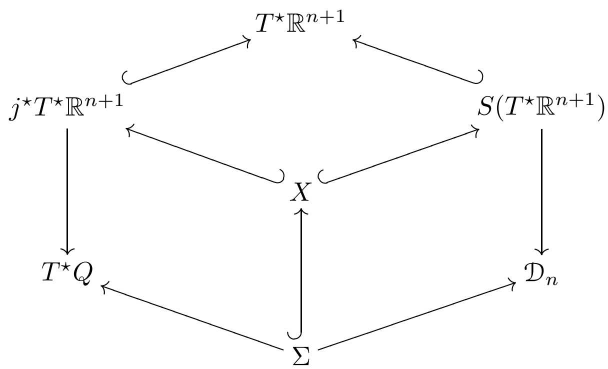
(to be written in terms of p).

[11] An overview of some integrable systems on moduli spaces can be found in [**9**].

[12] See [**2**] for its construction.

Geodesics on a Hypersurface and the Space of Lines. Let $Q \subset \mathbb{R}^{n+1}$ be a hypersurface equipped with the metric induced by the Euclidean metric. We can study the geodesic flow on Q by using a system defined on the space of lines obtained by associating to each geodesic (a curve on Q) the curve of its tangents.

The essential tool for doing this is the "Melrose" hexagonal diagram



where

- the top of the diagram is the ambient symplectic manifold, here $T^\star\mathbb{R}^{n+1}$ (as usual, (p, q) denotes a point in this space);
- the second line contains the two hypersurfaces $j^\star T^\star\mathbb{R}^{n+1}$ (defined by $q \in Q$ where $j : Q \subset \mathbb{R}^{n+1}$ is the inclusion map) and $S(T^\star\mathbb{R}^{n+1})$ (defined by $\|p\|^2 = 1$);
- the third line contains their intersection X;
- the next line contains the two spaces of characteristics (see Exercise I.9) of the two hypersurfaces, i.e., the cotangent bundle to Q and the space $\mathcal{D}_n$ of (geodesic!) lines of $\mathbb{R}^{n+1}$;
- finally at the bottom of the diagram, Σ is the unitary cotangent bundle of Q, a hypersurface in $T^\star Q$ whose space of characteristics is the space of geodesics on Q, viewed here as
$$\Sigma = \left\{(p, q) \mid \|p\|^2 = 1 \text{ and } p \in T_q^\star Q\right\}.$$
The map $\Sigma \to \mathcal{D}_n$ sends (p, q) to the line in the direction p and passing through q.

Thus we have described Σ as a hypersurface in the two symplectic manifolds $T^\star Q$ and $\mathcal{D}_n$ as well as a submanifold of codimension 3 of the ambient space $T^\star\mathbb{R}^{n+1}$, from which come all the symplectic structures that we utilize here. Hence the characteristics of Σ are the same whether we consider Σ

- as a submanifold of $\mathcal{D}_n$, the space of lines tangent to Q,
- or as the unitary bundle in $T^\star Q$—the space of characteristics is that of the geodesics on Q.

In other words, any characteristic of the space of lines tangent to Q in the space of lines consists of lines tangent to a geodesic in Q. Or, in the case of a hypersurface of

$\mathbb{R}^{n+1}$, we can consider the geodesic flow as a Hamiltonian system on the symplectic manifold $\mathcal{D}_n$ of lines.

I.4.a. Geodesics on Surfaces of Revolution. We consider a surface of revolution in the Euclidean space $\mathbb{R}^3$. We parametrize it by a cylinder $(a,b)\times\mathbb{R}/2\pi\mathbb{Z}$

$$(s,\theta)\longmapsto \big(f(s)\cos\theta, f(s)\sin\theta, g(s)\big),$$

where the meridian curve is parametrized by the arc length s. A tangent vector ξ written as

$$\xi=\sigma\frac{\partial}{\partial s}+\tau\frac{\partial}{\partial\theta}$$

has norm

$$\|\xi\|^2_{(s,\theta)}=\frac{1}{2}(\sigma^2+f(s)^2\tau^2)=L\big((s,\theta),(\sigma,\tau)\big).$$

