Preface to the English Edition

We thank the American Mathematical Society, and Edward G. Dunne in particular, for commissioning a translation of our original book and for his willingness to print the whole book in four-color. We seized the opportunity to make infinitely many infinitesimal corrections that have been observed since the first German edition, all of which are not worth listing here. We thank all our colleagues and students who helped to identify them. The book is intended for both undergraduate mathematics students, to introduce them to an advanced point of view on geometry, and for mathematics teachers, as a reference and source book.

This edition, like the original, has its own homepage,

http://www.ams.org/bookpages/stml-43

and any further corrections to errors, mathematical and typographical, will be posted there as they come to our attention. We also intend to present there additional material and a collection of related Web links that we hope the reader may find useful. We will be happy to receive and respond to any comments. In particular, any student who encounters difficulties in solving the Exercises is invited to outline the problem to us by e-mail.

For teachers in schools and universities we have prepared a small volume with hints for solutions, available from us on request.

The Bibliography has been chosen to signpost other material which may be helpful to our readers, and to whet their appetites for geometry and its ramifications. It is not a list of prerequisites.

We thank our translator, Dr. Philip G. Spain, for aiding us in making our book available to a wider public. In its present form the book owes a lot to his expertise, and we are very much indebted to him for an exceptionally pleasant and intensive collaboration.

Berlin, August 2007

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Acknowledgment

The impressions of the ornaments on pages 145 and 165 are printed from Owen Jones, *Grammatik der Ornamente* (unchanged reprint from the First Edition of 1856), 1987, by gracious permission of Greno Verlag, Nördlingen, Germany. The original English edition appeared under the title *The Grammar of Ornament* and has often been reprinted.

Preface to the German Edition

for Julius

Geometry occupies an extensive part of the mathematical syllabus in German schools. In middle school, one starts with properties of the elementary plane figures (line, triangle, circle), elementary transformations of the plane, and surfaces and bodies in space. In high school one comes to analytic geometry, trigonometry, advanced transformations, special curves and the conic sections. Elements of non-Euclidean geometry can be covered in further optional courses. Altogether we have a broad spectrum of geometrical themes that the mathematics teacher can present to his pupils. During the study for the teacher's diploma at university the syllabus starts with lectures on linear algebra and analytic geometry in the first year, followed by lectures on elementary geometry in the second year. This is to present these geometrical themes to the prospective teacher in a mathematically systematic form. If one considers the university education in Germany over a longer time span, it is easy to recognize that in the lectures on linear algebra the geometric themes are reduced step by step, often almost completely masked out.

In all, one gains the impression that the course on elementary geometry, with its clearly defined contents, forms the main part of the geometric education for teachers in training.

This book arose from a one-semester lecture course on “elementary geometry” for future teachers in their second year of study at the Humboldt-Universität in Berlin. The students had already attended the first-year courses on linear algebra and calculus; in the first chapter we present a summary of some aspects of these lectures. Our treatment of elementary geometry assumes this fundamental knowledge, although in a large part of the text they will hardly be needed. Accordingly, this text is intended as a companion book to such a course and seminars. Further, we hope that the book will be used by working teachers as a compendium of the curriculum for elementary geometry. Selected parts of the text are also suitable for good high school pupils and ideally might be used as a foundation for independent study or projects.

Chapter 2 is devoted to the elementary geometric figures and their properties. We begin with the incidence theorems for lines and then turn our attention to the triangle. After the congruence and similarity theorems, we apply in particular the theorems of Menelaus and Ceva in order to treat the intersection points of the special lines in a triangle. Further, we discuss the incircle, circumcircle and excircle of the triangle, its area, and its relation to the radii of the circle. We treat the circle similarly, and discuss in particular the Feuerbach circle, and the Simson and Steiner lines. With a view to the underlying hyperbolic geometry in Chapter 4 we already introduce a section on inversion in the circle here. The conic sections follow, with the derivation of their general equation, their eccentricity and parameters, as well as the determination of the focus and directrix. Some striking properties of the conic sections are proved directly in the text; the reader will find some other properties in the Exercises at the end of Chapter 2. Then we turn to surfaces and bodies in space. We derive the formulae for the surface area of a surface of revolution and also the formula for the volume of a body of revolution: we prove Euler’s polyhedron theorem (for convex polyhedra) and finish Chapter 2 with the classification of the Platonic bodies.

