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Chapter 2

Curves in $\mathbb{R}^n$

In the practical world, curves arise in many different ways, for example as the profile curves or contours of technical objects. On white drawing paper, curves appear as the trace of the pencil or other drawing medium used to draw it. For physicists, curves arise naturally in the *motion of a particle* in time t. From this point of view the association of the parameter t to the position $c(t)$ is important, and this process is called a *parametrization* of the curve; the curve is then called a *parametrized curve*. This notion is the most appropriate for a formal mathematical treatment of curves. In this formulation, one passes from the real-world notion of a "thin" object to one which has no width whatsoever: a one-dimensional or "infinitely thin" object. Here both the parametrization and the curve are supposed to have reasonable properties, which allow an acceptable mathematical treatment. A short introduction to the theory of curves can be found in [**27**], X, §5, but we will not assume any familiarity with this on the part of the reader.

2A Frenet curves in $\mathbb{R}^n$

Mathematically one can define a *curve* most easily as a continuous mapping from an interval $I \subseteq \mathbb{R}$ to $\mathbb{R}^n$. Unfortunately, the assumption of continuity is so weak that curves defined in this manner can look very complicated and have unexpected (pathological) properties. There are continuous curves which cover a whole square in the plane. Thus it is natural to take the point of view of analysis and require differentiability in addition to continuity. But still this assumption is not quite the right one. Differentiability of a map just means that it can be linearly approximated. For the image set, however, this no longer needs to be the case. From a geometrical point of view it makes sense to require that the image curve can be approximated by a line at each point, i.e., to require that the image curve has a tangent as a geometrical linearization at every point. This means that the derivative of the map from I to $I\!R^n$ must be non-vanishing. One calls a map with this property an *immersion*. This simply means that the derivative of the parametrization always has the highest possible rank, which in our case, where the domain is an interval, is one.

2.1. Definition. A *regular parametrized curve* is a continuously differentiable immersion $c\colon I \longrightarrow I\!R^n$, defined on a real interval $I \subseteq I\!R$. This means that $\dot{c} = \frac{dc}{dt} \neq 0$ holds everywhere.

The vector

$$\dot{c}(t_0) = \frac{dc}{dt}\Big|_{t=t_0}$$

is called the *tangent vector* to c at t_0, and the line spanned by this vector through $c(t_0)$ is called the *tangent* (line) to c at this point. This is a geometric approximation of the first order in a neighborhood of the point with $c(t_0 + t) = c(t_0) + t \cdot \dot{c}(t_0) + o(t)$.

A *regular curve* is an equivalence class of regular parametrized curves, where the equivalence relation is given by regular (orientation preserving) parameter transformations

$$\varphi\colon [\alpha, \beta] \longrightarrow [a, b], \quad \varphi' > 0, \quad \varphi \ \begin{array}{l}\text{bijective and}\\ \text{continuously differentiable;}\end{array}$$

c and $c \circ \varphi$ are then considered to be *equivalent*. The *length* of the curve

$$\int_a^b \Big|\Big|\frac{dc}{dt}\Big|\Big| dt$$

is invariant under the parameter transformations as just described. In the sciences one can view a curve as the motion of a particle, with the trajectory of the particle as a function of time. It is regular if the instantaneous speed $||\dot{c}||$ never vanishes.

2.2. Lemma. Every regular curve can be parametrized by its arc length (in other words, the tangent vector at every point has unit length).

PROOF: Let a curve $c\colon [a,b] \to I\!R^n$ be given, of total length $L = \int_a^b ||\frac{dc}{dt}||dt$. We then set $[\alpha, \beta] = [0, L]$ and introduce the *arc length parameter* s by the relation

$$s(t) := \psi(t) = \int_a^t \left\| \frac{dc}{dt}(\tau) \right\| d\tau.$$

This defines a map $\psi\colon [a,b] \to [0,L]$. Then one has $\frac{ds}{dt} = \frac{d\psi}{dt} = ||\frac{dc}{dt}|| \neq 0$, and consequently there is an inverse function $\varphi := \psi^{-1}$ such that $c \circ \varphi = c \circ \psi^{-1}$ is parametrized by arc length. This parametrization is unique up to a translation $s \mapsto s + s_0$ or $s \mapsto s_0 - s$. □

We will use the following notations in the sequel:

$c(t)$	denotes an arbitrary regular parametrization,
$c(s)$	denotes the parametrization by arc length,
$\dot{c} = \frac{dc}{dt}$	denotes the tangent vector,
$c' = \frac{dc}{ds}$	denotes the unit tangent vector.

In particular one then has $\dot{c} = \frac{ds}{dt}c' = ||\dot{c}||c'$ and $||c'|| = 1$.

2.3. Examples.

1. $c(t) = (at, bt)$, a *line* in standard parametrization. Since $\dot{c} = (a, b)$, the parameter is the arc length if and only if $a^2 + b^2 = 1$. The parametrization $c(t) = (at^3, bt^3)$ describes exactly the same line, but it is not regular for $t = 0$.
2. $c(t) = \frac{1}{2}(\cos 2t, \sin 2t)$, a *circle* of radius $\frac{1}{2}$. Since of course $\dot{c}(t) = (-\sin 2t, \cos 2t)$ one has $||\dot{c}|| = 1$. Hence t is the arc length, i.e., $t = s$.
3. $c(t) = (a\cos(\alpha t), a\sin(\alpha t), bt)$ with constants α, a, b. This is called a *(circular) helix.* Since

$$\dot{c}(t) = (-\alpha a \sin(\alpha t), \alpha a \cos(\alpha t), b),$$

one has $||\dot{c}|| = \sqrt{\alpha^2 a^2 + b^2}$. Therefore c is parametrized by arc length up to a constant multiple of t, i.e., one has $s = t \cdot \sqrt{\alpha^2 a^2 + b^2}$. Geometrically, the curve c arises as the trajectory of a point $(a, 0, 0)$ under the following one-parameter group of *screw-motions*:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \underbrace{\begin{pmatrix} \cos(\alpha t) & -\sin(\alpha t) & 0 \\ \sin(\alpha t) & \cos(\alpha t) & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}}_{\text{translation}} .$$

For appropriately chosen parameters, a motion of this kind maps every point on the curve to an arbitrary other point. Thus one expects that from a geometric point of view this curve will have good properties (a certain homogeneity in all scalar quantities which are geometrically relevant).

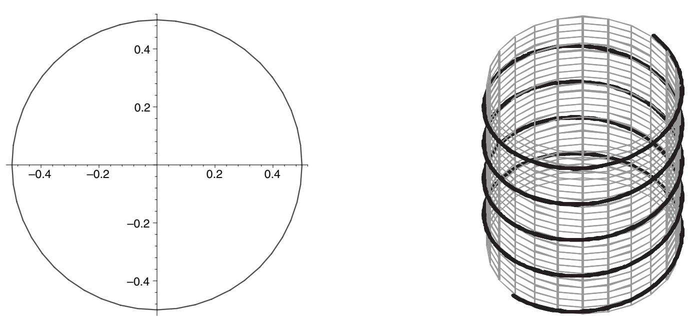


Figure 2.1. Circle, (circular) helix

4. $c(t) = (t^2, t^3)$, the so-called *Neil parabola* or *semicubical parabola.* The tangent vector is $\dot{c}(t) = (2t, 3t^2)$ with $\dot{c}(0) = (0, 0)$, hence at $t = 0$ there is no regular parametrization. In fact the curve doesn't have a tangent touching it at the point, as the curve has a "bend" by an angle π. This is no contradiction to the differentiability of the map c.

5. $c(t) = (t, a \cosh \frac{t}{a})$ with a constant a, the *catenary.* This curve arises as the stable position of a (heavy but infinitely supple)

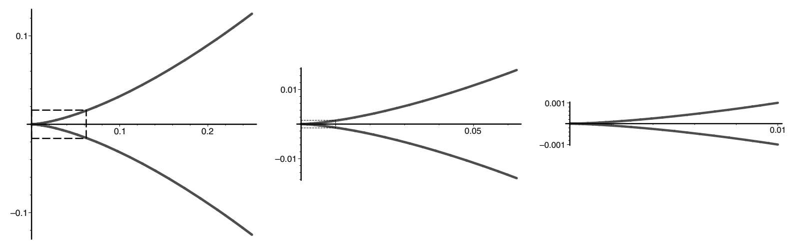


Figure 2.2. Neil parabola

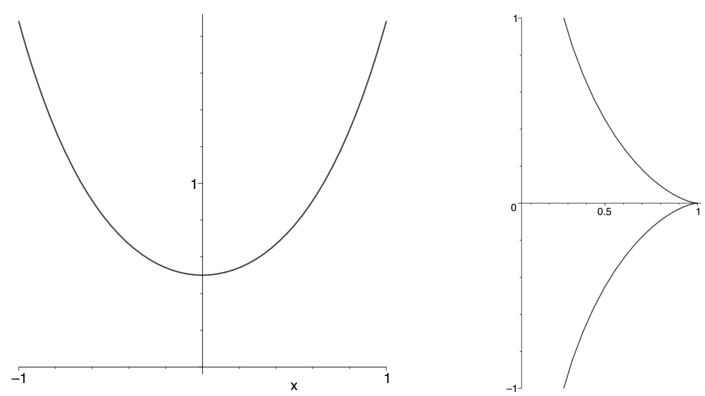


Figure 2.3. Catenary, tractrix

chain strung between two fixed points. Since $\dot{c}(t) = (1, \sinh \frac{t}{a})$, t is not the arc length.

6. The *tractrix* is characterized by the property that from every point p the tangent meets a fixed line (for example the y-axis) at a constant distance. For the case where the fixed line is the y-axis and the constant distance is 1, one can choose the parametrization

$$c(t) = \big(\exp(t), \int_0^t \sqrt{1 - \exp(2x)}dx\big)$$

for the upper part and

$$c(t) = a\big(\exp(-|t|), \int_0^t \sqrt{1-\exp(-2|x|)}dx\big)$$

for both parts together, see Figure 2.3.

Remark: The local behavior of a curve which has been parametrized by arc length can be studied by means of its *Taylor expansion*:

$$c(s) = c(0) + sc'(0) + \frac{s^2}{2}c''(0) + \frac{s^3}{6}c'''(0) + o(s^3).$$

The linearization $c(0) + sc'(0)$ is a line, which is the *tangent* of c at $s = 0$ (since $c'(0) \neq 0$). The quadratic part of the expansion, $c(0) + sc'(0) + \frac{s^2}{2}c''(0)$, is a parabola (if $c''(0) \neq 0$) which is referred to as the (Euclidean) *osculating conic*. It has contact of second order with the curve. Note that c'' is perpendicular to c', as can be seen by differentiating $\langle c', c'\rangle = 1$: $0 = \langle c', c'\rangle' = 2\langle c'', c'\rangle$. This is further explained and extended in the following definition.