The corresponding cotangent variables p_s and p_θ are

$$p_s=\frac{\partial L}{\partial\sigma}=\sigma \text{ and } p_\theta=\frac{\partial L}{\partial\tau}=f(s)^2\tau.$$

Thus the Hamiltonian is

$$H((s,\theta),(p_s,p_\theta))=\frac{1}{2}(\sigma^2+f(s)^2\tau^2)=\frac{1}{2}\Big(p_s^2+\frac{1}{f(s)^2}p_\theta^2\Big),$$

and the Hamiltonian system is

$$\begin{cases}\dot{s}=p_s\\ \dot{\theta}=\dfrac{1}{f(s)^2}p_\theta\\ \dot{p}_s=\dfrac{f'(s)}{f(s)^3}p_\theta^2\\ \dot{p}_\theta=0.\end{cases}$$

Since H does not depend on θ, it is clear that p_θ is a second independent first integral. This is the simplest way to describe the Clairaut integral, whose classical description[13] appears in Exercise I.23.

Exercise I.23.

REMARK I.4.1. As in the cases of the spherical pendulum and the top (see Sections I.3.e and I.3.g), the second first integral is a moment about the axis of revolution.

I.4.b. Geodesics on Quadrics. We now consider an ellipsoid in $\mathbb{R}^{n+1}$, that is, a hypersurface defined by

$$f(x)=\langle A^{-1}x,x\rangle-1=0.$$

where A is a positive definite symmetric matrix A of order $n+1$.

[13] See, for example, volume III of [**75**].

Let $t \mapsto x(t)$ be a geodesic on this hypersurface and $y(t) = \dot{x}(t)$ be its tangent vector. Saying that x is a geodesic is equivalent to saying that its acceleration $\dot{y}(t)$ has no tangent component or, in other words,

$$\dot{y}(t) = \lambda \operatorname{grad}_{x(t)} f.$$

Here $\operatorname{grad}_x f = 2A^{-1}x$ and we can easily compute λ by differentiating the relation $\langle A^{-1}x, y\rangle = 0$ (which says that y is tangent to the ellipsoid)

$$0 = \langle A^{-1}\dot{x}, y\rangle + \langle A^{-1}x, \dot{y}\rangle = \langle A^{-1}y, y\rangle + \lambda\langle A^{-1}x, A^{-1}x\rangle.$$

The differential system that yields the geodesics on the ellipsoid is thus

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\dfrac{\langle A^{-1}y, y\rangle}{\|A^{-1}x\|^2} A^{-1}x. \end{cases}$$

Integrability (the Classical Approach). Classical theorems due to Jacobi and Chasles allow us to show that this system is completely integrable[14].

I especially like the geometry of this example and will devote several pages to it. Readers pressed for time can skip it on first reading (but that would be unfortunate!).

The family of "confocal" quadrics to our ellipsoid plays a very important role. These are the quadrics Q defined by an equation

$$f_z(x) = \big\langle (A - z\operatorname{Id})^{-1}x, x\big\rangle - 1 = 0$$

(with the original ellipsoid Q_0 corresponding to the value 0 for the real parameter z). For the case of conics ($n = 1$), this is the family of conics having the same foci as the given ellipse, hence the term "confocal quadric." In higher dimensions, the remarkable characteristic of this family is that the dual family is a pencil, that is, it is *linear*.

The first result utilized here is a theorem due to Jacobi that states a generic point of the space is contained in $n+1$ quadrics of the family which are orthogonal at this point of intersection.

The second one, due to Chasles, states that a generic line of the space is tangent to n quadrics in the family and that the tangent hyperplanes to these n quadrics (at the points where the line is tangent to them) are pairwise orthogonal.

