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# Introduction to Topology

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# Foreword

This book arose from lecture notes of a course given to first and second year students at the Independent University of Moscow.

Topology is a very beautiful science. It is the bridge between geometry and algebra. Its ideas and images play a key role in almost all of modern mathematics: in differential equations, mechanics, complex analysis, algebraic geometry, functional analysis, mathematical and quantum physics, representation theory, and even—in a surprisingly modified form—in number theory, combinatorics, and complexity theory.

In recent years most of the new ideas in mathematics arose in topology from geometrical images and were then formalized and carried over to more algebraic fields. For this reason a sound knowledge of topology is necessary to any research mathematician. Unfortunately, in Russia and many other countries, topology is not included, even today, in the basic curriculum of mathematical departments in most universities. Serious teachers of the other disciplines must include various fragments of topology in their courses, but the student who studies Stokes' formula in the calculus, the argument principle and Riemann surfaces in complex analysis, the principle of contracting maps and the index of singular points of vector fields in differential equations, the Euler characteristic in combinatorics, stable regime

theorems in optimal control theory, and fixed point theorems in mathematical economics, usually does not understand that he/she is essentially studying the same things. And the student is led to studying basic topology individually. (An exceptional event, which apparently had a crucial influence on my generation of Moscow mathematicians and, undoubtedly, on my own mathematical education, was the special (i.e., nonobligatory) topology course given by D. B. Fuchs at the Mechanics and Mathematics Department of Moscow State University in 1976–77.)

For several years (in the late 80s and the early 90s), I gave informal introductory topology courses for undergraduates and high school students at specialized math schools. I would like to thank the administration of the Independent University of Moscow for the opportunity to give this course as part of the basic curriculum to IUM students in the second and third semesters in 1996.

I am also extremely grateful to V. V. Prasolov, who took down the lecture notes and carried out their initial editing, and to the director of Phasis Publishers, V. V. Filippov, for his initiative and support in their publication.

The lecture note origins of the book left a significant imprint on its style. It contains very few detailed proofs; I tried to give as many illustrations as possible and to show what really occurs in topology, not always explaining why it occurs. As a rule, only those proofs (or sketches of proofs) that are interesting *per se* and have important generalizations are presented.

In conclusion, here is a list of suggested references.

[1] J. W. Milnor, *Topology from the differentiable viewpoint*, The University Press of Virginia, Charlottesville, VA, 1965; Princeton Univ. Press, Princeton, NJ, 1997.

[2] A. H. Wallace, *Differential Topology*, W. A. Benjamin, New York, 1968.

[3] V. V. Prasolov, *Intuitive Topology*, American Mathematical Society, Providence, RI, 1995.

[4] C. Kosniowski, *A First Course in Algebraic Topology*, Cambridge University Press, 1980.

[5] A. T. Fomenko and D. B. Fuchs, *A Course in Homotopic Topology*, Nauka, Moscow, 1989; English transl. of a preliminary edition, A. T. Fomenko, D. B. Fuchs, and V. L. Gutenmakher, *Homotopic Topology*, Akademiai Kiado, Budapest, 1986.

[6] V. A. Rokhlin and D. B. Fuchs, *Beginner's Course in Topology. Geometric Chapters*. Nauka, Moscow, 1977; English transl., Springer-Verlag, Berlin–New York, 1984.

[7] M. M. Postnikov, *Lectures in Algebraic Topology, Homotopy Theory of Cell Spaces*, Nauka, Moscow, 1985 (in Russian).

[8] J. R. Munkres, *Elementary Differential Topology*, Ann. of Math. Studies, no. 54, Princeton University Press, 1966.

[9] J. W. Milnor, *Morse Theory*, Princeton University Press, 1963.

[10] J. W. Milnor, *Lectures on the  $h$ -Cobordism Theorem*, Princeton University Press, 1965.

[11] J. W. Milnor, *Characteristic Classes*, Notes by J. Stasheff, Princeton University, 1957.

[12] S. P. Novikov, *Topology-1, Contemporary Problems of Mathematics, Fundamental Directions*, Vol. 12, VINITI, Moscow, 1986; English transl., *Encyclopedia Math. Sci.*, Vol. 12, Springer-Verlag, Berlin–New York, 1988.

Books [1–4] provide a basis for topological geometric intuition; they are recommended as preliminary reading.

Chapters 1 and 2 of [5] cover such topics as homotopy groups, homotopy theory of cellular spaces, and basic (co)homology theory. The book [8] provides an introduction to smooth manifold theory, a nice explanation of Morse theory is contained in [9 and 10]. The book [6] is not easy reading for beginners and we recommend it with care; however, it can serve as an exhaustive handbook and dictionary for all topics studied in the first half of our book, and [7] helps in those rare cases when [6] is insufficient. The book [11] is one of the world's best textbooks in algebraic topology, and I hope that the reader will be able to handle it. Finally, [12] is a nice and very wide survey of the modern state of topology.