One says that two curves $c_1(s)$ and $c_2(s)$ (both assumed to be parametrized by arc length) are said to *have contact of the kth order*, if

$$c_1(0) = c_2(0),\ c_1'(0) = c_2'(0),\ c_1''(0) = c_2''(0),\ \ldots,\ c_1^{(k)}(0) = c_2^{(k)}(0);$$

that is, if the Taylor expansions of the two curves coincide up to terms of the kth order. This obviously is related to the phenomenon of the two curves touching each other. One could also say that a curve touches another to the kth order. For example, the osculating conic above touches the curve to the second order, at the apex of the parabola. At a point other than the apex, the parabola can touch a given curve to even third order (cf. Exercise 2 at the end of the chapter). Similarly, one can look for cubic and quartic curves which have contact with a given curve of the highest possible order. For example, cubic splines are an important tool in the computer treatment of curves.

In three-dimensional space and all the more in spaces of higher dimensions, one requires an adequate system of coordinates to describe curves, one which is adapted to the curve. Here one would expect that the vectors $c', c'', c''', \ldots$ describe the local behavior of the curve, at least as long as they do not vanish or – even better – if they are linearly independent. This motivates the following definition. Recall that an *n-frame* in Euclidean n-space is a basis of orthonormal vectors $e_1, \ldots, e_n$, in a specific order. For curves in n-space we take advantage of an adapted n-frame as follows.

2.4. Definition. (Frenet curve)
Let $c(s)$ be a *regular curve* in $\mathbb{R}^n$, which is parametrized by arc length and n-times continuously differentiable. Then c is called a *Frenet curve*, if at every point the vectors $c', c'', \ldots, c^{(n-1)}$ are linearly independent. The *Frenet n-frame* $e_1, e_2, \ldots, e_n$ is then uniquely determined by the following conditions:

(i) $e_1, \ldots, e_n$ are orthonormal and positively oriented.
(ii) For every $k = 1, \ldots, n-1$ one has $Lin(e_1, \ldots, e_k) = Lin(c', c'', \ldots, c^{(k)})$, where Lin denotes the linear span.
(iii) $\langle c^{(k)}, e_k \rangle > 0$ for $k = 1, \ldots, n-1$.

Note: In the case discussed most often, $n = 3$, the only restrictive condition on a Frenet curve is $c'' \neq 0$. This excludes only inflection points. For $n = 2$ there are no actual restrictions, cf. 2.5.

One obtains $e_1, \ldots, e_{n-1}$ from $c', \ldots, c^{(n-1)}$ by means of the *Gram–Schmidt orthogonalization procedure* as follows:

$$\begin{aligned}
e_1 &:= c', \\
e_2 &:= c''/ \parallel c'' \parallel, \\
e_3 &:= \Big(c''' - \langle c''', e_1\rangle e_1 - \langle c''', e_2\rangle e_2\Big) \Big/ \parallel \cdots \parallel, \\
&\vdots \\
e_j &:= \Big(c^{(j)} - \sum_{i=1}^{j-1} \langle c^{(j)}, e_i\rangle e_i\Big) \Big/ \parallel \cdots \parallel, \\
&\vdots \\
e_{n-1} &:= \Big(c^{(n-1)} - \sum_{i=1}^{n-2} \langle c^{(n-1)}, e_i\rangle e_i\Big) \Big/ \parallel \cdots \parallel .
\end{aligned}$$

The missing vector e_n is then uniquely determined by condition (i) in the above definition. One can say that every Frenet curve uniquely induces through its Frenet n-frame a curve in the Stiefel manifold of all n-frames in $\mathbb{R}^n$. The converse does not hold in general since, for example, for $n \geq 3$ a constant n-frame cannot correspond to any Frenet curve.

2B Plane curves and space curves

2.5. Plane curves. For $n = 2$ every regular curve is a *Frenet curve*, provided it is twice continuously differentiable. The *tangent vector* is $e_1 = c'$, the *normal vector* is e_2, which – if the orientation is positive – is the rotation by an angle of $\pi/2$ to the left of the vector e_1. From $0 = \langle c', c' \rangle' = 2\langle c', c'' \rangle = 2\langle e_1, c'' \rangle$, it follows that c'' and e_2 are linearly dependent, hence $c'' = \kappa e_2$ with some function κ. This function κ is said to the (oriented) *curvature* of c. Its sign indicates in which direction the curve (resp. its tangent) is rotating. Here $\kappa > 0$ indicates that the tangent goes to the left, while $\kappa < 0$ indicates that it rotates to the right. At an *inflection point* one has $\kappa = 0$, and the direction of the tangent is stationary.

One has the following equations for the derivatives, in which the second follows from the first, since e_2 and e_1 differ by a rotation of $\pi/2$:

$$\begin{aligned} e_1' &= c'' = \kappa e_2, \\ e_2' &= -\kappa e_1, \end{aligned}$$

or, using matrix notation,

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Note that the matrix on the right is skew-symmetric, which follows already from the relation $0 = \langle e_1, e_2 \rangle' = \langle e_1', e_2 \rangle + \langle e_2', e_1 \rangle$. These equations are also called the *Frenet equations*.

Exercise: If one describes a curve in an adapted coordinate system by $c(t) = (t, y(t))$ (t is not the arc length here), then one has

$$y(0) = \dot{y}(0) = 0, \quad \dot{c}(0) = (1, 0), \quad \ddot{c}(0) = (0, \ddot{y}(0)) = (0, \kappa(0)).$$

The curvature $\kappa(0)$ hence coincides with the opening of the parabola $t \mapsto (t, \frac{\ddot{y}(0)}{2}t^2)$, which is just the quadratic part of the Taylor expansion of c.

2.6. Theorem. (Plane curves with constant curvature)
A regular curve in $\mathbb{R}^2$ has constant curvature κ if and only if it is part of a circle of radius $\frac{1}{|\kappa|}$ (if $\kappa \neq 0$) or a line segment (if $\kappa = 0$).

Proof: The proof follows directly from the Frenet equations. Assume first that $\kappa(s_0) \neq 0$. Obviously the expression $c(s) + \frac{1}{\kappa(s_0)}e_2(s)$ is constant if and only if $c(s)$ is part of a circle of radius $|\frac{1}{\kappa(s_0)}|$, since the difference vector has constant length $|\frac{1}{\kappa(s_0)}|$. This is equivalent to $\kappa = \kappa(s_0)$ everywhere, because $c' + \frac{1}{\kappa(s_0)}e_2' = e_1 - \frac{1}{\kappa(s_0)}\kappa e_1$. The fact that $\kappa \equiv 0$ only holds for line segments is trivial, because $e_2' = -\kappa e_1$. $\square$

2.7. Remarks. 1. For every regular curve in the plane with non-vanishing curvature the circle centered at $c(s_0) + \frac{1}{\kappa(s_0)}e_2(s_0)$ with radius $|\frac{1}{\kappa(s_0)}|$ is called the *osculating circle* of c in the point $c(s_0)$. It has contact of order two with the curve and is uniquely determined by this property. The curve which is formed by all of the centers of these circles,

$$s \mapsto c(s) + \frac{1}{\kappa(s)}e_2(s),$$

is called the *evolute* or the *focal curve* of c. This curve is not necessarily regular. Typically one has cusps like that occuring in the Neil parabola. In fact, the evolute of the catenary has such a cusp.

2. Not only does every plane curve uniquely determine its curvature function $\kappa(s)$, but also conversely, the curvature function κ also determines the curve, up to Euclidean motions, i.e., up to the prescription of a point on the curve and the tangent of the curve at that point. We even have the following *explicit determination* of the curve in terms of its curvature. Let the curvature function $\kappa(s)$ be given. Then one can set

$$e_1 = \big(\cos(\alpha(s)), \sin(\alpha(s))\big)$$

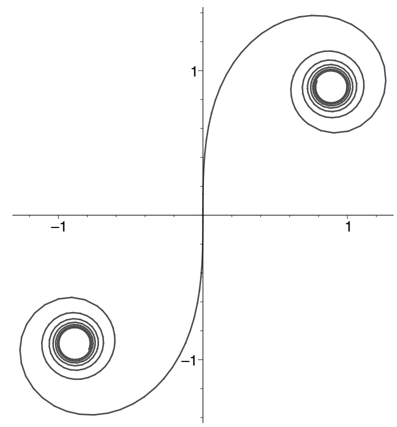


Figure 2.4. Cornu spiral with constant κ/s

with a function $\alpha(s)$ which is to be found. Necessarily one has

$$e_2 = \big(-\sin(\alpha(s)), \cos(\alpha(s))\big).$$

The Frenet equation says that $\kappa e_2 = e_1' = \alpha' e_2$, hence $\kappa = \alpha'$. By a judicious choice of adapted coordinate system we can assume that for $s = 0$, the curve passes through the origin with $e_1 = (1, 0)$; then $\alpha(0) = 0$, and hence $\alpha(s) = \int_0^s \kappa(t)dt$. The sought-for curve is then given by the relation

$$x(s) = \int_0^s \cos\Big(\int_0^\sigma \kappa(t)dt\Big)d\sigma, \quad y(s) = \int_0^s \sin\Big(\int_0^\sigma \kappa(t)dt\Big)d\sigma.$$

For constant κ this again leads to the solutions we already met in Theorem 2.6. If κ is a linear function[1] of s, then we obtain the so-called *Cornu spiral*, see Figure 2.4.

