Preface

This is a sequel to our books *Problems in Mathematical Analysis I, II* (Volumes 4 and 12 in the Student Mathematical Library series). The book deals with the Riemann-Stieltjes integral and the Lebesgue integral for real functions of one real variable. The book is organized in a way similar to that of the first two volumes, that is, it is divided into two parts: problems and their solutions. Each section starts with a number of problems that are moderate in difficulty, but some of the problems are actually theorems. Thus it is not a typical problem book, but rather a supplement to undergraduate and graduate textbooks in mathematical analysis. We hope that this book will be of interest to undergraduate students, graduate students, instructors and researches in mathematical analysis and its applications. We also hope that it will be suitable for independent study.

The first chapter of the book is devoted to Riemann and Riemann-Stieltjes integrals. In Section 1.1 we consider the Riemann-Stieltjes integral with respect to monotonic functions, and in Section 1.3 we turn to integration with respect to functions of bounded variation. In Section 1.6 we collect famous and not so famous integral inequalities. Among others, one can find Opial's inequality and Steffensen's inequality. We close the chapter with the section entitled "Jordan measure". The Jordan measure, also called content by some authors, is not a measure in the usual sense because it is not countably additive. However, it is closely connected with the Riemann integral, and we hope that this section will give the student a deeper understanding of the ideas underlying the calculus.

Chapter 2 deals with the Lebesgue measure and integration. Section 2.3 presents many problems connected with convergence theorems that permit the interchange of limit and integral; L^p spaces on finite intervals are also considered here. In the next section, absolute continuity and the relation between differentiation and integration are discussed. We present a proof of the theorem of Banach and Zarecki which states that a function f is absolutely continuous on a finite interval $[a, b]$ if and only if it is continuous and of bounded variation on $[a, b]$, and maps sets of measure zero into sets of measure zero. Further, the concept of approximate continuity is introduced. It is worth noting here that there is a certain analogy between two relationships: the relationship between Riemann integrability and continuity, on the one hand, and the relationship between approximate continuity and Lebesgue integrability, on the other hand. Namely, a bounded function on $[a, b]$ is Riemann integrable if and only if it is almost everywhere continuous; and similarly, a bounded function on $[a, b]$ is measurable, and so Lebesgue integrable, if and only if it is almost everywhere approximately continuous. The last section is devoted to the Fourier series. Given the existence of extensive literature on the subject, e.g., the books by A. Zygmund "Trigonometric Series", by N. K. Bari "A Treatise on Trigonometric Series", and by R. E. Edwards "Fourier Series", we found it difficult to decide what material to include in a book which is primarily addressed to undergraduate students. Consequently, we have mainly concentrated on Fourier coefficients of functions from various classes and on basic theorems for convergence of Fourier series.

All the notation and definitions used in this volume are standard. One can find them in the textbooks [**27**] and [**28**], which also provide the reader with the sufficient theoretical background. However, to avoid ambiguity and to make the book self-contained we start almost every section with an introductory paragraph containing basic definitions and theorems used in the section. Our reference conventions are best explained by the following examples: 1.2.13 or I, 1.2.13 or II, 1.2.13, which denote the number of the problem in this volume, in Volume I or in Volume II, respectively. We also use notation and terminology given in the first two volumes.

Many problems have been borrowed freely from problem sections of journals like the American Mathematical Monthly and Mathematics Today (Russian), and from various textbooks and problem books; of those only books are listed in the bibliography. We would like to add that many problems in Section 1.5 come from the book of Fichtenholz [**10**] and Section 1.7 is influenced by the book of Rogosinski [**26**]. Regrettably, it was beyond our scope to trace all the original sources, and we offer our sincere apologies if we have overlooked some contributions.

Finally, we would like to thank several people from the Department of Mathematics of Maria Curie-Skłodowska University to whom we are indebted. Special mention should be made of Tadeusz Kuczumow and Witold Rzymowski for suggestions of several problems and solutions, and of Stanisław Prus for his counseling and TeX support. Words of gratitude go to Richard J. Libera, University of Delaware, for his generous help with English and the presentation of the material. We are very grateful to Jadwiga Zygmunt from the Catholic University of Lublin, who has drawn all the figures and helped us with incorporating them into the text. We thank our students who helped us in the long and tedious process of proofreading. Special thanks go to Paweł Sobolewski and Przemysław Widelski, who have read the manuscript with much care and thought, and provided many useful suggestions. Without their assistance some errors, not only typographical, could have passed unnoticed. However, we do accept full responsibility for any mistakes or blunders that remain. We would like to take this opportunity to thank the staff at the AMS for their long-lasting cooperation, patience and encouragement.

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