Because this is very beautiful, I give a sketch of the proofs of these results below. Already, notice that the line $\{u + tp | t \in \mathbb{R}\}$ is tangent to the quadric Q_z exactly when the quadratic equation in t

$$f_z(u + tp) = 0$$

has a double root. Concretely, here the space Σ from the Melrose diagram is the hypersurface in $\mathcal{D}_n$ defined by $\Psi_z(p, u) = 0$, where

$$\Psi_z(p,u) = \big\langle (A - z\operatorname{Id})^{-1}u, p\big\rangle^2 - \big(\langle (A - z\operatorname{Id})^{-1}p, p\rangle\big)\big(\langle (A - z\operatorname{Id})^{-1}u, u\rangle - 1\big).$$

[14] The proof is also classical and can be found, for example, in [**76**]. It was known to Jacobi [**45**]. There are more modern proofs as well, using techniques of algebraic geometry. For more about geodesics on quadrics, see, for example, [**79, 50, 65, 6**].

This is a function of z of the form

$$\Psi_z(p,u) = \frac{Q_{p,u}(z)}{\prod(\alpha_i - z)},$$

where Q is a polynomial of degree n and where we supposed that A is a diagonal with eigenvalues α_i.

The condition that the line (p,u) is tangent to the quadric Q_z is thus equivalent to the fact that z is a root of a certain polynomial $Q_{(p,u)}(z)$ of degree n whose roots, Chasles tells us, are real. Locally, we have functions

$$z_1(p,u), \ldots, z_n(p,u)$$

describing the n quadrics to which the line $\{u + tp | t \in \mathbb{R}\}$ is tangent.

If we consider the Hamiltonian system describing the geodesics on the hypersurface Q as a system on the space $\mathcal{D}_n$ of lines, the functions $z_1, \ldots, z_n$ (locally) define functions on this space, and we now show (following Jacobi and Chasles) that these functions are in involution.

Let X_i be the Hamiltonian field associated to the function z_i. It is tangent to the characteristics of the level hypersurfaces of z_i, that is, the space of lines tangent to the quadric Q_{z_i} (for a fixed value of z_i). Thanks to the Melrose diagram discussed at the beginning of Section I.4, we know that this hypersurface consists of lines tangent to a geodesic of Q_{z_i}. We also know that these lines are tangent to Q_{z_j} for $j \neq i$. Thus z_j is constant along the flow of X_i, or in other words, $\{z_i, z_j\} = 0$.

Two Theorems on Confocal Families. Let us now prove the Jacobi and Chasles theorems that we have just used.

Proof of the Jacobi Theorem. We begin by translating the statement into one in terms of the dual family. We will show that every affine hyperplane of $(\mathbb{R}^{n+1})^\star$ not passing through the origin[15] is tangent to n quadrics of the pencil and that the vectors corresponding to the points of tangency are pairwise orthogonal. The equation of quadrics of the pencil is

$$\langle (A - z\,\mathrm{Id})x, x\rangle = 1.$$

Consider the quadratic form $B(x) = \langle Ax, x\rangle - \langle x, y\rangle$ and a line ℓ passing through the eigenvector corresponding to the eigenvalue z. Thus the quadratic form $B - z\,\mathrm{Id}$ and its polar form are zero on ℓ. Let y be a point in the space $\mathbb{R}^{n+1}$. The dual hyperplane is determined by $\langle x, y\rangle = 1$; it intersects the line ℓ at a point x_0 on the quadric $\langle (A - z\,\mathrm{Id})x, x\rangle = 1$, since $(B - z\,\mathrm{Id})x_0 = 0$. Moreover, since the polar form on $B - z\,\mathrm{Id}$ is also zero on ℓ, the quadric defined by $A - z\,\mathrm{Id}$ is tangent to the hyperplane at x_0.

The points of tangency of the hyperplane dual to y with the quadrics in the pencil are the eigenvectors of the quadratic form B. Since B is a real quadratic form, it has n independent orthogonal eigenvectors. □

[15] This is the condition of being generic in the original statement.

