Contents

IAS/Park City Mathematics Institute vii

Preface ix

Lecture 1. Simple Random Walk and Stirling's Formula 1

Lecture 2. Simple Random Walk in Many Dimensions 9

Lecture 3. Self-Avoiding Walk 15

Lecture 4. Brownian Motion 21

Lecture 5. Shuffling and Random Permutations 27

Lecture 6. Seven Shuffles are Enough (Sort of) 33

Lecture 7. Markov Chains on Finite Sets 39

Lecture 8. Markov Chain Monte Carlo 47

Lecture 9. Random Walks and Electrical Networks 53

Lecture 10. Uniform Spanning Trees 63

Lecture 11. Random Walk Simulations 69

Lecture 12. Other Simulations 75

Lecture 13. Simulations in Finance 81

Problems 85

Bibliography 99

IAS/Park City Mathematics Institute

The IAS/Park City Mathematics Institute (PCMI) was founded in 1991 as part of the "Regional Geometry Institute" initiative of the National Science Foundation. In mid 1993 the program found an institutional home at the Institute for Advanced Study (IAS) in Princeton, New Jersey. The PCMI will continue to hold summer programs alternately in Park City and in Princeton.

The IAS/Park City Mathematics Institute encourages both research and education in mathematics, and fosters interaction between the two. The three-week summer institute offers programs for researchers and postdoctoral scholars, graduate students, undergraduate students, high school teachers, mathematics education researchers, and undergraduate faculty. One of PCMI's main goals is to make all of the participants aware of the total spectrum of activities that occur in mathematics education and research: we wish to involve professional mathematicians in education and to bring modern concepts in mathematics to the attention of educators. To that end the summer institute features general sessions designed to encourage interaction among the various groups. In-year activities at sites around the country form an integral part of the High School Teacher Program.

Each summer a different topic is chosen as the focus of the Research Program and Graduate Summer School. Activities in the Undergraduate Program deal with this topic as well. Lecture notes from the Graduate Summer School are published each year in the IAS/Park City Mathematics Series. Course materials from the Undergraduate Program, such as the current volume, are now being published as part of the IAS/Park City Mathematical Subseries in the Student Mathematical Library. We are happy to make available more of the excellent resources which have been developed as part of the PCMI.

At the summer institute late afternoons are devoted to seminars of common interest to all participants. Many deal with current issues in education; others treat mathematical topics at a level which encourages broad participation. The PCMI has also spawned interactions between universities and high schools at a local level. We hope to share these activities with a wider audience in future volumes.

Robert Bryant and Dan Freed, Series Editors

May, 1999

Preface

These notes summarize two of the three classes held for undergraduates at the 1996 Park City/IAS Institute in Probability. There were twenty undergraduates participating, who were divided into an advanced group and a beginning group. Both groups participated in a class on computer simulations in probability. The beginner's class, taught by Emily Puckette, discussed Markov chains and random walks, covering material in the first few chapters of [**L3**]. This book gives notes from the advanced class and the computer class. The first ten lectures are those given to the advanced class by Greg Lawler, and the last three summarize the material in the computer class led by Lester Coyle. The material was coordinated so that some of the major simulations done in the computer class related to topics discussed in the advanced class. For this reason we have decided to combine these notes into one book.

The title of the advanced lecture series (and of these notes) is taken from a recent book [**S**], *Topics in Contemporary Probability and its Applications,* edited by J. Laurie Snell, which contains a number of survey articles that are accessible to advanced undergraduates and beginning graduate students. The lectures were based loosely on three of the papers in that book: "Random Walks: Simple and Self-Avoiding" by Greg Lawler, "How Many Times Should You Shuffle a Deck of Cards?" by Brad Mann, and "Uniform Spanning Trees" by

Robin Pemantle. The idea was to present some topics which are accessible to advanced undergraduates yet are areas of current research in probability.

The first lecture discusses simple random walk in one dimension and is anything but contemporary. It leads to a derivation of Stirling's formula. The second lecture discusses random walk in several dimensions and introduces the notion of power laws. Standard results about probability of return to the origin and the intersection exponent are discussed. The latter is a simply stated exponent whose value is not known rigorously today, and it is a natural exponent to study by simulation. The third lecture discusses the self-avoiding walk, which is a very good example of a simply stated mathematical problem for which most of the interesting questions are still open problems. The fourth lecture considers the continuous limit of random walk, Brownian motion. This topic was included to help those students who were involved in simulations related to finance.

The next two lectures consider the problem of shuffling a deck of cards. Lecture 5 discusses the general idea of random permutations and introduces the notion of a random walk on a symmetric group. The case of random riffle shuffles and the time (number of shuffles) needed to get close to the uniform distribution was analyzed in a paper of Bayer and Diaconis [**BD**], and Mann's paper [**M**] is an exposition of this result. We give a short discussion of this result, although we do not give all the details of the proof. This topic leads naturally to the discussion of Markov chains and rates of convergence to equilibrium. Lecture 7 is a standard introduction to Markov chains; it outlines a proof of convergence to equilibrium that emphasizes the importance of the size of the second eigenvalue for understanding the rate of convergence. Lecture 8 disucsses a recent important technique to sample from complicated distributions, Markov Chain Monte Carlo.

Lecture 9 discusses a very beautiful relationship between random walks and electrical networks (see [**DS**] for a nice exposition of this area). The basic ideas in this section are used in more sophisticated probability; this is basically the discrete version of Dirichlet forms. The work on electrical networks leads to the final lecture on uniform spanning trees. We discuss one result that relates three initially quite different objects: uniform spanning trees, random walks on graphs, and electrical networks.

The purpose of the computer class was to introduce students to the idea of Monte Carlo simulations and to give them a chance to do some nontrivial projects. The previous computer experience of the students varied widely, some having significant programming backgrounds and some having never computed. We first used Maple and then C as the languages for simulations. While these sections are labeled as "lectures" they actually represent a summary of many lectures, and the topics were not really presented in the order that they appear here. Lecture 11 discusses simulations for random walks and includes some basic material on curve fitting to estimate exponents. It ends with a discussion of the most serious project done in this area, the estimate of the intersection exponent. Lecture 12 discusses simulation topics other than random walk that were discussed in the class, including sampling from continuous distributions, random permutations, and finally a more difficult project — using Markov Chain Monte Carlo as discussed in Lecture 8 to estimate the number of matrices with certain conditions. The last lecture discusses a different area, simulations of stochastic differential equations for applications in finance.

We conclude the book with a number of problems that were presented to the students. The difficulty of these problems varies greatly; some are routine, but many were given more to stimulate thought than with the expectation that the students would completely solve them. They are numbered to indicate which lecture they refer to. Of particular note are the problems from Lectures 11 and 12. These are representative of the simpler projects that we gave to the students as they were learning how to do simulations, and are typical of simulation problems that we give to students when we teach undergraduate probability.

We would like to thank a number of people who helped with the program for undergraduates, including: Emily Puckette, the third member of our team; Chad Fargason, who helped write some of the software used in the labs; David Levin, who helped in the preparation of these notes; Brad Mann and Robin Pemantle, for providing copies of their papers; Persi Diaconis, who suggested one of the ideas that is used in Lecture 8; Jennifer Chayes, Thad Dankel, Ingemar Kaj, Ariel Landau, and Vladimir Vinogradov, who gave special lectures to the undergraduates; and most of all to the students who participated.

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