2.8. Space curves. For $n = 3$ a regular three-times continuously differentiable curve is called a *Frenet curve*, if $c'' \neq 0$ everywhere. The accompanying three-frame is then given by

[1]Pictures of the curves for which κ is quadratic in s can be found for example in F. Dillen, *The classification of hypersurfaces of a Euclidean space with parallel higher order fundamental form*, Math. Zeitschrift **203**, 635–643 (1990).

$$\begin{aligned} e_1 &= c', & &(\textit{tangent vector}) \\ e_2 &= \tfrac{c''}{\|c''\|}, & &(\textit{principal normal vector}) \\ e_3 &= e_1 \times e_2. & &(\textit{binormal vector}) \end{aligned}$$

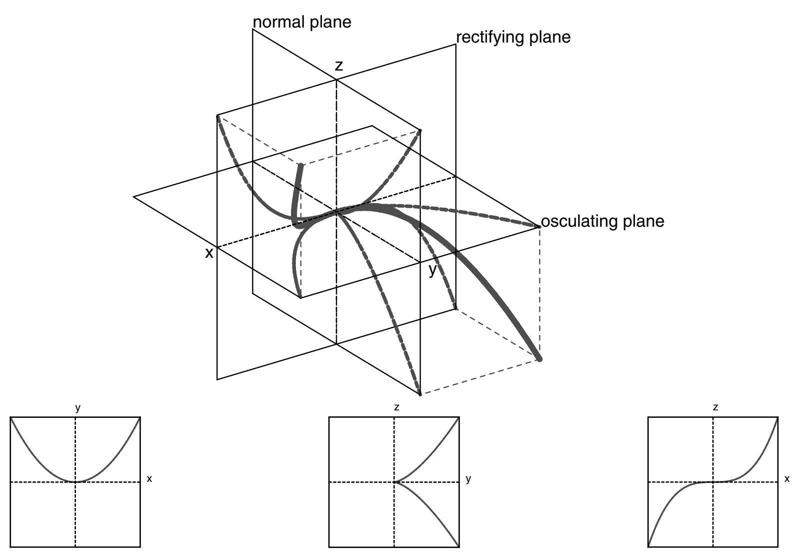
The function $\kappa := ||c''||$ is called the *curvature* of c. By assumption this number is always positive. The equations for the derivatives are

$$\begin{aligned} e_1' &= c'' = \kappa e_2, \\ e_2' &= \langle e_2', e_1\rangle e_1 + \underbrace{\langle e_2', e_2\rangle}_{=0} e_2 + \langle e_2', e_3\rangle e_3 \\ &= \langle -e_2, e_1'\rangle e_1 + \underbrace{\langle e_2', e_3\rangle}_{=:\tau} e_3 \\ &= -\kappa e_1 + \tau e_3, \\ e_3' &= \langle e_3', e_1\rangle e_1 + \langle e_3', e_2\rangle e_2 + \underbrace{\langle e_3', e_3\rangle}_{=0} e_3 \\ &= -\underbrace{\langle e_3, e_1'\rangle}_{=0} e_1 - \underbrace{\langle e_3, e_2'\rangle}_{=\tau} e_2 \\ &= -\tau e_2. \end{aligned}$$

The function $\tau := \langle e_2', e_3\rangle$ is called the *torsion* of c. It indicates how the (e_1, e_2)-plane changes along the curve. These three equations for the derivatives are called the *Frenet equations*, and in matrix notation they take the following form:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

Remark: A *plane curve* (viewed as a space curve) with $c'' \neq 0$ is also a Frenet curve in $\mathbb{R}^3$. The torsion of this curve is $\tau \equiv 0$, because e_3 is constant. The converse of this is also true: $\tau \equiv 0$ implies that e_3 is constant, and in addition that c lies in a plane which is perpendicular to e_3. This follows easily from the Frenet equations. If $c''(s) = 0$ at a single point, one gets a Frenet three-frame from the left and a different one from the right, with an orthogonal "jump matrix" at the point where $c''(s) = 0$. On the other hand, it is possible to join two curves lying in different planes in such a way that $\tau \equiv 0$ still holds everywhere, with the exception of this one point, see Problem 24 at the end of this chapter. If $\tau \neq 0$, then the sign of τ indicates a certain sense of rotation of the curve (in the sense of orientation). In earlier times, differential geometers had special names for these orientations ("weinwendig" and "hopfenwendig" in German because of the different growth behavior of grapevine and hops). For more on space curves with constant curvature and constant torsion, see Section 2.12.



(a) Osculating plane (b) Normal plane (c) Rectifying plane

Figure 2.5. Three projections of the space curve $xe_1 + ye_2 + ze_3$

2.9. Corollary. (Taylor expansion of the accompanying three-frame) The ordinary Taylor expansion around the point $s = 0$ is

$$c(s) = c(0) + sc'(0) + \frac{s^2}{2}c''(0) + \frac{s^3}{6}c'''(0) + o(s^3)$$

and can be translated into an expansion for the three-frame of the following form:

$$c(s) = c(0) + \alpha(s)e_1(0) + \beta(s)e_2(0) + \gamma(s)e_3(0) + o(s^3)$$

with unknown coefficients α, β, γ. This is seen as follows.

First one has, by the Frenet equations,

$$\begin{array}{lllllll} c' & = & e_1, & & & & \\ c'' & = & e_1' & = & \kappa e_2, & & \\ c''' & = & (\kappa e_2)' & = & \kappa' e_2 + \kappa e_2' & = & \kappa' e_2 + \kappa(-\kappa e_1 + \tau e_3), \end{array}$$

which implies

$$\begin{aligned} c(s) &= c(0) + se_1 + \frac{s^2}{2}\kappa e_2 \\ &\quad + \frac{s^3}{6}\Big(\kappa' e_2 - \kappa^2 e_1 + \kappa\tau e_3\Big) + o(s^3) \\ &= c(0) + \Big(s - \frac{s^3\kappa^2}{6}\Big)e_1 + \Big(\frac{s^2\kappa}{2} + \frac{s^3\kappa'}{6}\Big)e_2 \\ &\quad + \frac{s^3\kappa\tau}{6}e_3 + o(s^3). \end{aligned}$$

The projections in the various (e_i, e_j)-planes are the following, see Figure 2.5:

(e_1, e_2)*-plane* (*osculating plane*):

$$c(s) = c(0) + se_1(0) + \frac{s^2\kappa(0)}{2}e_2(0) + o(s^2).$$

The projection onto the osculating plane has the form of a *parabola* (up to $o(s^2)$).

(e_2, e_3)*-plane* (*normal plane*):

$$c(s) = c(0) + \Big(\frac{s^2\kappa(0)}{2} + \frac{s^3\kappa'(0)}{6}\Big)e_2(0) + \frac{s^3\kappa(0)\tau(0)}{6}e_3(0) + o(s^3).$$

The projection onto the normal plane has the form of a *Neil parabola* in case $\tau(0) \neq 0$ (up to $o(s^3)$).

(e_1, e_3)*-plane* (*rectifying plane*):

$$c(s) = c(0) + \Big(s - \frac{s^3\kappa^2(0)}{6}\Big)e_1(0) + \frac{s^3\kappa(0)\tau(0)}{6}e_3(0) + o(s^3).$$

The projection onto the rectifying plane is of the type of a *cubical parabola*, in case $\tau(0) \neq 0$ (up to $o(s^3)$).

2C Relations between the curvature and the torsion

We have seen in Section 2.6 that a Frenet curve in $\mathbb{R}^3$ with constant κ and vanishing τ is necessarily an arc of a circle (because it is contained in a plane). A *circular helix* has constant κ and τ, since it is a trajectory of a fixed point under a one-parameter group of helicoidal rotations or screw-motions, see Section 2.3. On the other hand, every Frenet curve with constant κ and τ is such a helix, as will be shown in Section 2.12. More generally, one would expect that every equation between the curvature and the torsion will lead to a similar characterization of the corresponding curve. Conversely one can attempt to classify the different possible classes of curves by means of the equations which hold between the curvature and torsion of these classes of curves. This is particularly interesting for spherical curves, i.e., curves which lie entirely on a sphere.

2.10. Theorem. (Osculating sphere and spherical curves)

(i) Let c be a Frenet curve in $\mathbb{R}^3$ with $\tau(s_0) \neq 0$. Then the surface of the sphere centered at the point

$$c(s_0) + \frac{1}{\kappa(s_0)} e_2(s_0) - \frac{\kappa'(s_0)}{\tau(s_0)\kappa^2(s_0)} e_3(s_0),$$

which passes through the point $c(s_0)$, has a point of contact with the curve at the point s_0 of the third order. This sphere is uniquely determined by these properties and is called the *osculating sphere*.

(ii) Let c be a Frenet curve of class C^4 in $\mathbb{R}^3$ with $\tau \neq 0$ everywhere. Then c lies on a sphere if and only if the following equation holds:

$$\frac{\tau}{\kappa} = \left(\frac{\kappa'}{\tau\kappa^2}\right)'.$$

(iii) Let c be a C^3-curve that is parametrized by arc length and whose image lies on the unit sphere $S^2 \subset \mathbb{R}^3$. Set $J := \mathrm{Det}(c, c', c'')$. Then c is a Frenet curve with curvature $\kappa = \sqrt{1 + J^2}$ and torsion $\tau = J'/(1 + J^2)$. The great circles are characterized by the condition $J \equiv 0$, other circles by constant J.

PROOF: For part (i) we start with the center $m(s_0)$ of the hypothetical osculating sphere

$$m(s_0) = c(s_0) + \alpha e_1(s_0) + \beta e_2(s_0) + \gamma e_3(s_0),$$

with coefficients α, β, γ which are to be determined. For the function $r(s) = \big\langle m - c(s), m - c(s)\big\rangle$ we calculate the derivatives:

$$\begin{aligned}
r' &= -2\big\langle m - c(s), c'(s)\big\rangle,\\
r'' &= -2\big\langle m - c(s), c''(s)\big\rangle + 2\big\langle c'(s), c'(s)\big\rangle,\\
r''' &= -2\big\langle m - c(s), c'''(s)\big\rangle.
\end{aligned}$$

The optimal contact of the sphere with the curve simply means that as many derivatives of $r(s)$ as possible vanish at the point $s = s_0$:

$$\begin{aligned}
r'(s_0) = 0 \quad &\Longleftrightarrow \big\langle m - c(s_0), c'(s_0)\big\rangle = 0 \\
&\Longleftrightarrow \big\langle m - c(s_0), e_1(s_0)\big\rangle = 0 \Longleftrightarrow \alpha = 0, \\
r''(s_0) = 0 \quad &\Longleftrightarrow \big\langle m - c(s_0), c''(s_0)\big\rangle - \big\langle c'(s_0), c'(s_0)\big\rangle = 0 \\
&\Longleftrightarrow \beta\kappa - 1 = 0 \Longleftrightarrow \beta = \frac{1}{\kappa(s_0)}, \\
r'''(s_0) = 0 \quad &\Longleftrightarrow \big\langle m - c(s_0), c'''(s_0)\big\rangle = 0 \\
&\Longleftrightarrow \big\langle m - c(s_0), \kappa' e_2 - \kappa^2 e_1 + \kappa\tau e_3\big\rangle = 0 \\
&\Longleftrightarrow \frac{\kappa'}{\kappa} + \kappa\tau\gamma = 0 \Longleftrightarrow \gamma = -\frac{\kappa'(s_0)}{\kappa^2(s_0)\tau(s_0)}.
\end{aligned}$$

Part (ii) follows similarly, if one considers $m(s)$ for variable s and puts on it the condition that $m(s)$ is constant, i.e., $m' \equiv 0$. This condition is

$$(m(s))' = \Big(c(s) + \frac{1}{\kappa(s)} e_2(s) - \frac{\kappa'(s)}{\tau(s)\kappa^2(s)} e_3(s)\Big)' = \left[\frac{\tau}{\kappa} - \Big(\frac{\kappa'}{\tau\kappa^2}\Big)'\right] e_3(s);$$

hence $m(s)$ is constant if and only if the differential equation in (ii) is satisfied. Then one also has $r'(s) = 0$, and the statement of part (ii) follows from this.