Proof of the Chasles Theorem. We choose a line ℓ and project (orthogonally) $\mathbb{R}^{n+1}$ onto the hyperplane $\ell^\perp$. We consider the apparent contours[16] of the quadrics of the family onto $\ell^\perp$.

This family of hypersurfaces of $\ell^\perp$ is still a confocal family. We utilize the duality that transforms the apparent contour of Q on $\ell^\perp$ into the intersection of the dual quadric with $\ell^\perp$. Since the intersection of a linear family of quadrics with a hyperplane is certainly a linear family of quadrics, our family of apparent contours is indeed a confocal family.

We now apply Jacobi's theorem to the point of intersection of ℓ with $\ell^\perp$. It lies on n quadrics of the family of apparent contours which is equivalent to saying the line ℓ is tangent to n quadrics of the original confocal family. □

I will come back to this system and its integrability at length in Chapter IV.

Exercise I.24.

I.5. Appendix: The Theorem of Darboux

This is a theorem that asserts that, locally, all symplectic forms are isomorphic.

Theorem I.5.1 ([**64, 80**]). *Let ω_0 and ω_1 be two symplectic forms on W that are equal at x. Then there is a neighborhood U_0 of x in W and a map*

$$\psi : (U_0, x) \longrightarrow (W, x)$$

such that $\psi^\star\omega_1 = \omega_0$.

Remark I.5.2. The map ψ is then a local diffeomorphism since $\omega_0^{\wedge n} = \psi^\star\omega_1^{\wedge n}$ and these two $2n$-forms are volume forms.

Corollary I.5.3 (Theorem of Darboux). *Let x be a point on a symplectic manifold W with symplectic form ω. Then there exist local coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$ centered at x such that $\omega = \sum dp_i \wedge dq_i$.*

Proof of the Corollary. Let U be a neighborhood of 0 in $\mathbb{R}^{2n}$ and $f : U \to W$ a coordinate chart centered at x. We consider two symplectic forms on U:

- the form $f^\star\omega$
- the constant form $(f^\star\omega)_0$.

By definition, these two forms are equal at 0, so the Theorem asserts that they are diffeomorphic. The coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ in a symplectic basis for $(f^\star\omega)_0$, transported by ψ, have the desired property. □

Proof of the Theorem. We apply the path method of Moser (see [**64**]). Namely, for each $t \in [0, 1]$, we consider the form $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. It is clearly closed. Since ω_0 and ω_1 are equal at x, it is nondegenerate at x and thus on some neighborhood of x since the property of being nondegenerate is an open property. Then, since the interval $[0, 1]$ is compact, we can find a neighborhood of x on which all the ω_t are symplectic forms.

[16]The apparent contour of one of the quadrics is the set of critical values of the projection. It consists of the images of the points where ℓ is tangent to the quadric.

Since ω_0 and ω_1 are closed, the same is true for $\omega_0 - \omega_1$, and thus we can find a 1-form α such that $d\alpha = \omega_0 - \omega_1$ in a neighborhood of x. Let f be a function defined on a neighborhood of x such that $(df)_x = \alpha_x$. Then $(\alpha - df)_x = 0$ and $d(\alpha - df) = d\alpha = \omega_0 - \omega_1$. By replacing α by $\alpha - df$, we can thus assume that $\alpha_x = 0$.

Since the 2-form ω_t is symplectic, it defines a duality between the tangent and cotangent bundles, yielding for each t, a vector field X_t dual to α by ω_t (i.e., such that $i_{X_t}\omega_t = \alpha$). We now consider X_t as a vector field (depending on the time t) that is zero at x for every t. The flow φ_t of X_t fixes x, and thus we can find a neighborhood U of x on which φ_t is defined and satisfies $\varphi_t(U) \subset U$.

