Preface

Generally acknowledged as India's greatest mathematician, Srinivasa Ramanujan is most often thought of as a number theorist, although he made substantial contributions to analysis and several other areas of mathematics. For most number theorists, when Ramanujan's name is mentioned, the partition and tau functions immediately come to mind. His interest in these arithmetic functions was inextricably intertwined with his primary interests of theta functions and other q-series. In fact, most of Ramanujan's research in number theory arose out of q-series and theta functions. Theta functions are the fundamental building blocks in the theory of elliptic functions, and Ramanujan independently developed his own theory of elliptic functions, which is quite unlike the classical theory. We do not formally define an elliptic function, but, roughly, elliptic functions are meromorphic functions with two linearly independent periods over the real numbers. The concept of double periodicity is not used in this book, and, to the best of our knowledge, Ramanujan never utilized this idea.

The purpose of this book is to provide an introduction to this large expanse of Ramanujan's work in number theory. Needless to say, we shall be able to cover only a very small fraction of Ramanujan's work on theta functions and q-series and their connections with number theory. However, after developing only a few facts about

q-series and theta functions, we will be equipped to prove many interesting theorems. The arithmetic functions on which we focus are the partition function $p(n)$, Ramanujan's tau function $\tau(n)$, the number of representations of a positive integer n as a sum of $2k$ squares denoted by $r_{2k}(n)$, and other arithmetic functions closely allied to $r_{2k}(n)$. Most of the material upon which we draw can be found in Ramanujan's published papers on $p(n)$ and $\tau(n)$, the later chapters in his second notebook, his lost notebook, and his handwritten manuscript on $p(n)$ and $\tau(n)$ published with his lost notebook. We emphasize that Ramanujan left behind few of his proofs, especially for his claims in his notebooks and lost notebook. Thus, for many of the theorems that we discuss, we do not know Ramanujan's proofs. This is particularly true for the theorems on sums of squares and similar arithmetic functions that we prove in Chapter 3.

The requirements for reading and understanding the material in this book are relatively modest. An undergraduate course in elementary number theory is advisable. For some of the analytic arguments, a solid undergraduate course in complex analysis is essential. However, the occasions when deep analytical rigor is needed are few, and so readers who do not have a strong background in analysis can simply verify formally the needed manipulations. Our intent here is not to give a rigorous course in analysis but to emphasize the most important ideas about q-series and theta functions and how they interplay with number theory. This book should be suitable for junior and senior undergraduates and beginning graduate students.

Since many readers may not be familiar with Ramanujan's life, we begin with a short account of his life where readers learn about the notebooks and lost notebook in which he recorded his theorems over several years. We provide brief histories, first of the "ordinary" notebooks, and second of the lost notebook. After these biographical and historical narrations, we provide short summaries of the book's seven chapters.

Ramanujan was born on December 22, 1887 in the home of his maternal grandmother in Erode, located in the southern Indian state of Tamil Nadu. After a few months, his mother, Komalatammal, returned with her son to her home in Kumbakonam approximately 160

miles south-southwest of Madras, where her husband was a clerk in the office of a cloth merchant. At the age of twelve, Ramanujan borrowed a copy of the second part of Loney's *Plane Trigonometry* [**149**] from an older student and worked all the problems in it. This longtime popular textbook in India has much more in it than its title suggests. For example, infinite series and elementary functions of a complex variable are two of its topics. At the age of about fifteen, he borrowed from the Kumbakonam College library a copy of G. S. Carr's *A Synopsis of Elementary Results in Pure Mathematics* [**64**], which served as his primary source for learning mathematics. Carr was a tutor in London and compiled this compendium of 4417 results (with very few proofs) to facilitate his tutoring. At the age of sixteen, Ramanujan entered the Government College in Kumbakonam. By that time, Ramanujan was completely devoted to mathematics and consequently failed his examinations at the end of his first year, because he would not study any other subject. He therefore lost his scholarship and, because his family was poor, was forced to terminate his formal education. He later twice tried to obtain an education at Pachaiyappa's College in Madras, but each time he failed his examinations.