It is not surprising that for the condition just considered a differential equation in only the two variables κ and τ arises. Still it is interesting that the property in question can be verified just from this differential equation, without even knowing the position of the sphere.

(iii) By assumption the vectors $c, c', c \times c'$ form an orthonormal three-frame along the curve. From this fact we get

$$c'' = \langle c'', c\rangle c + \langle c'', c'\rangle c' + \langle c'', c \times c'\rangle c \times c'.$$

Now one has $\langle c'', c\rangle = -\langle c', c'\rangle = -1$, hence $c'' = -c + Jc \times c'$ and from this

$$\kappa^2 = \langle c'', c''\rangle = 1 + J^2 > 0.$$

Moreover one has $e_2 = \frac{1}{\kappa}c'', e_3 = c' \times e_2$ and, by differentiating $\langle c'', c\rangle = -1$, also $\langle c''', c\rangle = 0$, from which it follows that

$$\begin{aligned}\tau = -\langle e_3', e_2\rangle &= -\Big\langle (\frac{1}{\kappa}c' \times c'')', \frac{1}{\kappa}c''\Big\rangle \\ &= -\frac{1}{\kappa^2}\Big\langle c' \times c''', c''\Big\rangle + \frac{\kappa'}{\kappa^3}\Big\langle c' \times c'', c''\Big\rangle \\ &= -\frac{1}{\kappa^2}\Big\langle c' \times c''', -c + Jc \times c'\Big\rangle = \frac{J'}{\kappa^2}.\end{aligned}$$

Here the last equality follows from the fact that c''' is perpendicular to c (see the discussion above) and consequently $c' \times c'''$ is perpendicular to $c' \times c$. Notice that $J' = \langle c' \times c'', c\rangle' = \langle c' \times c''', c\rangle'$. □

Remarks: 1. The determinant J is itself an interesting invariant of the curve, which is just the curvature inside of the sphere. The vector $c \times c'$ is the unit vector which is perpendicular to the curve but tangent to the sphere (as it is perpendicular to the vector in space determined by the points of c). Then $J = \langle c'', c \times c'\rangle$ is the part of c'' which is tangent to the sphere. One also calls this the *geodesic curvature* of the curve; see in this respect also 4.37 in Chapter 4. One has $J = 0$ precisely for the great circles, and J is a non-vanishing constant for all other circles (exercise), compare Figure 4.1.

2. Condition (ii) yields an equation between the curvature and the torsion, with the help of which it is easy to check if a curve lies on a sphere. If the condition is satisfied, then it is in principle clear that only one of the functions is necessary for a complete description of the curve, the other being itself a function of the first. If one prescribes κ, then (iii) gives an explicit way of expressing this dependency by introducing the function $J = \pm\sqrt{\kappa^2 - 1}$. To see this, one considers the system of equations

$$\kappa^2 = 1 + J^2, \quad \tau\kappa^2 = J'.$$

Even the case $\tau = 0$ is taken account of in this relation; this case can only occur in conjunction with the relation $\kappa' = 0$, for example the great or lesser circles. The case $\kappa = 0$ cannot occur. As a test of this statement, set $\kappa = \sqrt{1 + J^2}$ and $\tau = J'/(1 + J^2)$ in the equation in (ii).

2.11. Theorem. (Slope lines)
For a Frenet curve in $\mathbb{R}^3$, the following conditions are equivalent:
(i) There is a vector $v \in \mathbb{R}^3 \setminus \{0\}$ with the property that $\langle e_1, v\rangle$ is constant.
(ii) There is a vector $v \in \mathbb{R}^3 \setminus \{0\}$ with $\langle e_2, v\rangle = 0$.
(iii) There is a vector $v \in \mathbb{R}^3 \setminus \{0\}$ such that $\langle e_3, v\rangle$ is constant.
(iv) The quotient $\dfrac{\tau}{\kappa}$ is constant.

In particular, in this case all rectifying planes contain a fixed vector v. Such curves are called *slope lines*, because they run up or down a surface with a constant slope. Also spherical curves can be slope lines – see Figure 2.6 as well as the exercises at the end of the chapter.

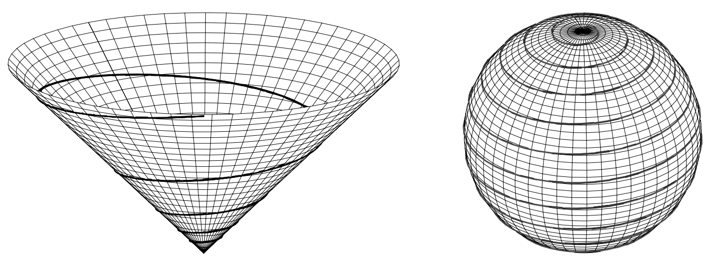


Figure 2.6. Slope line in a cone and in a sphere

Proof: For a plane curve this is trivial, since $\tau = 0$ and v can be chosen as a normal to that plane. So let us assume $\tau \neq 0$.

(i) $\Leftrightarrow$ (ii): $0 = \langle e_1, v\rangle' = \langle e_1', v\rangle$ implies that $\langle e_2, v\rangle = 0$, since $e_1' = \kappa e_2$ and conversely (note that $\kappa \neq 0$). (iii) $\Leftrightarrow$ (ii) follows similarly from $0 = \langle e_3, v\rangle' = -\tau\langle e_2, v\rangle$.

(i), (ii), (iii) together imply $v = \alpha e_1 + \beta e_3$ with constants α, β. Since in addition v is constant, one has $0 = \alpha e_1' + \beta e_3' = (\alpha\kappa - \beta\tau)e_2$, hence $\frac{\tau}{\kappa} = \frac{\alpha}{\beta}$. Conversely, if $\frac{\tau}{\kappa}$ is constant, this implies that also

$$v := \tfrac{\tau}{\kappa} e_1 + e_3$$

is constant, because $v' = \frac{\tau}{\kappa} e_1' + e_3' = \frac{\tau}{\kappa}\kappa e_2 - \tau e_2 = 0$. This implies (i), (ii), (iii) because $\langle e_2, v\rangle = 0$.

The vector $\kappa v = \tau e_1 + \kappa e_3$, which points in the same direction, is also interesting for other curves, and is called the *Darboux rotation vector*, see 2.12. $\square$

In particular, a curve is a slope line whenever both κ and τ are constant. This case can be completely classified as follows.

2.12. Example. (Curves with constant Frenet curvature in $I\!R^3$)
For given constants a, b, α, the circular helix

$$c(t) = \big(a\cos(\alpha t), a\sin(\alpha t), bt\big)$$

is a Frenet curve in $I\!R^3$ when $a > 0, \alpha \neq 0$, see Figure 2.1. This curve is parametrized by arc length when

$$1 = \alpha^2 a^2 + b^2.$$

It then has constant Frenet curvature κ and constant torsion τ with

$$\kappa^2 = \alpha^4 a^2,$$

$$\tau^2 = \alpha^2 b^2.$$

Conversely, for given constants κ, τ the system of the three equations above has the unique solution

$$\begin{aligned} \alpha^2 &= \kappa^2 + \tau^2, \\ a &= \kappa/(\kappa^2 + \tau^2), \\ b^2 &= \tau^2/(\kappa^2 + \tau^2). \end{aligned}$$

Consequence: Every Frenet curve in $I\!R^3$ with constant curvature κ and constant torsion τ is a part of a circular helix. The special case in which $\tau = 0$ is the case of a circle.

REMARKS: 1. The angular velocity α occurs in the normal form of the skew-symmetric matrix

$$K = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix};$$

one can calculate $-\alpha^2$ as the unique non-zero eigenvalue of the squared (and hence symmetric) matrix

$$K^2 = \begin{pmatrix} -\kappa^2 & 0 & \kappa\tau \\ 0 & -\kappa^2 - \tau^2 & 0 \\ \kappa\tau & 0 & -\tau^2 \end{pmatrix}.$$

The characteristic polynomial of this matrix is

$$\mathrm{Det}(K^2 - \lambda \cdot \mathrm{Id}) = -\lambda(\kappa^2 + \tau^2 + \lambda)^2,$$

hence $-\alpha^2 = -(\kappa^2 + \tau^2)$ is the sole non-vanishing eigenvalue. This determines α (up to sign), and we obtain from the above equations the result $a = \kappa/\alpha^2$ and $b = \pm\tau/\alpha$.

2. For every Frenet curve in $\mathbb{R}^3$ and every point p on that curve, there is a uniquely determined *accompanying helix* such that both curves have the same Frenet three-frame at the point p as well as having the same curvature and torsion. The screw-motion itself can be viewed as an accompanying motion to the curve. The *Darboux rotation vector*

$$D = \tau e_1 + \kappa e_3$$

should be seen in this context as well. It is contained in the rectifying plane and describes the accompanying screw-motion given by its direction (this is the axis of motion) and its length (this is the angular velocity of the motion). The *Darboux equations*

$$e_i' = D \times e_i \quad \text{for } i = 1, 2, 3$$

are then just a variant of the Frenet equations. See the exercises at the end of the chapter. For the helix considered above with $\alpha > 0$, the value of D is given by $D = (0, 0, \sqrt{\kappa^2 + \tau^2}) = (0, 0, \alpha)$, as is easily verified.

Chapter 5

Riemannian Manifolds

In this chapter we want to introduce the notion of an "intrinsic geometry" without making reference to an ambient space $\mathbb{R}^{n+1}$, not only locally, but also as a global notion. This continues the considerations of Chapter 4. The most important tools for this are on the one hand, from a local point of view, a notion of "first fundamental form" independent of an ambient space $\mathbb{R}^{n+1}$ (similar to the notion of intrinsic geometry in the previous chapter), and on the other hand, from a global point of view, the notion of a "manifold". The local notion goes back essentially to the famous lecture of Riemann[1], which explains the modern notions *Riemannian geometry*, *Riemannian manifold* and *Riemannian space*. From the point of view of the development in the book up to now, this is motivated on the one hand by the intrinsic geometry of surfaces, including the Gauss-Bonnet theorem, and on the other hand by the natural occurrence of such spaces which can *not* in any meaningful way be embedded as hypersurfaces in some $\mathbb{R}^n$, as for example the Poincaré upper half-plane as a model of non-Euclidean geometry. Furthermore, the space-times of $3+1$ dimensions which are considered in general relativity do not admit an ambient space in a natural way. This motivates the intention of explaining all geometric quantities in a purely intrinsic manner.

[1] B. Riemann, *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, edited by H. Weyl, Springer, 1921; see also [**7**], Vol. II, Chapter 4.