Hence we have

$$\frac{d}{dt}\left[\varphi_t^\star \omega_t\right] = \varphi_t^\star \left[\frac{d\omega_t}{dt} + \mathcal{L}_{X_t}\omega_t\right] = \varphi_t^\star \left[\omega_1 - \omega_0 + \omega_0 - \omega_1\right] = 0,$$

since $\mathcal{L}_{X_t}\omega_t = di_{X_t}\omega_t + i_{X_t}d\omega_t = d\alpha$ (from Cartan's formula and the definition of α). Thus the form $\varphi_t^\star\omega_t$ does not depend on t and equals ω_0 at $t = 0$. The desired result follows by taking $\psi = \varphi_1$. □

REMARK I.5.4. The statement analogous to Theorem I.5.1 for Riemannian metrics is definitely false. In the case of a Riemannian metric, there is a *local* invariant distinguishing a neighborhood of a point from its tangent space, namely, the curvature. We have just shown that there is *no* local invariant in symplectic geometry.

A more general theorem can be obtained by essentially the same method [**80**]. It describes the tubular neighborhoods of isotropic submanifolds of W (those on which ω is identically zero). In particular, it states that if L is an isotropic submanifold of maximal dimension (so $\dim L = \frac{1}{2}\dim W$, and L is called *Lagrangian*), a tubular neighborhood of L in W is (symplecticly) isomorphic to a neighborhood of the zero section of $T^\star L$ (with the canonical symplectic form).

Exercises

EXERCISE I.1. We consider $\mathbb{C}^n$ as a *real* vector space of dimension $2n$. Show that the (real) bilinear form

$$\omega(X, Y) = \operatorname{Im}\langle X, Y\rangle$$

where the bracket denotes the standard Hermitian form on $\mathbb{C}^n$, is alternating and nondegenerate. Compare this with the form defined on $\mathbb{R}^n \times \mathbb{R}^n$ in Section I.1.a. Construct a symplectic basis from the unitary complex basis of $\mathbb{C}^n$.

EXERCISE I.2. Let ω be a skew-symmetric bilinear form on a vector space E. Show that there exists a basis $e_1, \dots, e_r, f_1, \dots, f_r, g_1, \dots, g_r$ of E such that

$$\omega(e_i, f_j) = \delta_{i,j}, \quad \omega(e_i, e_j) = \omega(e_i, g_\ell) = \omega(f_i, g_\ell) = \omega(g_\ell, g_m) = 0$$

for all indices i, j, ℓ, and m.

Let $\dim E = 2n$. Show that $\omega^{\wedge n} \neq 0$ if and only if ω is nondegenerate.

EXERCISE I.3. Let F be a k-dimensional isotropic subspace of a symplectic vector space E. Show that there exists a symplectic basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ of E such that $(e_1, \ldots, e_k)$ is a basis of F.

EXERCISE I.4 (Symplectic Reduction). Let F be an isotropic subspace of a symplectic vector space E. Show that the symplectic form on E defines a symplectic form on the quotient space F°/F.

Let L be a Lagrangian subspace of E such that $L + F^\circ = E$. Show that $L \cap F^\circ$ *embeds*[17] as a Lagrangian subspace of F°/F.

EXERCISE I.5 (The Symplectic Group). Show that $\mathrm{Sp}(2;\mathbb{R}) = \mathrm{SL}(2;\mathbb{R})$.

Compute the dimension of the kernel of the linear map $A \mapsto {}^tAJ + JA$, and deduce that the dimension of $\mathrm{Sp}(2n;\mathbb{R})$ is $2n^2 + n$.

Given a symplectic basis of $\mathbb{R}^{2n}$, we write a $2n \times 2n$ matrix as an $n \times n$ block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

What conditions on A, B, C, and D make this matrix an element of the Lie algebra $\mathfrak{sp}(2n)$? Of the Lie group $\mathrm{Sp}(2n)$?[18]

EXERCISE I.6. Let L be a Lagrangian subspace of a symplectic vector space E. Show that the group of symplectic automorphisms of E that restrict to the identity on L is isomorphic to the group of matrices of the form

$$\begin{pmatrix} \mathrm{Id} & B \\ 0 & \mathrm{Id} \end{pmatrix},$$

where B is a symmetric matrix. Show that this group is Abelian.

