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Preface

Background. These notes are based on an undergraduate course given in July 1995 at the Park City Mathematics Institute's summer program on Nonlinear Waves. The undergraduate course on Linear and Nonlinear Waves consisted of a series of lectures, problem sessions, and computer labs centered around the mathematical modeling and analysis of wave phenomena. This book was written to reflect the content and nature of the course, intermixing discussion with exercises and computer experiments.

The intent of this book is to provide an introduction to basic terminology and concepts found in mathematical studies of wave phenomena. The level of this material is aimed at someone who has completed a basic calculus sequence through multi-variable calculus, and preferably completed a beginning course in ordinary differential equations. Concepts from partial differential equations are introduced as needed and no prior experience with this topic is assumed.

Companion Software. Several problems within these notes are best analyzed with mathematical software to perform plots and visualizations of functions of two variables. Most mathematical software packages are sufficient, although selected exercises refer to particular files which are to be used with MATLAB[®]. These supplemental files were originally written for use at the PCMI summer program and

have been incorporated here as exercises. Readers who use MATLAB are encouraged to obtain these supplemental files through The MathWorks anonymous FTP site at the address

<ftp://ftp.mathworks.com/pub/books/knobel>.

Further Reading. The following books were influential in the writing of these notes and are recommended for readers wishing to supplement or expand the material presented here. The text *Partial Differential Equations for Engineers and Scientists* by Stanley Farlow provides a basic introduction to partial differential equations. Much of David Logan's book *An Introduction to Nonlinear Partial Differential Equations* pertains to the wave behavior arising from nonlinear partial differential equations discussed here. A full collection of material and exercises for the models of traffic flow discussed later in these notes can be found in the textbook *Mathematical Models: Mechanical Vibrations, Population Dynamics, and Traffic Flow* by Richard Haberman. His text *Elementary Applied Partial Differential Equations* expands upon much of the material of these notes, with many sections devoted to various aspects of wave theory. Finally, a more advanced reference on the mathematical theory of linear and nonlinear wave phenomena can be found in the book *Linear and Nonlinear Waves* by G.B. Whitham. See the bibliography for these and other suggested sources of further reading.

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