Chapter 2

# Lebesgue Measure

## 2.1. Introduction

In the previous section we studied two definitions of integration that were based on two important facts: (1) There is only one obvious way to define the integral of step functions assuming we want it to satisfy certain basic properties, and (2) these properties force the definition for the integral for more general functions which are uniformly approximated by step functions (regulated integral) or squeezed between step functions whose integrals are arbitrarily close (Riemann integral).

To move to a more general class of functions we first find a more general notion to replace step functions. For a step function $f$ there is a partition of $I = [0, 1]$ into intervals on each of which $f$ is constant. We now would like to allow functions for which there is a finite partition of $I$ into sets on each of which $f$ is constant, but with the sets *not necessarily intervals.* For example, we will consider functions such as

$$f(x) = \begin{cases} 3, & \text{if } x \text{ is rational;} \\ 2, & \text{otherwise.} \end{cases} \tag{2.1.1}$$

The interval $I$ is partitioned into two sets, $A = I \cap \mathbb{Q}$ and $B = I \cap \mathbb{Q}^c$, i.e., the rational points of $I$ and the irrational points. Clearly, the integral of this function should be $3\,\mathrm{len}(A) + 2\,\mathrm{len}(B)$, but only if we can make sense of $\mathrm{len}(A)$ and $\mathrm{len}(B)$. That is the problem to which this chapter is devoted. We want to generalize the concept of length to include as many subsets of $\mathbb{R}$ as we can. We proceed in much the same way as in previous chapters. We first decide what are the "obvious" properties this generalized length must satisfy to be of any use, and then try to define it by approximating with simpler sets where the definition is clear, namely sets of intervals.

The generalization of length we want is called *Lebesgue measure.* Ideally, we would like it to work for *any* subset of the interval $I = [0, 1]$, but it turns out that this is not possible to achieve.

There are several properties which we want any notion of "generalized length" to satisfy. These are analogous to the basic properties we required for a definition of integral in Chapter 1. For each bounded subset $A$ of $\mathbb{R}$ we would like to be able to assign a non-negative real number $\mu(A)$ that satisfies the following:

**I. Length:** If $A = (a, b)$ or $[a, b]$, then $\mu(A) = \mathrm{len}(A) = b - a$, i.e., the measure of an open or closed interval is its length.

**II. Translation invariance:** If $A \subset \mathbb{R}$ is a bounded subset of $\mathbb{R}$ and $c \in \mathbb{R}$, then $\mu(A + c) = \mu(A)$, where $A + c$ denotes the set $\{x + c \mid x \in A\}$.

**III. Countable additivity:** If $\{A_n\}_{n=1}^{\infty}$ is a countable collection of bounded subsets of $\mathbb{R}$, then

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \le \sum_{n=1}^{\infty} \mu(A_n)$$

and if the sets are *pairwise disjoint,* then

$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$

Note that for a finite collection $\{A_n\}_{n=1}^{m}$ of bounded sets the same conclusion applies (just let $A_i = \emptyset$ for $i > m$).

**IV. Monotonicity:** If $A \subset B$, then $\mu(A) \le \mu(B)$. Actually, this property is a consequence of additivity since $A$ and $B \backslash A$ are disjoint and their union is $B$.

It should be fairly clear why most of these properties are absolutely necessary for any sensible notion of length. The only exception is property III, which deserves some comment. We might ask that additivity holds only for finite collections of sets, but that is too weak. For example, if we had a collection of pairwise disjoint intervals of length $1/2, 1/4, 1/8, \dots 1/2^n, \dots$, etc., then we would certainly like to be able say that the measure of their union is the sum $\sum 1/2^n = 1$ which would not follow from finite additivity. Alternatively, one might wonder why additivity is only for *countable* collections of pairwise disjoint sets. But it is easy to see why it would lead to problems if we allowed uncountable collections. Suppose $A_x = \{x\}$ is the set consisting of a single point $x \in [0, 1]$. Then $\mu(A_x) = 0$ by property I. But $[a, b]$ is an uncountable set and hence an uncountable union of pairwise disjoint sets each containing a single point, namely each of the sets $A_x$ for $x \in [a, b]$. Hence, "uncountable additivity" would imply that $\mu([a, b]) = b-a$ is an *uncountable* sum of zeroes. This is the main reason the concept of uncountable sums isn't very useful. Indeed, we will see that the concept of countability is intimately related to the concept of measure.