After leaving the Government College in Kumbakonam, Ramanujan devoted all of his time to mathematics, recording his results without proofs in notebooks. It was probably around the age of sixteen that Ramanujan began to record his mathematical discoveries in notebooks, although the entries on magic squares in Chapter 1 in both his first and second notebooks likely emanate from his school days. Living in poverty with no means of financial support, suffering at times from serious illnesses, and working in isolation, Ramanujan devoted all of his efforts to mathematics and continued to record his discoveries without proofs in notebooks for the next five years. In 1909, he married Janaki, who was only nine or ten years old. With mounting pressure to find a job, Ramanujan visited V. Ramaswami Ayyar, the founder of the Indian Mathematical Society. Ayyar contacted R. Ramachandra Rao, who agreed to give Ramanujan, who now had moved to Madras, a monthly stipend so that Ramanujan could continue his mathematical research unabated.

After being supported for about fifteen months, for reasons that are unclear, Ramanujan refused further financial assistance and became a clerk in the Madras Port Trust Office. This turned out to be a watershed in Ramanujan's career. Several people, including S. Narayana Iyer, the Chief Accountant, and Sir Francis Spring, the Chairman, offered support, and Ramanujan was persuaded to write English mathematicians about his mathematical discoveries. Two of them, H. F. Baker and E. W. Hobson, evidently did not reply. M. J. M. Hill replied but was not encouraging. But on January 16, 1913, Ramanujan wrote G. H. Hardy, who responded immediately and encouragingly, inviting Ramanujan to come to Cambridge to develop his mathematical gifts. Ramanujan and his family were Iyengars, a conservative branch in the Brahmin tradition. Travelling to a distant land would make a person unclean, and so Ramanujan's mother was particularly adamant about her son's not accepting Hardy's invitation. After a pilgrimage to Namakal with S. N. Iyer and after Goddess Namagiri appeared in a dream to Komalatammal, Ramanujan received permission to travel. So on March 17, 1914, Ramanujan boarded a passenger ship for England.

At about this time, Ramanujan evidently stopped recording his theorems in notebooks, although a few entries in his third notebook were undoubtedly recorded in England. That Ramanujan no longer concentrated on logging entries in his notebooks is evident from two letters that he wrote to friends in Madras during his first year in England. In a letter of November 13, 1914 to his friend R. Krishna Rao [**51**, pp. 112–113], Ramanujan confided, "I have changed my plan of publishing my results. I am not going to publish any of the old results in my notebooks till the war is over." And in a letter of January 7, 1915 to S. M. Subramanian [**51**, pp. 123–125], Ramanujan admitted, "I am doing my work very slowly. My notebook is sleeping in a corner for these four or five months. I am publishing only my present researches as I have not yet proved the results in my notebooks rigorously."

Ramanujan soon became famous for the papers he published in England, some of them coauthored with Hardy. One of his most important papers is [**186**], [**192**, pp. 136–162], in which he introduced

his famous tau function $\tau(n)$, discussed in Chapter 2 of this book, and the Eisenstein series, P, Q, and R, which are introduced here in Chapter 4 and which were key players in so much of his research. In their paper [**109**], [**192**, pp. 262–275], Hardy and Ramanujan launched the new field of probabilistic number theory, which became an important branch in number theory. In another paper [**110**], [**192**, pp. 276–309], in the course of obtaining an asymptotic series for the partition function $p(n)$, Hardy and Ramanujan introduced the *circle method*, which still today is the primary tool for analytically attacking problems in additive number theory. The genesis of the circle method can be found in Ramanujan's notebooks [**193**], but unfortunately it is most frequently called the Hardy–Littlewood circle method today. In the latter part of his stay in England, Ramanujan wrote his famous papers on congruences for $p(n)$ [**188**], [**192**, pp. 210–213] and [**190**], [**192**, p. 230], about which much is written in this book.