EXERCISE I.7 (The Liouville Form is Canonical). A differential 1-form on a manifold V is a section of the cotangent bundle $T^\star V$

$$\eta : V \longrightarrow T^\star V \qquad \eta(x) \in T_x^\star V.$$

(Do you agree?) Letting α be the Liouville form, show that

$$\eta^\star \alpha = \eta.$$

EXERCISE I.8 (Symplectic Form on the Torus). Show that the standard symplectic form $\omega = \sum dp_i \wedge dq_i$ on $\mathbb{R}^{2n}$ defines a symplectic form on the torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ when we pass to the quotient.

EXERCISE I.9 (Symplectic Reduction—Continuation). Let (W, ω) be a symplectic manifold. If $V \subset W$ is a hypersurface, what can be said about the rank of ω on V? Suppose that the integral curves of the kernel of $\omega_{|V}$ (the "characteristics") form a manifold Z. Show that ω defines a symplectic structure[19] on Z.

[17] A minor miracle of symplectic geometry.

[18] This is another way of computing the dimension of the symplectic group.

[19] The symplectic manifold obtained is called "the space of characteristics".

- Determine the symplectic manifold obtained when $W = \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and $V = S^n \times \mathbb{R}^{n+1}$.
- Consider the set of geodesics (oriented but non-parametrized curves) on a Riemannian manifold, which we suppose is a manifold. Show that it can be equipped with a symplectic structure.

EXERCISE I.10 (Hamiltonian Vector Field and the Poisson Bracket in Coordinates). Show that on $W = \mathbb{R}^n \times \mathbb{R}^n$ with the standard symplectic form, we have

$$X_H(q_1, \dots, q_n, p_1, \dots, p_n) = \left(\frac{\partial H}{\partial p_1}, \dots, \frac{\partial H}{\partial p_n}, -\frac{\partial H}{\partial q_1}, \dots, -\frac{\partial H}{\partial q_n}\right)$$

for any function H and

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i}\right)$$

for all functions f and g.

What are the values of the Poisson brackets $\{p_i, p_j\}$, $\{p_i, q_j\}$, $\{q_i, q_j\}$?

EXERCISE I.11. Show that any Hamiltonian vector field on a compact symplectic manifold has a zero.

EXERCISE I.12. Consider the phase space $\mathbb{R} \times \mathbb{R}$, and fix a real number $a > 0$. Write out and solve the Hamiltonian system associated to the function $H = \frac{1}{2}p^2 + \frac{1}{2}aq^2$, and draw its trajectories.

EXERCISE I.13. Show that the standard symplectic form on $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ restricts to a symplectic form on the submanifold

$$TS^n = \left\{(q, p) \mid \|q\|^2 = 1 \text{ and } q \cdot p = 0\right\}.$$

EXERCISE I.14. Let W be a symplectic manifold. Suppose that V is a symplectic submanifold of W, that is, that the symplectic form on W defines a *symplectic* form on V.

- Show that, for any $v \in V$,
$$T_v V \oplus (T_v V)^\circ = T_v W.$$
- Let $H : W \to \mathbb{R}$ be a function. Show that the Hamiltonian vector field of the restriction of H to V is the projection of the Hamiltonian vector field X_H to the tangent bundle of V (for the decomposition obtained above).
- Use this method to compute the Hamiltonian vector field for the spherical pendulum.

EXERCISE I.15. Let X and Y be two vector fields that are *locally* Hamiltonian, i.e., such that $i_X\omega$ and $i_Y\omega$ are closed forms. Show that their bracket $[X, Y]$ is (globally) Hamiltonian, i.e. that the form $i_{[X,Y]}\omega$ is exact.

EXERCISE I.16. In defining Hamiltonian vector fields and the Poisson bracket, the *only* property of the symplectic form that we have used is that it is nondegenerate. Thus let ω be a nondegenerate 2-form. Determine $(d\omega)_x(X, Y, Z)$ when X, Y, and Z are vectors in T_xW that are the values at x of the Hamiltonian vector fields associated to three function f, g, and h. Show that ω is closed if and only if the Poisson bracket it defines satisfies the Jacobi identity[20].