Unfortunately, as mentioned above, it turns out that it is impossible to find a $\mu$ which satisfies I–IV and which is defined for *all* bounded subsets of the reals; but we can do it for a very large collection which includes all the open sets and all the closed sets. The measure we are interested in using is called *Lebesgue measure.* Its actual construction is slightly technical and we have relegated that to an appendix. Instead, we will focus on some of the properties of Lebesgue measure and how it is used.

## 2.2. Null Sets

One of our axioms for the regulated integral was, "Finite sets don't matter." Now we want to generalize that to say that a set doesn't matter if its "generalized length," or measure, is zero. It is a somewhat surprising fact that even without defining Lebesgue measure in general we can easily define those sets whose measure must be 0 and investigate the properties of these sets.

**Definition 2.2.1. (Null set).** *A set* $X \subset \mathbb{R}$ *is called a* null set *if for every* $\varepsilon > 0$ *there is a collection of open intervals* $\{U_n\}_{n=1}^{\infty}$ *such that*

$$\sum_{n=1}^{\infty} \operatorname{len}(U_n) < \varepsilon \quad \textit{and} \quad X \subset \bigcup_{n=1}^{\infty} U_n.$$

Notice that this definition makes no use of the measure $\mu$. Indeed, we have not yet defined the measure $\mu$ for *any* set $X$! However, it is clear that if we can do so in a way that satisfies properties I-IV above, then if $X$ is a null set, $\mu(X) < \varepsilon$ for every positive $\varepsilon$. This, of course, implies $\mu(X) = 0$.

If $X$ is a null set, we will say that its complement $X^c$ has *full measure.*

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**Exercise 2.2.2.**

(1) Prove that a finite set is a null set.

(2) Prove that a countable union of null sets is a null set (and hence, in particular, countable sets are null sets).

(3) Assuming that a measure $\mu$ has been defined and satisfies properties I-IV above, find the numerical value of the integral of the function $f(x)$ defined in equation (2.1.1). Prove that the Riemann integral of this function does not exist.

(4) Prove that if $X$ is a countable compact subset of $\mathbb{R}$, then for any $\varepsilon > 0$ there is a *finite* collection of pairwise disjoint open intervals $\{U_k\}_{k=1}^{n}$ such that

$$\sum_{k=1}^{n} \operatorname{len}(U_k) < \varepsilon \quad \text{and} \quad X \subset \bigcup_{k=1}^{n} U_k.$$

Use this to prove that any closed interval $[a, b]$ with $b > a$ is uncountable.

It is not true that countable sets are the only sets which are null sets. We give an example in Exercise 2.5.4 below, namely, the Cantor middle third set, which is an uncountable null set.

## 2.3. Sigma Algebras

As mentioned before there does not exist a function $\mu$ satisfying properties I-IV from Section 2.1 and which is defined for every subset of $I = [0, 1]$. In this section we want to consider what is the best we can do. Is there a collection of subsets of $I$ for which we can define a "generalized length" or *measure* $\mu$ which satisfies properties I–IV and which is large enough for our purposes? And what properties would such a collection need to have?

Suppose we have somehow defined $\mu$ for all the sets in some collection $\mathcal{A}$ of subsets of $I$ and it satisfies properties I–IV. Property I only makes sense if $\mu$ is defined for open and closed intervals, i.e., we need open and closed intervals to be in $\mathcal{A}$. For property III to make sense we will need that any countable union of sets in $\mathcal{A}$ is also in $\mathcal{A}$. Finally, it seems reasonable that if $A$ is a set in the collection $\mathcal{A}$, then the set $A^c$, its complement in $I$, should also be in $\mathcal{A}$.

All of this motivates the following definition.

**Definition 2.3.1. (Sigma algebra).** *Suppose $X$ is a set and $\mathcal{A}$ is a collection of subsets of $X$. $\mathcal{A}$ is called a* $\sigma$-algebra *of subsets of $X$ provided it contains the set $X$ and is closed under taking complements (with respect to $X$), countable unions, and countable intersections.*

In other words, if $\mathcal{A}$ is a $\sigma$-algebra of subsets of $X$, then any complement (with respect to $X$) of a set in $\mathcal{A}$ is also in $\mathcal{A}$, any countable union of sets in $\mathcal{A}$ is in $\mathcal{A}$, and any countable intersection of sets in $\mathcal{A}$ is in $\mathcal{A}$. In fact, the property concerning countable intersections follows from the other two and Proposition A.5.3 which says that the intersection of a family of sets is the complement of the union of the complements of the sets. Also note that, if $A, B \in \mathcal{A}$, then their set difference $A \setminus B = \{x \in A \mid x \notin B\}$ is in $\mathcal{A}$ because $A \setminus B = A \cap B^c$.