In the previous Chapters 3 and 4 we have basically been considering surface elements $f\colon U \to \mathbb{R}^{n+1}$, where $U \subset \mathbb{R}^n$ was a given open set. From a geometric point of view, we are really more interested in the image set $f(U)$ than we are in the map f itself. Nonetheless, for a description and for local calculations we do use the parameter set U and the parametrization f:

$$U \ni u \overset{f}{\longmapsto} p = f(u) \in f(U).$$

If we decide that the basic object we are considering is the image $f(U)$, then we come to view the inverse mapping

$$f(U) \ni p \overset{f^{-1}}{\longmapsto} u \in U$$

as an image which is "thrown" from $f(U)$, in order to carry out calculations in U. This map is called a "chart" in what follows, which should be thought of as creating a "map" (but the word "map" has a fixed, different meaning in mathematics, so that one uses "chart" instead), and a set of charts which cover the object of interest forms an "atlas", just as a world atlas contains a map containing an arbitrary location on the earth. For the mathematical notion this means that every point has a neighborhood which is contained in one of the charts, in which local computations near that point can be carried out in the corresponding set U. What we have to be able to guarantee is that all defined notions are *independent* of the choice of charts used, just as the Gaussian curvature in the theory of surfaces was independent of the parametrization. In particular, we need to carefully consider the transformations which map us from one chart into a different, nearby one.

5A The notion of a manifold

We have already met submanifolds of $\mathbb{R}^n$ in the form of zero sets of differentiable maps, cf. Chapter 1. If there is no ambient space to begin with, this definition no longer makes any sense. Instead, one uses a description in terms of local coordinates in the form of parametrizations or *charts*, just as one considers maps of the earth to approximate that round object by flat pictures. Note that the chart maps go in the opposite direction from the usual parametrization we have been using up to now.

5.1. Definition. (Abstract differentiable manifold)
A *k-dimensional differentiable manifold* (briefly: a *k-manifold*) is a set M together with a family $(M_i)_{i\epsilon I}$ of subsets such that

1. $M = \bigcup_{i\in I} M_i$ (union),
2. for every $i \in I$ there is an injective map $\varphi_i \colon M_i \to \mathbb{R}^k$ so that $\varphi_i(M_i)$ is open in $\mathbb{R}^k$, and
3. for $M_i \cap M_j \neq \emptyset$, $\varphi_i(M_i \cap M_j)$ is open in $\mathbb{R}^k$, and the composition
$$\varphi_j \circ \varphi_i^{-1} \colon \varphi_i(M_i \cap M_j) \to \varphi_j(M_i \cap M_j)$$
is differentiable for arbitrary i, j.

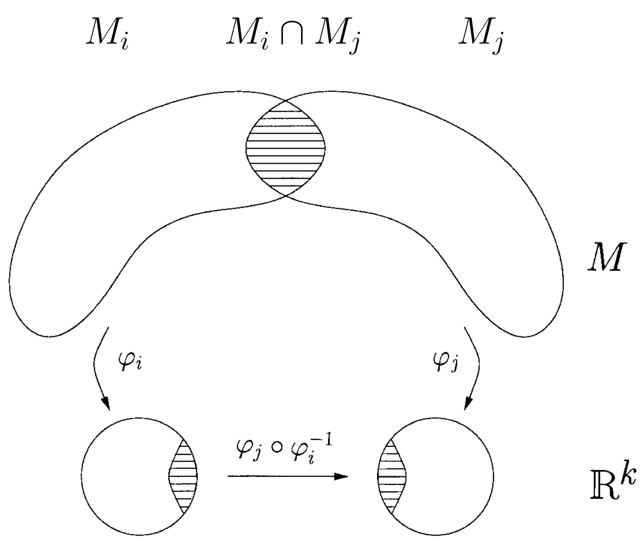


Figure 5.1. Charts on a manifold

Each φ_i is called a *chart*, φ_i^{-1} is referred to as the *parametrization*, the set $\varphi_i(M_i)$ is called the *parameter domain*, and $(M_i, \varphi_i)_{i\in I}$ is called an *atlas*. The maps $\varphi_j \circ \varphi_i^{-1} \colon \varphi_i(M_i \cap M_j) \to \varphi_j(M_i \cap M_j)$, defined on the intersections of two such charts, are called *coordinate transformations* or *transition functions*. Without restriction of generality, we may assume that the atlas is *maximal* with respect to adding more charts satisfying the conditions 2 and 3 above. A maximal atlas in this sense is then referred to as a *differentiable structure*.

EXAMPLES:

1. Every open subset U of $\mathbb{R}^k$ is a k-manifold, where a single chart is sufficient for the entire manifold, namely the inclusion map $\varphi\colon U \to \mathbb{R}^k$. Condition 3 is trivially satisfied in this case.
2. Every k-dimensional submanifold M of $\mathbb{R}^n$ (cf. Chapter 1) is also a k-dimensional manifold in the sense of the above definition. If M is given locally by $M = \{x \in \mathbb{R}^n \mid F(x) = 0\}$, where $F : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is a continuously differentiable submersion (i.e., the differential DF is surjective, or in other words $\mathrm{Rank}(DF) = n - k$), then according to the implicit functions theorem one can locally solve the equation
$$F(x^1, \dots, x^n) = 0$$
(perhaps after a renumbering) in the explicit form
$$\begin{aligned} x^{k+1} &= x^{k+1}(x^1, \dots, x^k), \\ &\vdots \\ x^n &= x^n(x^1, \dots, x^k). \end{aligned}$$
By making the association
$$(x^1, \dots, x^k) \longmapsto (x^1, \dots, x^k, x^{k+1}, \dots, x^n),$$
we get a parametrization, while the association $(x^1, \dots, x^n) \longmapsto (x^1, \dots, x^k)$ gives us a chart.
3. The (abstract) torus $\mathbb{R}^2/\mathbb{Z}^2$ is defined as the quotient (group) of these two Abelian groups. To give it a differentiable structure, one defines charts by starting with arbitrary open sets M_i in $\mathbb{R}^2$ (more precisely, take their images in the quotient) which are contained in the open square $(x_0 - \frac{1}{2}, x_0 + \frac{1}{2}) \times (y_0 - \frac{1}{2}, y_0 + \frac{1}{2})$ for an arbitrary point $(x_0, y_0) \in \mathbb{R}^2$. Then set $\varphi(x, y) := (x - x_0, y - y_0)$ to obtain one chart (depending on the choice of (x_0, y_0)). It

follows that the coordinate transformations are just translations in $\mathbb{R}^2$. One sees without difficulty that three of these charts suffice to cover the image, namely the just mentioned squares centered at the points $(0,0)$, $(\frac{1}{3}, \frac{1}{3})$, $(\frac{2}{3}, \frac{2}{3})$. Two such sets do *not* suffice.

Similar results, with appropriate modifications, hold also for the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$.

4. The (abstract) *Klein bottle* is a quotient of the two-dimensional torus by the involution $(x,y) \mapsto (x+\frac{1}{2}, -y)$. We may take any square in the (x,y)-plane whose length in the x-direction is at most $\frac{1}{2}$ and whose length in the y-direction is at most 1, as charts.

5. The real projective plane is the quotient of the two-sphere

$$\mathbb{R}P^2 := S^2/\sim,$$

where the equivalence relation is given by $x \sim -x$. We may take any open set in S^2 as M_i, provided it is contained in a hemisphere (by which we mean half a sphere), and in particular contains no antipodal points. φ can be defined as a projection to a hemisphere, followed by a projection of this onto a disc.
A model of this is the closed disc modulo the identification of the antipodal pairs of points on the boundary. On the other hand, the "classical" model of projective geometry is all of $\mathbb{R}^2$ with an added "line at infinity".
An atlas of the projective plane containing three charts can be constructed as the charts induced by the centrally symmetric atlas on S^2, which consists of the six hemispheres in the three directions (x^1, x^2, x^3).

6. The *rotation group* $\mathbf{SO}(3)$ is defined as the set of all (real) orthogonal (3×3)-matrices with determinant equal to 1. We show that it is a 3-manifold by defining the *Cayley map*

$$CAY\colon \mathbb{R}^3 \to \mathbf{SO}(3), \quad CAY(A) := (\mathbf{1}+A)(\mathbf{1}-A)^{-1}.$$

Here $\mathbf{1}$ denotes the unit matrix, and A denotes the skew-symmetric matrix

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

with real parameters a, b, c, which can also be viewed as an element of $\mathbb{R}^3$. The Cayley map is injective, and the inverse map can be used as a chart of $\mathbf{SO}(3)$ and determined as follows:

$$CAY(A) = B \Longleftrightarrow B(\mathbf{1} - A) = \mathbf{1} + A$$

$$\Longleftrightarrow (B + \mathbf{1})A = B - \mathbf{1} \Longleftrightarrow A = (B + \mathbf{1})^{-1}(B - \mathbf{1}).$$

Note that $B + \mathbf{1}$ is always invertible, except when -1 is an eigenvalue of B. The matrices B for which this last condition holds are precisely the rotations by π. In fact, the image of the Cayley map is all of $\mathbf{SO}(3)$ with the exception of the set of rotation matrices by a rotational angle of π.
The set of all such rotations by π is naturally bijective to the set of all possible axes of rotation, hence bijective to a projective plane $\mathbb{R}P^2$. To get charts covering this exceptional set of $\mathbf{SO}(3)$, we require three more charts, just as in the above example of an atlas for the projective plane. If we define $\mathbf{E_i}$ as the rotation matrix by an angle of π around the ith axis, and if we formally set $\mathbf{E_0} = \mathbf{1}$, then the following four maps (resp. their inverses) define an atlas of $\mathbf{SO}(3)$:[2]

$$A \mapsto \mathbf{E_i} \cdot CAY(A), \quad i = 0, 1, 2, 3.$$

The four parametrizations of the atlas thus consist of the Cayley maps "centered at" $\mathbf{1}, \mathbf{E_1}, \mathbf{E_2}, \mathbf{E_3}$. The transformations from one chart to another are given by matrix multiplication and are therefore differentiable.

[2] I am indebted to Prof. E. Grafarend for a question giving rise to this, which arose from applications in geodesy. Traditionally one considers in geodesy only a single chart for the rotation group, yielding the *Euler angles* or *Cardan angles*.

5.2. Definition. (Structures on a manifold)

Given a k-dimensional manifold, one gets *additional structure* by placing additional requirements on the transformation functions $\varphi_j \circ \varphi_i^{-1}$, which belong to the atlas of the manifold; if all $\varphi_j \circ \varphi_i^{-1}$ are (left-hand side), then one speaks of (right-hand side) as follows:

continuous	$\leftrightarrow$	topological manifold
differentiable	$\leftrightarrow$	differentiable manifold
C^1-differentiable	$\leftrightarrow$	C^1-manifold
C^r-differentiable	$\leftrightarrow$	C^r-manifold
C^∞-differentiable	$\leftrightarrow$	C^∞-manifold
real analytic	$\leftrightarrow$	real analytic manifold
complex analytic	$\leftrightarrow$	complex manifold of dimension $\frac{k}{2}$
affine	$\leftrightarrow$	affine manifold
projective	$\leftrightarrow$	projective manifold
conformal	$\leftrightarrow$	manifold with a conformal structure
orientation-preserving	$\leftrightarrow$	orientable manifold

Convention: In what follows we shall understand by the term "manifold" a C^∞-manifold, and "differentiable" will always mean C^∞. One can show that a C^k-atlas always contains a C^∞ one, so that this convention is not a real restriction.