EXERCISE I.17. Let $\mathcal{Q}$ be the vector space of quadratic forms on $\mathbb{R}^{2n}$. What is its dimension?

Consider the standard symplectic form $\omega = \sum dp_i \wedge dq_i$ on $\mathbb{R}^{2n}$. Show that $\mathcal{Q}$ is invariant under the Poisson bracket and that the Lie algebra $\big(\mathcal{Q}, \{\ ,\ \}\big)$ is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(2n;\mathbb{R})$[21].

EXERCISE I.18. In the (q,p) plane, draw the level sets of the Hamiltonian $H = \frac{1}{2}p^2 - \cos q$ for the simple pendulum and the trajectories of the vector field (see also Figure 4 in Chapter II).

EXERCISE I.19. Consider the Hénon–Heiles Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) - q_2^2(A + q_1) - \frac{\lambda}{3}q_1^3$$

(see Section I.2.d) with $\lambda \neq 0$.

- Draw the trajectories of X_H contained in the plane $q_2 = p_2 = 0$.
- Determine explicitly the solution $(q_1(t), p_1(t))$ such that

$$\frac{1}{2}p_1^2(t) - \frac{\lambda}{3}q_1^3(t) = 0.$$

EXERCISE I.20 (The Rigid Body). Show that the bilinear forms $\omega_{(\Gamma,M)}$ defined by

$$\omega_{(\Gamma,M)}\big((\xi,\eta),(\xi',\eta')\big) = (\xi \times M + \Gamma \times \eta)\cdot \xi' + (\Gamma \times \xi)\cdot \eta'$$

define a symplectic form ω on the 4-dimensional manifold

$$W_a = \Big\{(\Gamma, M) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|\Gamma\|^2 = 1 \text{ and } \Gamma\cdot M = a\Big\}.$$

Using the notation from Section I.2.g, check that the Hamiltonian vector field associated to $H = \frac{1}{2}M\cdot\Omega + \Gamma\cdot L$ is

$$X_H(\Gamma, M) = (\Gamma \times \Omega, M \times \Omega + \Gamma \times L).$$

Let $L = 0$ (this is the Euler–Poinsot case). Show that the function K defined by

$$K(M) = \|M\|^2$$

commutes with H.

Finally, suppose that $M - \Omega$ is collinear with L (this is the case of the top). Show that the function K defined by

$$K(M) = M\cdot\Gamma$$

commutes with H.

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[20]This is the best possible justification for requiring the closure condition in the definition of a symplectic form: the Poisson bracket should define a Lie algebra structure.

[21]This is a third way to compute the dimension of the symplectic group.

EXERCISE I.21. Consider the function K defined on $\mathbb{R}^2 \times \mathbb{R}^2$ by

$$K = q_2^4 + 4q_1^2 q_2^2 + 4p_2(p_2 q_1 - p_1 q_2) + 8A(q_2^2 q_1 - p_2^2 + 2Aq_2^2).$$

Show that the Hénon–Heiles Hamiltonian (see Section I.2.d) commutes[22] with K if and only if $\lambda = 6$.

EXERCISE I.22. Consider the Hamiltonian H of the spherical pendulum and the moment

$$K(q,p) = q_1 p_2 - q_2 p_1.$$

By calculating the Poisson bracket $\{H, K\}$, show that K is indeed a first integral.

EXERCISE I.23 (The Clairaut Integral). Consider a surface of revolution. For a geodesic $c(t)$ on the surface, let $\alpha(t)$ be the angle the vector $\dot{c}(t)$ makes with the meridian passing through $c(t)$, and let $r(t)$ be the distance from $c(t)$ to the axis of rotation. Show that $r(t)\sin\alpha(t)$ does not depend on t (by showing that, in the notation of Section I.4.a, we have $p_\theta(t) = r(t)\sin\alpha(t)$).