Although Ramanujan never doubted his decision to accept Hardy's invitation to Cambridge, not all was well with Ramanujan. World War I began shortly after his arrival, and being a strict vegetarian, he could not always obtain familiar food and spices from India. On March 24, 1915, near the end of his first winter in Cambridge, Ramanujan wrote his friend E. Vinayaka Row in Madras [**51**, pp. 116–117], "I was not well till the beginning of this term owing to the weather and consequently I couldn't publish any thing for about 5 months." By the end of his third year in England, Ramanujan was critically ill, and, for the next two years, he was confined to sanitariums and nursing homes.

Ramanujan's health turned slightly upward when in 1918 he became the second Indian to be elected as a Fellow of the Royal Society and the first Indian to be chosen as a Fellow of Trinity College. After World War I ended, in 1919, Ramanujan returned home, but his health continued to deteriorate, and on April 26, 1920 Ramanujan died at the age of 32.

Doctors in both England and India had difficulty diagnosing Ramanujan's illness. He was treated for tuberculosis, but a severe vitamin deficiency, liver cancer, lead poisoning purportedly from not properly cleaning his cooking vessels, and a rare tropical disease were

other diagnoses. However, D. A. B. Young [**52**, pp. 65–75] made a careful examination of all extant records and recorded symptoms of Ramanujan's illness and convincingly concluded that Ramanujan suffered from hepatic amoebiasis (a parasitic infection of the liver). Not only do all of Ramanujan's symptoms suggest this disease, but Ramanujan's medical history in India also favors this diagnosis. Amoebiasis is a protozoal infection of the large intestine that gives rise to dysentery. In 1906 Ramanujan left home to attend Pachaiyappa's College in Madras, where he contracted a severe case of dysentery and had to return home for three months. Unless adequately treated, the infection is permanent, although the patient may go for long periods without exhibiting any symptoms. Relapses occur when the host–parasite relationship is disturbed, which likely happened when he endured a colder climate and perhaps inadequate nutrition after his arrival in England. The illness is difficult to diagnose.

Our description of Ramanujan's life has been necessarily brief. For several years, the standard sources about Ramanujan's life have been the obituaries of P. V. Seshu Aiyar, R. Ramachandra Rao, and Hardy, found in Ramanujan's *Collected Papers* [**192**] and Chapter 1 of Hardy's book [**107**]. By far, the most comprehensive biography of Ramanujan has been written by R. Kanigel [**134**]. The letters from and to Ramanujan are also a source of both mathematical and personal information about Ramanujan, and most of the extant letters have been compiled with commentary by R. A. Rankin and the author [**51**].

After Ramanujan died, Hardy strongly urged that Ramanujan's notebooks be edited and published. By "editing," Hardy meant that each claim made by Ramanujan in his notebooks should be examined and proved, if it cannot be found in the literature. Ramanujan, in fact, had left his first notebook with Hardy when he returned to India in 1919, and in 1923 Hardy wrote a paper [**106**], [**108**, pp. 505–516] about a chapter on hypergeometric series found in the first notebook. In this paper, Hardy pointed out that Ramanujan had independently discovered most of the important classical results in the subject while also discovering several new theorems as well. For the definition of a hypergeometric series, see Chapter 5 of this monograph.

Hardy sent the first notebook to the University of Madras where Ramanujan's other notebooks and papers were being preserved. Plans were undertaken to publish Ramanujan's collected papers and, possibly, his notebooks and other manuscripts. Handwritten copies of the notebooks were sent to Hardy along with other manuscripts and papers in 1923, but the papers were never returned to the University of Madras. It transpired that Ramanujan's *Collected Papers* [**192**] were published in 1927, but his notebooks and other manuscripts were not published.

Sometime in the late 1920s, G. N. Watson and B. M. Wilson began the task of editing Ramanujan's notebooks. The second notebook, being a revised, enlarged edition of the first, was their primary focus. Wilson was assigned Chapters 2–14, and Watson was to examine Chapters 15–21. Wilson devoted his efforts to this task until 1935, when he died from an infection at the early age of 38. Watson wrote over 30 papers inspired by the notebooks before his interest evidently waned in the late 1930s. Thus, the project was never completed.