Chapter 3 deals with the symmetries of Euclidean space. We describe affine mappings briefly, also the linear mappings corresponding to them, and the centroid of a finite weighted point system. Parallel projections onto a plane along a line and onto a line along a plane are the first examples of affine mappings. Then we treat central dilations and translations exhaustively. First we characterize them through a common geometric property and deduce that together they form a nonabelian group of transformations of space to itself. Then, for the plane, we determine in detail their compositions and discuss as an application the dilation centers of two circles, with whose help one can construct the common tangents to two circles. Next follows the study of isometries of the plane. First come examples of axis reflections, translations and rotations, and again we study their compositions. Fixed points are important: An isometry of the plane with three noncollinear fixed points is the identity. Analogously we characterize all isometries with exactly two fixed points, with one fixed point, and also the fixed point free isometries. The group generated by all isometries and central dilations consists of the similarity transforms of the plane. In a similar way we treat the transformations of three-dimensional space. First we study the composition of distinct such mappings and then turn again to the description of the fixed point sets of spatial isometries. These fixed point sets yield a classification of the isometries of $\mathcal{E}^3$. The last two sections of this chapter are devoted to the study of the discrete isometry groups of Euclidean space. In the case of the plane we treat the cyclic rotation groups, the dihedral group and lattice. We deduce a necessary condition for the point group of a discrete isometry group of the plane and finally obtain a classification of all the groups in question. In the case of space we restrict ourselves to classifying the finite isometry groups. These are the invariance groups of the Platonic bodies and of the symmetry group of a pyramid or of a cylinder with regular polygonal base. The tetrahedron group, the cube group, and also the dodecahedron group are described completely.

We begin Chapter 4 with the axiomatics of elementary geometry and the significance of the parallel axiom. We construct hyperbolic geometry in the upper half plane, in which lines are Euclidean circular arcs or half lines. We treat various expressions for the hyperbolic distance of two points. This can as well be represented by a cross ratio as by a direct formula. In particular, the triangle inequality holds, and the hyperbolic plane is a metric space. We determine its isometry group and derive the formulae for the hyperbolic length of a curve and for the hyperbolic area of a region. By means of the Cayley transform we pass to the disc model of hyperbolic geometry. We then treat selected properties of geometric figures in the hyperbolic plane. We compute the perimeter of a circle, its hyperbolic area, and derive the hyperbolic Pythagoras theorem as well as other formulae from trigonometry. The formula for the area of a triangle and its angle defect is proved completely. In the Exercises the reader will find a number of results in hyperbolic elementary geometry that are analogous to those of Euclidean geometry. These concern pairs of hyperbolic lines, triangles and their notable points, the incircle and circumcircle of a triangle, and also the horocycle. In a further section we present the classification of the isometries into elliptic, parabolic and hyperbolic transformations both by means of the Jordan normal form and also through their fixed point sets. We study in detail the question of the type of the commutator of two isometries. The last section of this chapter is devoted to Fuchsian groups. Here we are dealing with discrete subgroups of the isometry group of the hyperbolic plane. As well as a series of examples of such groups we introduce their limit sets and prove that these sets have either $0, 1, 2$ or infinitely many points. Fuchsian groups with no more than two limit points are called elementary. We classify all elementary Fuchsian groups.

Spherical geometry is treated in the last chapter in imitation of hyperbolic geometry. We consider the set of all points of the two-dimensional sphere $\mathbb{S}^2$. The great circles play the rôle of spherical lines and realize the shortest distance between two points in spherical space. We determine the isometry group and also the group of all conformal mappings completely. To conclude we prove the most important formulae of spherical trigonometry and study the polar triangle associated to each spherical triangle. From this we obtain the formulae for the areas of spherical lunes and triangles and various inequalities between the side lengths and angles.

At the end of each chapter the reader will find a selection of Exercises that have regularly been assigned to our auditors as homework. Any student who encounters difficulties in solving these Exercises is warmly invited to outline his problem to us by e-mail. We will endeavor to help. For teachers in schools and universities we have prepared a small volume with hints for solutions, which is available from us on request. Moreover, the German edition of the book has its own Internet page,

http://www-irm.mathematik.hu-berlin.de/˜agricola/elemgeo.html

One will find there a list of all known typographical errors, and pdf-files of all the pages on which pictures appear that are multicolored in the original but are printed here in black-and-white on grounds of cost. There is also a collection of www-links on elementary geometry, though it makes no claim to completeness.

We thank the participants in our seminars for numerous suggestions that have led to extending and improving the text. Dr. sc. Hubert Gollek and Dr. Christof Puhle have read through the whole manuscript and have indicated necessary corrections in many chapters. Not last we thank Frau Schmickler-Hirzebruch of Vieweg Verlag for her willingness to print some pages of this book in two-tone. We are aware that this is a rare (if also much desired) privilege. We hope that this will not remain an isolated case in the mathematical literature and that the reader will appreciate and enjoy this enrichment of the text, which was not to be taken for granted.

Berlin, December 2004

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