EXERCISE I.24. Using the notation of Section I.4.b, show that the Hamiltonian system associated to the function $\frac{1}{2}\Psi_z$ on the space $\mathcal{D}_n$ of lines is

$$\begin{cases} \dot{p} = \Gamma p \\ \dot{u} = \Gamma u - (A - z\,\mathrm{Id})^{-1} p, \end{cases}$$

where Γ is the matrix with entries $\Gamma_{i,j} = \dfrac{p_i u_j - p_j u_i}{(\alpha_i - z)(\alpha_j - z)}$.

EXERCISE I.25. The complex[23] vector space $\mathbb{C}^{2n}$ is equipped with the symplectic form $\sum dp_i \wedge dq_i$. Consider a matrix $A \in \mathfrak{sp}(2n;\mathbb{C})$.

Show that the characteristic polynomial of A has the form

$$P(\lambda) = \lambda^{2r} Q(\lambda) Q(-\lambda)$$

for some integer r and some polynomial Q.

Show that there exist symplectic subspaces $E_1, \ldots, E_k$ of $\mathbb{C}^{2n}$, pairwise orthogonal and invariant under A, such that $\mathbb{C}^{2n} = E_1 \oplus \cdots \oplus E_k$ and such that each E_i

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[22] The second first integral comes from [**20**].

[23] One can obtain analogous results in the real case. As always in linear algebra, this case is somewhat more delicate but not much more difficult.

has a symplectic basis in which the matrix $A_{|E_i}$ has one the following forms

$$\left[\begin{array}{cccc|cccc}
\alpha & & & & & & & \\
-1 & \ddots & & & & & & \\
 & \ddots & \ddots & & & 0 & & \\
 & & -1 & \alpha & & & & \\
\hline
 & & & & -\alpha & 1 & & \\
 & & & & & \ddots & \ddots & \\
 & 0 & & & & & \ddots & 1 \\
 & & & & & & & -\alpha
\end{array}\right], \quad
\left[\begin{array}{cccc|cccc}
0 & & & & & & & \\
-1 & \ddots & & & & & & \\
 & \ddots & \ddots & & & 0 & & \\
 & & -1 & 0 & & & & \\
\hline
0 & & & & 0 & 1 & & \\
 & \ddots & & & & \ddots & \ddots & \\
 & & \ddots & & & & \ddots & 1 \\
 & & & 1 & & & & 0
\end{array}\right],$$

where α is complex number and where all the entries not written are zero.

Now consider the vector space $\mathbb{C}^{2r}$ with a matrix B of one of the above forms. Show that the subspace

$$Z_B = \{M \in \mathfrak{sp}(2r;\mathbb{C}) \mid [M,B] = 0\}$$

is a Lie subalgebra of $\mathfrak{sp}(2r;\mathbb{C})$ containing an Abelian Lie subalgebra of dimension r.

Exercise I.26 (Normal Form for Quadratic Forms). This exercise is a simple reformulation of the one above. Let Q be a quadratic form on the complex vector space $\mathbb{C}^{2n}$ equipped with the complex symplectic form $\sum dp_i \wedge dq_i$. Show that there is a decomposition of $\mathbb{C}^{2n}$ into a sum of symplectic subspaces $E_1, \dots, E_k$, pairwise orthogonal for the symplectic form *and* for the quadratic form Q, such that

- $\mathbb{C}^{2n} = E_1 \oplus \cdots \oplus E_k$ and
- each E_i has a symplectic basis in which the quadratic form $Q_{|E_i}$ is either of the form

$$\alpha(p_1q_1 + \cdots + p_rq_r) - (p_2q_1 + \cdots + p_rq_{r-1})$$

  or

$$p_2q_1 + \cdots + p_rq_{r-1} + \frac{1}{2}q_r^2.$$

Show that a quadratic Hamiltonian on $\mathbb{C}^{2n}$ must have n independent first integrals in involution. In particular, it is integrable[24].

[24]These results (including the real case) are due to Williamson, in [**82**] for the normal forms and in [**83**] for the integrability.