5.3. Definition. (Topology)

A subset $O \subseteq M$ is called *open*, if $\varphi_i(O \cap M_i)$ is open in $\mathbb{R}^k$ for every i. This defines a *topology* on M as the set of all open sets. Then all φ_i are continuous, since the inverse images under them of open sets are again open. M is said to be *compact*, if every open covering contains a finite sub-covering (Heine-Borel covering property). In particular, every compact manifold can be covered with finitely many charts.

Running assumption: In what follows we will always assume that the manifolds which occur satisfy the *Hausdorff separation axiom* (T_2-axiom), formulated as follows. Every two distinct points p, q have disjoint open neighborhoods U_p, U_q. Note that this property does not follow from Definition 5.1.

The important point here is that *locally* (or *in the small*) the topology of a manifold is the same as that of an $\mathbb{R}^k$. In particular this means that the inverse images of open ε-balls in $\mathbb{R}^k$ are again open in M, although one cannot necessarily make sense of the notion of ε-balls there, as there is no distance function (metric) defined. But this suffices to define the notion of convergence of sequences just as in $\mathbb{R}^k$. In addition, the topology of every manifold is locally compact, which means that every point has a compact neighborhood, for example the inverse image of a closed ε-ball in $\mathbb{R}^k$.

5.4. Definition. (Differentiable map)
Let M be an m-dimensional differentiable manifold, and let N be an n-dimensional differentiable manifold; furthermore, let $F\colon M \to N$ be a given map. F is said to be *differentiable*, if for all charts $\varphi : U \to \mathbb{R}^m, \psi : V \to \mathbb{R}^n$ with $F(U) \subset V$ the composition $\psi \circ F \circ \varphi^{-1}$ is also differentiable.

In particular this defines the concept of a *differentiable function* $f\colon M \to \mathbb{R}$, where in this case $\mathbb{R}$ carries the (identity) standard chart.

This definition is independent of the choice of φ and ψ. A *diffeomorphism* $F : M \to N$ is defined to be a bijective map which is differentiable in both directions. One then calls the two manifolds M and N *diffeomorphic*. Two diffeomorphic manifolds necessarily have the same dimension. This is because for $\mathbb{R}^m$ and $\mathbb{R}^n$ with $n \neq m$, there is no bijective mapping which is differentiable in both directions, since the corresponding Jacobi matrix is not square and hence cannot have non-vanishing determinant (i.e., cannot be invertible).

Remark: With respect to additional structures on our manifold, one can similarly define when a map is analytic or complex analytic or affine, etc. For example, let us consider here the *Riemann sphere* $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. By means of the inclusion $\mathbb{C} \to \widehat{\mathbb{C}}$ one has a chart, and a second is given by $z \mapsto \frac{1}{z}$. These two charts define a *complex structure* on the Riemann sphere, if one adds all compatible charts. Then all meromorphic maps of the Riemann sphere to itself are differentiable maps in the sense of the above definition, for example, also the map $z \mapsto z^{-k}$. Furthermore, this defines a conformal structure on S^2 since every complex-analytic function $f(z)$ with $f' \neq 0$ in one variable z is conformal, cf. Section 3D.

Convention: For a chart φ we will denote by $(u^1, \ldots, u^k)$ the standard coordinates of $\mathbb{R}^k$, and by $(x^1, \ldots, x^k)$ the corresponding coordinates in M. Thus, $x^i(p)$ is the function given by the ith coordinate of $\varphi(p)$, $x^i(p) = u^i(\varphi(p))$. The functions $(u^1, \ldots, u^k)$ as well as $(x^1, \ldots, x^k)$ are thus on the one hand the coordinates of the points considered, while on the other hand $(u^1, \ldots, u^k)$ and $(x^1, \ldots, x^k)$ are also viewed as variables, with respect to which we can form derivatives. For a real-valued function $f : M \to \mathbb{R}$ we set

$$\left.\frac{\partial f}{\partial x^i}\right|_p := \left.\frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\right|_{\varphi(p)}$$

and emphasize this notation by thinking of the partial derivatives as infinitesimal changes of a function in the directions x^i or u^i.

5B The tangent space

Let M be an n-dimensional differentiable manifold and $p \in M$ a fixed point. The tangent space of M at the point p is going to be thought of as the n-dimensional set of "directional vectors", which – starting at p – point in all directions of M, cf. for example [**27**]. Since there is no ambient space, this notion has to be intrinsically defined and constructed. For this, there are three possible definitions, all of which we describe here.

5.5. Definition. (Tangent vector, tangent space)
Geometric Definition:
A *tangent vector* at p is an equivalence class of differentiable curves $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = p$, where $c \sim c^* \Leftrightarrow (\varphi \circ c)\dot{}(0) = (\varphi \circ c^*)\dot{}(0)$ for every chart φ containing p.

Briefly: *tangent vectors are tangents to curves lying on the manifold.*

Unfortunately there is no privileged representative of such an equivalence class, and such a representative would depend on the choice of chart (for example, a line in the parameter domain).

Algebraic Definition:
A *tangent vector* X at p is a derivation (derivative operator) defined on the set of *germs of functions*

$$\mathcal{F}_p(M) := \{f : M \to \mathbb{R} \mid f \text{ differentiable}\}/\sim \quad ,$$

where the equivalence relation $\sim$ is defined by declaring $f \sim f^*$ if and only if f and f^* coincide in a neighborhood of p. The value $X(f)$ is also referred to as the *directional derivative* of f in the direction X.

This definition means more precisely the following. X is a map $X : \mathcal{F}_p(M) \to \mathbb{R}$ with the two following properties:

1. $X(\alpha f + \beta g) = \alpha X(f) + \beta X(g),\ \ \alpha, \beta \in \mathbb{R},\ \ f, g \in \mathcal{F}_p(M)$ (*$\mathbb{R}$-linearity*);
2. $X(f \cdot g) = X(f) \cdot g(p) + f(p) \cdot X(g)$ for $f, g \in \mathcal{F}_p(M)$ (*product rule*).
   (For this to make sense, both f and g have to be defined in a neighborhood of p.)

Briefly: *tangent vectors are derivations acting on scalar functions.*

Physical Definition:
A *tangent vector* at the point p is defined as an n-tuple of real numbers $(\xi^i)_{i=1,\ldots,n}$ in a coordinate system $x^1, \ldots, x^n$ (that is, in a chart), in such a way that in any other coordinate system $\widetilde{x}^1, \ldots, \widetilde{x}^n$ (i.e., in any other chart) the same vector is given by a corresponding n-tuple $(\widetilde{\xi}^i)_{i=1,\ldots,n}$, where

$$\widetilde{\xi}^i = \sum_j \frac{\partial \widetilde{x}^i}{\partial x^j}\Big|_p \xi^j.$$

Briefly: *tangent vectors are elements of $\mathbb{R}^n$ with a particular transformation behavior.*

The *tangent space* T_pM of M at p is defined in all cases as the set of all tangent vectors at the point p. By definition T_pM and T_qM are disjoint if $p \neq q$.

For the special case of an open subset $U \subset \mathbb{R}^n$, the tangent space can be identified with $T_pU := \{p\} \times \mathbb{R}^n$ endowed with the standard basis $(p, e_1), \ldots, (p, e_n)$. The vector e_i corresponds to the curve $c_i(t) := p + t \cdot e_i$ (geometric definition) and to the derivation given by the partial derivative $f \longmapsto \frac{\partial f}{\partial u^i}\big|_p$ (algebraic definition). Therefore 5.5 is compatible with the previous definitions given in 1.7 and 3.1. The directional derivative coincides in $\mathbb{R}^n$ with the directional derivative which was already defined in 4.1.

Special (geometric) tangent vectors are those given by the parameter lines (lines along which parameter values are constant), formally meaning the equivalence classes of them. The corresponding special tangent vectors in the algebraic definition are the partial derivatives $\frac{\partial}{\partial x^i}\big|_p$ defined by

$$\frac{\partial}{\partial x^i}\Big|_p (f) := \frac{\partial f}{\partial x^i}\Big|_p = \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}\Big|_{\varphi(p)}$$

in a chart φ which contains p. As a notational convenience one also writes $\partial_i\big|_p$ instead of $\frac{\partial}{\partial x^i}\big|_p$. The special tangent vectors in the sense of the physical definition are in this case simply the tuples which consist of zeros except in the ith place.

The geometric definition is probably the most intuitive (*a tangent vector is a tangent to a curve*), but not easy to work with. In this definition it is not even clear that the tangent space is a real vector space. The algebraic definition is most convenient for doing computations, and by its very definition it is independent of any chart. The physical definition will be further clarified below. The art of doing computations with the geometric quantities of the physical definition goes back to G. Ricci and is called the *Ricci calculus*, cf. [**16**]. A vector is simply written as ξ^i, and the very fact that the notion involves a superscript indicates the transformation behavior, in this case, for example, as a vector (or 1-contravariant tensor), cf. Section 6.1. This aspect will be of importance in what follows, but for all definitions we will give a coordinate-independent formulation as far as this is feasible. The equivalence of these three definitions is explained for example in [**39**], Chapter 2. In what follows we base our analysis on the algebraic definition and will therefore not require this equivalence.

5.6. Theorem. The (algebraic) tangent space at p on an n-dimensional differentiable manifold is an n-dimensional $\mathbb{R}$-vector space and is spanned in any coordinate system $x^1, \ldots, x^n$ in a given chart by

$$\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p.$$

For every tangent vector X at p one has

$$X = \sum_{i=1}^{n} X(x^i)\frac{\partial}{\partial x^i}\Big|_p.$$

Looking at the last equation, we see that the components ξ^i of a tangent vector X in the Ricci calculus are nothing but the $X(x^i)$, that is, the directional derivatives of the coordinate functions x^i in the direction X. To prove the statement of the theorem we require the following lemma.