It was not until 1957 that the notebooks were made available to the public when the Tata Institute of Fundamental Research in Bombay published a photocopy edition [**193**], but no editing was undertaken. The first notebook was published in volume 1, and volume 2 comprises the second and third notebooks. The present author undertook the task of editing Ramanujan's notebooks in 1977. With the help of several mathematicians, the author completed his work with the publication of his fifth volume [**38**] on the notebooks in 1998.

In the spring of 1976, George Andrews of Pennsylvania State University visited Trinity College, Cambridge, to examine the papers left by Watson. Among Watson's papers, he found a manuscript containing 138 pages in the handwriting of Ramanujan. In view of the fame of Ramanujan's notebooks [**193**], it was natural for Andrews to call this newly found manuscript "Ramanujan's lost notebook." How did this manuscript reach Trinity College?

Watson died in 1965 at the age of 79. Shortly thereafter, on separate occasions, J. M. Whittaker and R. A. Rankin visited Mrs. Watson. Whittaker was a son of E. T. Whittaker, who coauthored with

Watson probably the most popular and frequently used text on analysis in the 20th century [**221**]. Rankin had succeeded Watson as Mason Professor of Mathematics at the University of Birmingham, where Watson served for most of his career, but was now Professor of Mathematics at the University of Glasgow. Both Whittaker and Rankin went to Watson's attic office to examine the papers left by him, and Whittaker found the aforementioned manuscript by Ramanujan. Rankin suggested to Mrs. Watson that he might sort her late husband's papers and send those worth preserving to Trinity College Library, Cambridge. During the next three years, Rankin sorted through Watson's papers sending them in batches to Trinity College Library, with Ramanujan's manuscript being sent on December 26, 1968. Not realizing the importance of Ramanujan's papers, neither Rankin nor Whittaker mentioned them in their obituaries of Watson [**195**], [**222**]. The next question is: How did Watson come into possession of this sheaf of 138 pages of Ramanujan's work?

We mentioned above that in 1923 the University of Madras had sent a package of Ramanujan's papers to Hardy. Most likely, this shipment contained the "lost notebook." Of the over 30 papers that Watson wrote on Ramanujan's work, two of his last papers were devoted to Ramanujan's mock theta functions, which Ramanujan discovered in the last year of his life, which he described in a letter to Hardy only about three months before he died [**51**, pp. 220–223], and which are also found in the lost notebook. In these two papers, Watson made some conjectures about the existence of certain mock theta functions. If he had the lost notebook at that time, he would have seen that his conjectures were correct. Thus, probably sometime after Watson's interest in Ramanujan's work declined in the late 1930s, Hardy passed Ramanujan's papers to Watson.

In early 1988, just after the centenary of Ramanujan's birth, Narosa Publishing House in New Delhi published a photocopy edition of the lost notebook [**194**]. Included in this publication are partial manuscripts, loose papers, and fragments by Ramanujan, as well as letters from Ramanujan to Hardy written from nursing homes during the last two years of Ramanujan's sojourn in England.

The first chapter of this book is devoted to basic facts about q-series and theta functions, including the q-binomial theorem, the Jacobi triple product identity, the pentagonal number theorem, Ramanujan's $_1\psi_1$ summation theorem, and the quintuple product identity. Many of the theorems proved in Chapter 1 can be found in Chapter 16 of Ramanujan's second notebook [**193**], [**34**].

Chapter 2 focuses on congruences for the partition function $p(n)$ and Ramanujan's tau function $\tau(n)$. Much of this material is taken from Ramanujan's handwritten manuscript on $p(n)$ and $\tau(n)$, which was first published in 1988 along with Ramanujan's lost notebook [**194**]. Adding details to many of Ramanujan's proofs and discussing Ramanujan's theorems in light of the literature written after Ramanujan's death, the present author and K. Ono [**50**] published an expanded version of this manuscript.

