Preface

This book provides an introduction to the growing field of ergodic theory, also known as measurable dynamics. It covers topics such as recurrence, ergodicity, the ergodic theorem and mixing. It is aimed at students who have completed a basic course in undergraduate real analysis covering topics such as basic compactness properties and open and closed sets in the real line. Measure theory is not assumed and is developed as needed. Readers less familiar with these topics will find a discussion of the relevant material from real analysis in the appendices.

I have used early versions of this book in courses that are designed as capstone courses for the mathematics major, including students with a variety of interests and backgrounds. The study of measurable dynamics can be used to reinforce and apply the student's knowledge of measure theory and real analysis while introducing some beautiful mathematics of relatively recent vintage. Measure theory is developed as needed and applied to study notions in dynamics. While it has less emphasis, some metric space topology, including the Baire category theorem, is presented and applied to topological dynamics. Several examples are developed in detail to illustrate concepts from measurable and topological dynamics.

This book can be used as a special-topics course for upper-level mathematics students. It can also be used as a short introduction to Lebesgue measure and integration, as an introduction to ergodic theory, and for independent study. The Bibliographical Notes provide some guidelines for further reading.

An introductory course could start with a short review of the topology on the real line and basic properties of metric spaces as covered in Appendix B or with the construction of Lebesgue measure on the real line in Chapter 2. The reader who wants to get to ergodic theory quickly needs to cover only Sections 2.1 through 2.4 and could then start with Chapter 3, perhaps omitting Sections 3.10 through 3.12. Topological dynamics is closely related to measurable dynamics, and the book introduces some topics from topological dynamics. This is not necessary for the main development of the book, and the reader has the option of omitting the topological dynamics topics or of using them to learn some metric space topology and some elegant ideas from topological dynamics. A more advanced course could cover Chapters 2 and 3 in more detail. Some of the measure theory notions that are covered include the Carathéodory extension theorem, product measures and L^p spaces. Lebesgue integration is introduced in Chapter 4, and some of these notions are used to study the eigenvalues of measure-preserving transformations. The chapter on the ergodic theorem, in addition to being of intrinsic interest, provides a beautiful example for applications of various theorems of Lebesgue integration. The final chapter on mixing uses ideas from all the other chapters.

The book contains both simple exercises, called questions, designed to test the reader's immediate grasp of the new material, and more challenging exercises at the end of each section. Harder exercises are marked with a star ($\star$). Partial solutions and hints for some of the exercises will be available at the book's webpage listed on the back cover. Some sections also contain open questions designed to suggest to the reader some avenues of research. The bibliography is not intended to be exhaustive; it is there to provide suggestions for additional reading and to acknowledge the sources I have used.

I am indebted to many people who through their conversations and writings have taught me measure theory and ergodic theory. I first learned analysis from César Carranza. I was introduced to ergodic theory by Dorothy Maharam and later was influenced by Shizuo Kakutani and John Oxtoby. I have also learned much from all my coauthors and the students I have supervised in research and in courses. I am indebted to several anonymous reviewers and readers at various stages of this work who have provided advice and suggestions. In particular I would like to thank my Williams colleagues Ollie Beaver, Ed Burger, Satyan Devadoss, Frank Morgan and Mihai Stoiciu, and my editor, Sergei Gelfand. I also thank Blaire Madore and Karin Reinhold, who used an early version with their students and sent me helpful suggestions.

I have used early versions of this book in courses, tutorials and summer SMALL REU projects and would like to thank the many students who corrected errors, discovered typos and made suggestions, in particular Katherine Acton, Ali Al-Sabah, Amie Bowles, John Bryk, John Chatlos, Tegan Cheslack-Postava, Alexandra Constantin, Jon Crabtree, Michael Daub, Sarah Day, Chris Dodd, Jason Enelow, Lukasz Fidkowski, Thomas Fleming, Artur Fridman, Ilya Grigoriev, Brian Grivna, Kate Gruher, Fred Hines, Sarah Iams, Catalin Iordan, Nate Ince, On Jesakul, Anne Jirapattanakul, Jeff Kaye, Eric Katerman, Brian Katz, Min Kim, James Kingsbery, Dave Kleinschmidt, Thomas Koberda, Ross Kravitz, Gary Lapon, Alex Levin, Amos Lubin, Amy Marinello, Earle McCartney, Abe Menon, Erich Muehlegger, Karl Naden, Nara Narasimhan, Deepam Patel, Ravi Purushotham, Andy Raich, Hyejin Rho, Becky Robinson, Richard Rodriguez, David Roth, Charles Samuels, Brian Simanek, Peter Speh, Anita Spielman, John Spivack, Noah Stein, Joseph Stember, Andrea Stier, Brian Street, Daniel Sussman, Mike Touloumtzis, Paul Vichyanond, Robert Waelder, Kirsten Wickelgren, Alex Wolfe, and Wenhuan Zhao. Thanks are due most especially to Darren Creutz, Daniel Kane, Jennifer James, Kathryn Lindsey, and Anatoly Preygel.

Finally, I would like to dedicate this book to my wife and two daughters and the memory of my parents.

Cesar E. Silva