5.7. Lemma. If X is a tangent vector and f is a constant function, then $X(f) = 0$.

Proof: First suppose $f = 1$ everywhere. Then by the product rule 5.5.2 we have

$$X(1) = X(1 \cdot 1) = X(1) \cdot 1 + 1 \cdot X(1) = 2 \cdot X(1),$$

hence $X(1) = 0$. Now suppose that f has the constant value $f = c$. Then by the linearity 5.5.1 we have

$$X(c) = X(c \cdot 1) = c \cdot X(1) = c \cdot 0 = 0.$$

□

Proof of 5.6: The proof utilizes an adapted representation of the transition functions in local coordinates. We calculate in a chart $\varphi : U \longrightarrow V$, where without restricting generality we may assume V is an open ε-ball with $\varphi(p) = 0$, hence $x^1(p) = \cdots = x^n(p) = 0$. Let $h : V \to \mathbb{R}$ be a differentiable function, and $f := h \circ \varphi$. We set

$$h_i(y) := \int_0^1 \frac{\partial h}{\partial u^i}(t \cdot y)dt \quad (\text{note: } h \in C^\infty \Rightarrow h_i \in C^\infty)$$

and perform the following computation:

$$\sum_{i=1}^{n} \frac{\partial h}{\partial u^i}(t \cdot y) \cdot \underbrace{\frac{d(tu^i)}{dt}}_{=u^i} = \frac{\partial h}{\partial t}(t \cdot y),$$

which implies

$$\sum_{i=1}^{n} h_i(y) \cdot u^i = \int_0^1 \frac{\partial h}{\partial t}(t \cdot y)dt = h(y) - h(0).$$

From this we get, using the identities $f = h \circ \varphi,\ f_i = h_i \circ \varphi,\ x^i = u^i \circ \varphi$, the equation

$$f(q) - f(p) = \sum_{i=1}^{n} f_i(q) \cdot x^i(q)$$

for a variable point q. Taking derivatives, we get

$$\left.\frac{\partial f}{\partial x^i}\right|_p = f_i(p).$$

Now if we are given a tangent vector X at p, then it follows from properties 1 and 2 in 5.5 that

$$X(f) = X\Big(f(p) + \sum_{i=1}^{n} f_i x^i\Big) = 0 + \sum_{i=1}^{n} X(f_i) \cdot \underbrace{x^i(p)}_{=0} + \sum_{i=1}^{n} f_i(p) \cdot X(x^i)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\Big|_p \cdot X(x^i) = \Big(\sum_{i=1}^{n} X(x^i) \cdot \frac{\partial}{\partial x^i}\Big|_p\Big)(f)$$

for every f. It remains to show that the vectors $\frac{\partial}{\partial x^i}\big|_p$ are linearly independent. But this is easy to see, since $\frac{\partial}{\partial x^i}\big|_p(x^j) = \frac{\partial x^j}{\partial x^i} = \delta_i^j$.

Note that this proof only works for C^∞-manifolds, as otherwise the degree of differentiability of h_i is one less that of h. In fact, the algebraic tangent space on a C^k-manifold is infinite-dimensional. But there are no difficulties in simply passing to the subspace spanned by $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ and performing the same calculations there. □

5.8. Definition and Lemma. (Derivative, chain rule)
Let $F : M \to N$ be a differentiable map, and let p, q be two fixed points with $F(p) = q$. Then the *derivative* or the *differential* of F at p is defined as the map

$$DF|_p : T_pM \longrightarrow T_qN$$

whose value at $X \in T_pM$ is given by $(DF|_p(X))(f) := X(f \circ F)$ for every $f \in \mathcal{F}_q(N)$ (which automatically implies the relation $f \circ F \in \mathcal{F}_p(M)$). For the derivative as defined in this manner, one has the *chain rule* in the form

$$D(G \circ F)|_p = DG|_{F(p)} \circ DF|_p$$

for every composition $M \xrightarrow{F} N \xrightarrow{G} Q$ of maps, or, more briefly, $D(G \circ F) = DG \circ DF$.

PROOF: By definition we have

$$D(G \circ F)|_p(X)(f) = X(f \circ G \circ F)$$

$$= (DF|_p(X))(f \circ G) = \Big(DG|_q\big(DF|_p(X)\big)\Big)(f).$$

□

REMARK: One can view $DF|_p$ as a linear approximation of F at p, just as in vector analysis on $\mathbb{R}^n$. In coordinates $x^1, \ldots, x^m$ on M and $y^1, \ldots, y^n$ on N, $DF|_p$ is represented by the Jacobi matrix, for which we have the more precise relation

$$DF|_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \sum_i \frac{\partial(y^i \circ F)}{\partial x^j}\Big|_p \frac{\partial}{\partial y^i}\Big|_q.$$

In the physical definition of tangent spaces, the chain rule consists essentially of the product of the Jacobi matrices, applied to the tangent vector. In the geometric definition of the tangent space (i.e., for equivalence classes of curves through the point p), the differential is simply described by the transport of curves, as follows:

$$DF|_p([\ c\ ]) := [\ F \circ c\ ],$$

and the chain rule $DG\big(DF([c])\big) = [G \circ F \circ c]$ is then quite obvious. Note the action on the tangent of a curve:

$$\dot{c}(0) \mapsto (F \circ c)^{\cdot}(0) = DF|_p(\dot{c}(0)).$$

EXAMPLES:

(i) In case $F : U \to \mathbb{R}^{n+1}$ $(U \subset \mathbb{R}^n)$ is a surface element in the sense of Chapter 3 with $u \mapsto F(u) = p$, then the differential of F acts in the following way on the basis $\frac{\partial}{\partial u^1}\big|_u, \ldots, \frac{\partial}{\partial u^n}\big|_u$ of T_uU resp. $\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^{n+1}}\big|_p$ of $T_p\mathbb{R}^{n+1}$:

$$DF\Big|_u\Big(\frac{\partial}{\partial u^j}\Big|_u\Big) = \sum_i \frac{\partial x^i}{\partial u^j}\Big|_u \cdot \frac{\partial}{\partial x^i}\Big|_p,$$

where the matrix $\frac{\partial x^i}{\partial u^j}$ is the familiar *Jacobi matrix* of the mapping F. Here, x^i is the ith component of $F(u^1, \ldots, u^n)$, also written as the function $x^i(u^1, \ldots, u^n)$.

(ii) If $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ are two coordinate systems on a single manifold, then one has similarly, for F equal to the identity,

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j}\ \frac{\partial}{\partial y^i}.$$

(iii) For the components ξ^i and η^j, respectively, of a tangent vector $X = \sum_j \xi^j \frac{\partial}{\partial x^j} = \sum_i \eta^i \frac{\partial}{\partial y^i}$, one has similarly $X = \sum_j \xi^j \frac{\partial}{\partial x^j} = \sum_{i,j} \xi^j \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}$; hence $\eta^i = \sum_j \xi^j \frac{\partial y^i}{\partial x^j}$. This is precisely the transformation behavior of tangent vectors in Ricci calculus (Definition 5.5).

The following *summation convention* is used in Ricci calculus, and is usually referred to as the Einstein summation convention: sums are formed over indices which occur in formulas as both an upper (in the numerator) and a lower (in the denominator) subscript, without the explicit summation symbol, for example

$$h_{ik} = h_i^j g_{jk} \quad \text{instead of} \quad h_{ik} = \sum_j h_i^j g_{jk} \text{ and}$$

$$\eta^i = \xi^j \frac{\partial y^i}{\partial x^j} \quad \text{instead of} \quad \eta^i = \sum_j \xi^j \frac{\partial y^i}{\partial x^j}.$$

5.9. Definition. (Vector field)
A differentiable *vector field* X on a differentiable manifold is an association $M \ni p \longmapsto X_p \in T_pM$ such that in every chart $\varphi \colon U \to V$ with coordinates $x^1, \ldots, x^n$, the coefficients $\xi^i \colon U \to I\!R$ in the representation (valid at a point)

$$X_p = \sum_{i=1}^n \xi^i(p) \frac{\partial}{\partial x^i}\bigg|_p$$

are differentiable functions.

Another common notation for this is $X = \sum_i \xi^i \frac{\partial}{\partial x^i}$ or, in Ricci calculus, $X = \xi^i$. Note that in the physical definition, a vector field is identified with the n-tuple $(\xi^1, \ldots, \xi^n)$ of functions of the coordinates $x^1, \ldots, x^n$.

As to the notations used in conjunction with vector fields, for a scalar function $f : M \to I\!R$, the symbol fX denotes the vector field $(fX)_p := f(p) \cdot X_p$ (one can say that *the set of vector fields is a module over the ring of functions f on M*), while the symbol $Xf = X(f)$

denotes the function $(Xf)(p) := X_p(f)$ (in other words, Xf is the derivative of f in the direction of X).

5C Riemannian metrics

The first fundamental form of a surface element is a scalar product, which is defined by restricting the Euclidean scalar product to each tangent space, as we have explained in Chapter 3. In our present endeavor, we have to find a way to do this without the ambient space, that is, defining (intrinsically) a scalar product on each tangent space. Recall the following fact from linear algebra, which we will require in this regard.

The space $L^2(T_pM; I\!R) = \{\alpha\colon T_pM \times T_pM \to I\!R \mid \alpha \text{ bilinear}\}$ has the basis

$$\{dx^i|_p \otimes dx^j|_p \mid i, j = 1, \ldots, n\},$$

where the dx^i form the *dual basis* in the dual space

$$(T_pM)^* = L(T_pM; I\!R),$$

defined as follows:

$$dx^i\Big|_p\Big(\frac{\partial}{\partial x^j}\Big|_p\Big) = \delta^i_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

The bilinear forms $dx^i|_p \otimes dx^j|_p$ are defined in terms of their action on the basis (this action being then extended by linearity):

$$(dx^i|_p \otimes dx^j|_p)\Big(\frac{\partial}{\partial x^k}\Big|_p, \frac{\partial}{\partial x^l}\Big|_p\Big) := \delta^i_k \delta^j_l = \begin{cases} 1 & \text{if } i = k \text{ and } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

By inserting the basis, for the coefficients of the representation

$$\alpha = \sum_{i,j} \alpha_{ij} \cdot dx^i \otimes dx^j$$

one obtains the expression

$$\alpha_{ij} = \alpha\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big).$$

In Ricci calculus, the form α is just represented by the symbol α_{ij}; one also refers to this as a *tensor of degree two*, cf. 6.1.

5.10. Definition. (Riemannian metric)
A *Riemannian metric* g on M is an association $p \longmapsto g_p \in L^2(T_pM; \mathbb{R})$ such that the following conditions are satisfied:

1. $g_p(X,Y) = g_p(Y,X)$ for all X, Y, (*symmetry*)
2. $g_p(X,X) > 0$ for all $X \neq 0$, (*positive definiteness*)
3. The coefficients g_{ij} in every local representation (i.e., in every chart)
$$g_p = \sum_{i,j} g_{ij}(p) \cdot dx^i|_p \otimes dx^j|_p$$
are differentiable functions. (*differentiability*)

The pair (M, g) is then called a *Riemannian manifold*. One also refers to the Riemannian metric as the *metric tensor*. In local coordinates the metric tensor is given by the matrix (g_{ij}) of functions. In Ricci calculus this is simply written as g_{ij}.