In his notebooks [**193**], Ramanujan recorded a large number of entries on Lambert series. These identities for Lambert series were used by Ramanujan to establish theta function identities and formulas for the number of representations of an integer as a sum of a certain numbers of squares or of triangular numbers. We introduce readers to Lambert series in Chapter 3 and establish many identities leading to formulas for sums of squares and triangular numbers. A manuscript with no proofs on precisely this subject is another of those manuscripts published with Ramanujan's lost notebook [**194**], [**19**, Chapter 18]. His second notebook also contains a large number of such theorems.

Eisenstein series permeate Ramanujan's notebooks [**193**] and lost notebook [**194**]. Much of our exposition on Eisenstein series in Chapter 4, however, is taken from Ramanujan's epic paper [**186**], [**192**, 136–162]. One of Ramanujan's approaches to congruences for $p(n)$ is based on Eisenstein series, which we demonstrate at the close of Chapter 4.

In Chapter 5, we introduce readers to hypergeometric functions and elliptic integrals. Our goal in this chapter is to prove one of the most fundamental theorems of elliptic functions relating hypergeometric functions and elliptic integrals to theta functions. This theorem enables us to express theta functions and Eisenstein series

at various arguments in terms of certain elliptic parameters. Our exposition is derived from Chapter 17 of Ramanujan's second note-book [**193**], [**34**, Chapter 17], where, through a series of preliminary lemmas, Ramanujan leads us to the aforementioned key theorem.

Applications of the aforementioned representations for Eisenstein series and theta functions form the content of Chapter 6. First, we return to the topic of sums of squares and demonstrate how the formulas for Eisenstein series lead to short proofs of some of the results from Chapter 3. However, most of Chapter 6 is devoted to modular equations, a topic to which Ramanujan made more contributions than any other mathematician. Chapters 19–21 in his second note-book are devoted to modular equations, and our short introduction to this topic is drawn from our previous account of Ramanujan's work in these chapters [**34**].

One of Ramanujan's favorite topics was the Rogers–Ramanujan continued fraction, the focus of Chapter 7. Because we wish to share so much about this continued fraction with readers and because the length of the chapter would be prohibitive if we proved all theorems offered in this chapter, we forego some of the proofs. However, we do prove two key theorems relating the continued fraction with its reciprocal. These theorems are then used to give an alternative, cleaner proof of an identity of Ramanujan in Chapter 2, yielding immediately the congruence $p(5n + 4) \equiv 0 \,(\mathrm{mod}\,5)$. The famous Rogers–Ramanujan functions are also discussed, and, in particular, we prove that the Rogers–Ramanujan continued fraction can be represented as a quotient of the two Rogers–Ramanujan functions. Our fervent wish is that our sampling of the many beautiful properties satisfied by this continued fraction will motivate readers to turn to original sources to learn more about it.

Ubiquitous in this book are products of the form
$$(1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}) =: (a; q)_n$$
as well as their infinite versions
$$(1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^n) \cdots =: (a; q)_\infty, \qquad |q| < 1,$$
which are called q-products. Although we assume that readers of this book are familiar with infinite series, it may well be that some

are not familiar with infinite products. A reader desiring to learn a few basic facts about the convergence of infinite products may consult a good text on complex analysis, such as that of N. Levinson and R. Redheffer [**142**, pp. 382–385], for basic properties of infinite products. In particular, all the infinite products in the present text converge absolutely and uniformly on compact subsets of $|q| < 1$. In particular, taking logarithms of infinite products and differentiating the resulting series termwise is permitted. At first, you may find that working with the products $(a;q)_n$ and $(a;q)_\infty$ is somewhat tedious. In order to verify q-product identities or to manipulate q-products, it may be helpful to write out the first three or four terms of each q-product. This should provide the needed insight in order to justify a given step. After working with q-products for awhile, you will begin to handle them more quickly and adroitly, and no longer need to write out any of their terms longhand. When you reach this stage, you should feel quite comfortable in manipulating q-series. It is assumed throughout the entire book that $|q| < 1$. Over 50 exercises are interspersed within the exposition.