Remarks:

1. A Riemannian metric g defines at every point p an *inner product* g_p on the tangent space T_pM, and therefore the notation $\langle X, Y \rangle$ instead of $g_p(X,Y)$ is also used. The notions of angles and lengths are determined by this inner product, just as these notions are determined by the first fundamental form on surface elements. The length or norm of a vector X is given by $||X|| := \sqrt{g(X,X)}$, and the angle β between two tangent vectors X and Y can be defined by the validity of the equation $\cos\beta \cdot ||X|| \cdot ||Y|| = g(X,Y)$, cf. Chapter 1.

2. If the condition that g is positive definite is replaced by the weaker condition that it is *non-degenerate* (meaning that $g(X,Y) = 0$ for all Y implies $X = 0$), then one arrives at the notion of a *pseudo-Riemannian metric* or *semi-Riemannian metric*, in which all notions are defined in exactly the same way as for a Riemannian metric. In particular, a so-called *Lorentzian metric* is defined as one for which the signature of g is $(-,+,+,+)$; such metrics are basic to the general theory of relativity. In this case the tangent spaces are modeled after

Minkowski space $I\!R^4_1$ instead of Euclidean space (cf. Section 3E) with the metric

$$(g_{ij}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The difference compared with Euclidean space is that there are vectors $X \neq 0$ with $g(X, X) = 0$, so-called *null vectors*. We have already studied the three-dimensional Minkowski space in connection with curves and surfaces (compare sections 2E and 3E). The tensor g_{ij} is referred to in the theory of relativity as the *gravitational potential* or *gravitational field*, cf. [**25**], Section 1.3. It gives a metric form to the manifold (four-dimensional space-time) according to the gravity coming from the matter which is contained in the space.

Examples:

(i) The first fundamental form g of a hypersurface element in $I\!R^{n+1}$ is an example of a Riemannian metric.

(ii) The standard example is $(M, g) = (I\!R^n, g_0)$, where the metric $(g_0)_{ij} = \delta_{ij}$ (identity matrix) is the Euclidean metric in the standard chart of $I\!R^n$ (given by Cartesian coordinates). This space is also referred to as *Euclidean space* and denoted by $I\!E^n$. The metric is

$$(g_0)_{ij} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix},$$

so that $g_0(.,.) = \langle \cdot, \cdot \rangle$ is, not unexpectedly, nothing but the *Euclidean inner product*.

(iii) A different Riemannian metric on $I\!R^n$ is given for example by $g_{ij}(x_1, \dots, x_n) := \delta_{ij}(1 + x_i x_j)$:

$$(g_{ij}) = \begin{pmatrix} 1 + x_1^2 & 0 & \dots & 0 \\ 0 & 1 + x_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 + x_n^2 \end{pmatrix}$$

Similarly, one can define numerous Riemannian metrics simply by choosing the coefficients g_{ij} arbitrarily, provided only that one has positive definiteness or non-degeneracy of the metric.

(iv) After choosing constants $0 < b < a$, on $(0, 2\pi) \times (0, 2\pi) \subset \mathbb{R}^2$, $0 < r < 1$, one can define a Riemannian metric by

$$(g_{ij}(u, v)) = \begin{pmatrix} b^2 & 0 \\ 0 & (a + b\ \cos u)^2 \end{pmatrix}.$$

This coincides with the first fundamental form on an open subset of the torus of revolution (cf. Chapter 3).

(v) We can give the abstract torus $\mathbb{R}^2/\mathbb{Z}^2$ a uniquely defined Riemannian metric g with the property that the projection

$$(\mathbb{R}^2, g_0) \longrightarrow (\mathbb{R}^2/\mathbb{Z}^2, g)$$

is a local isometry in the sense of 5.11. This is called the *flat torus*. In the chart $(0, 1) \times (0, 1)$ the metric is $(g_{ij}) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, as in the Euclidean plane.

(vi) Similarly, the real projective plane $\mathbb{R}P^2 = S^2/\pm$ can be given a unique Riemannian metric g such that the projection $(S^2, g_1) \to (\mathbb{R}P^2, g)$ is a local isometry in the sense of 5.11, where g_1 is the standard metric on the unit sphere.

(vii) The Poincaré upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the metric

$$(g_{ij}(x, y)) := \frac{1}{y^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is a Riemannian manifold. In this metric, length is given by $||\frac{\partial}{\partial y}|| = \frac{1}{y}$; thus the half-lines in the y-direction have infinite length: $\int_\eta^1 \frac{1}{t} dt = -\log(\eta) \longrightarrow \infty$ for $\eta \to 0$ and $\int_1^\eta \frac{1}{t} dt = \log(\eta) \longrightarrow \infty$ for $\eta \to \infty$. In fact, every geodesic is of infinite length in both directions. We refer also to the Exercises at the end of Chapter 4 as well as Section 7A for more details.

5.11. Definition. (Maps which are compatible with the metric)
A differentiable map $F\colon M \longrightarrow \widetilde{M}$ between two Riemannian manifolds $(M,g), (\widetilde{M},\widetilde{g})$ is called a *(local) isometry*, if for all points p and tangent vectors X, Y we have

$$\widetilde{g}_{F(p)}(DF|_p(X), DF|_p(Y)) = g_p(X,Y);$$

more generally, F is called a *conformal mapping*, if there is a function $\lambda\colon M \to \mathbb{R}$ without zeros, such that for all p, X, Y, one has

$$\widetilde{g}_{F(p)}(DF_p(X), DF_p(Y)) = \lambda^2(p) g_p(X,Y).$$

See also Definitions 3.29 and 4.29.

By definition a local isometry preserves the length of a vector, angles, and areas and volumes, whereas a conformal mapping preserves angles but rescales the length of any vector by the factor λ.

Examples: The map $(x,y) \mapsto (\cos x, \sin x, y)$ is a local isometry of the plane onto a cylinder. Stereographic projection defines a conformal map between the plane and the punctured sphere.

Question: Does there exist a Riemannian metric on an arbitrary manifold M? Locally there is no problem in constructing one, as we choose any (g_{ij}) which is both positive definite and symmetric. To make this construction global, one can use the method of a *partition of unity*. To introduce this notion, we define the following

Notation: For a given function $f\colon M \to \mathbb{R}$, the topological closure

$$supp(f) := \overline{\{x \in M \mid f(x) \neq 0\}}$$

is called the *support of* f.

5.12. Definition and Lemma. (Partition of unity)

A differentiable *partition of unity* on a differentiable manifold M is a family $(f_i)_{i \in I}$ of differentiable functions $f_i\colon M \to \mathbb{R}$ such that the following conditions are satisfied:

1. $0 \le f_i \le 1$ for all $i \in I$,
2. every point $p \in M$ has a neighborhood which intersects only finitely many of the $supp(f_i)$, and

3. $\sum_{i \in I} f_i \equiv 1$ (locally this is always to be a finite sum).

If there is a partition of unity on M such that the support $supp(f_i)$ of each function is contained in a coordinate neighborhood, then there exists a Riemannian metric on M.

Proof: For each $i \in I$ choose $g_{kl}^{(i)}$ as a symmetric, positive definite matrix-valued function (in the chart associated with $supp(f_i)$). This locally defines a Riemannian metric $g^{(i)}$, and $f_i \cdot g^{(i)}$ is differentiable and well-defined on all of M, namely, it vanishes identically outside of $supp(f_i)$. Then we set

$$g := \sum_{i \in I} f_i \cdot g^{(i)}.$$

It follows that g is symmetric and positive semi-definite because $f_i \geq 0$ and $g^{(i)} > 0$, and from $\sum_i f_i \equiv 1$ we see that g is even positive definite at every point. $\square$

Warning: The same method does *not* show the existence of an indefinite metric $\widetilde{g}$ on M, because in this case $\tilde{g}$ can degenerate, even if all $\widetilde{g}^{(i)}$ are non-degenerate. In fact, there are topological obstructions to the existence of indefinite metrics. For example there is a Lorentz metric of type $(- + + \cdots +)$ on a compact manifold if and only if the Euler characteristic satisfies $\chi = 0$. This is because precisely in this case, a line element field exists[3]. Among the compact surfaces, only the torus and the Klein bottle satisfy this condition.

We mention the following result without proof.

Theorem: If the topology of M (i.e., the system of open sets, cf. 5.3) is locally compact (which always holds for manifolds) and the second countability axiom is satisfied (there exists a countable basis for the topology), then there exists in every open covering an associated partition of unity, in the sense that $supp(f_i)$ is always contained in one of the given open sets.

For a proof, see for example [**40**]. In fact it is sufficient to make the (weaker) assumption that the space is paracompact.

[3] L. Markus, *Line element fields and Lorentz structures on differentiable manifolds*, Annals of Mathematics (2) **62**, 411–417 (1955).

Under the same assumptions there exists a Riemannian metric. In particular, the compactness of M implies the topological assumptions required. Thus, on an arbitrary compact manifold there exists a Riemannian metric.

5D The Riemannian connection

Just as at the beginning of Chapter 4, we have here the problem of defining the derivative on an abstract differentiable manifold or abstract Riemannian manifold not only for scalar functions (this is sufficiently done in the algebraic Definition 5.5), but also for vector fields. What we have to define is the notion of the derivative of a (tangent) vector field with respect to a tangent vector, with a result which is again a tangent vector. This will be defined in 5.13 in such a way that a Riemannian metric is not necessary and both arguments X and Y are treated equally. The so-called *Riemannian connection*, defined in 5.15, is nearer to the notion of covariant derivative of Chapter 4; in fact, it is just a generalization. Here we also require a compatibility with the Riemannian metric. The fundamental lemma of Riemannian geometry, presented in 5.16, shows the existence of a unique Riemannian connection for an arbitrary Riemannian metric.

5.13. Definition. (The Lie bracket[4])
Let X, Y be (differentiable) vector fields on M, and let $f\colon M \to \mathbb{R}$ be a differentiable function. Through the relation

$$[X,Y](f) := X(Y(f)) - Y(X(f))$$

we define a vector field $[X,Y]$, which is referred to as the *Lie bracket* of X, Y (also called the *Lie derivative* $\mathcal{L}_X Y$ of Y in the direction X). At a point $p \in M$ we have $[X,Y]_p(f) = X_p(Yf) - Y_p(Xf)$.

The Lie bracket measures the degree of non-commutativity of the derivatives. In Section 4.5 above we made a similar definition, namely $[X,Y] := D_X Y - D_Y X$, which in $\mathbb{R}^n$ is equivalent to the above definition. For the definition of the Lie bracket, no Riemannian metric is required; the differentiable structure is sufficient. The exercises

[4] Named after Sophus Lie, the founder of the theory of transformation groups.