Since $X$ is in any $\sigma$-algebra of subsets of $X$ (by definition), so is its complement, the empty set. A trivial example of a $\sigma$-algebra of subsets of $X$ is $\mathcal{A} = \{X, \emptyset\}$, i.e., it consists of only the whole set $X$ and the empty set. Another example, at the other extreme, is $\mathcal{A} = \mathcal{P}(X)$, the power set of $X$, i.e., the collection of all subsets of $X$. Several more interesting examples are given in the exercises below. Also, in these exercises we ask you to show that any intersection of

$\sigma$-algebras is a $\sigma$-algebra. Thus, for any collection $\mathcal{C}$ of subsets of $\mathbb{R}$ there is a smallest $\sigma$-algebra of subsets of $\mathbb{R}$ which contains all sets in $\mathcal{C}$, namely the intersection of all $\sigma$-algebras containing $\mathcal{C}$ (there is a least one such $\sigma$-algebra, namely the power set $\mathcal{P}(\mathbb{R})$).

**Definition 2.3.2. (Borel sets).** *If $\mathcal{C}$ is a collection of subsets of $\mathbb{R}$ and $\mathcal{A}$ is the the smallest $\sigma$-algebra of subsets of $\mathbb{R}$ which contains all the sets of $\mathcal{C}$, then $\mathcal{A}$ is called the $\sigma$-algebra generated by $\mathcal{C}$. Let $\mathcal{B}$ be the $\sigma$-algebra of subsets of $\mathbb{R}$ generated by the collection of all open intervals. $\mathcal{B}$ is called the* Borel $\sigma$-algebra *and elements of $\mathcal{B}$ are called* Borel sets.

In other words, $\mathcal{B}$ is the collection of subsets of $\mathbb{R}$ which can be formed from open intervals by any finite sequence of countable unions, countable intersections, or complements. The $\sigma$-algebra $\mathcal{B}$ can also be described as the $\sigma$-algebra generated by open subsets of $\mathbb{R}$, or by closed intervals, or by closed subsets of $\mathbb{R}$ (see part (5) of Exercise 2.3.3 below).

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**Exercise 2.3.3.**

(1) Let $\mathcal{A} = \{A \subset I \mid A \text{ is countable, or } A^c \text{ is countable}\}$. Prove that $\mathcal{A}$ is a $\sigma$-algebra.

(2) Let $\mathcal{A} = \{A \subset I \mid A \text{ is a null set, or } A^c \text{ is a null set}\}$. Prove that $\mathcal{A}$ is a $\sigma$-algebra.

(3) Suppose $\mathcal{A}_\lambda$ is a $\sigma$-algebra of subsets of $X$ for each $\lambda$ in some indexing set $\Lambda$. Prove that
$$\mathcal{A} = \bigcap_{\lambda \in \Lambda} \mathcal{A}_\lambda$$
is a $\sigma$-algebra of subsets of $X$.

(4) Let $\mathcal{A}$ be a $\sigma$-algebra of subsets of $\mathbb{R}$ and suppose $I$ is a closed interval which is in $\mathcal{A}$. Let $\mathcal{A}(I)$ denote the collection of all subsets of $I$ which are in $\mathcal{A}$. Prove that $\mathcal{A}(I)$ is a $\sigma$-algebra of subsets of $I$.

(5) Suppose $\mathcal{C}_1$ is the collection of closed intervals in $\mathbb{R}$, $\mathcal{C}_2$ is the collection of all open subsets of $\mathbb{R}$, and $\mathcal{C}_3$ is the collection of all closed subsets of $\mathbb{R}$. Let $\mathcal{B}_i$ be the $\sigma$-algebra generated

by $\mathcal{C}_i$. Prove that $\mathcal{B}_1, \mathcal{B}_2$, and $\mathcal{B}_3$ are all equal to the Borel $\sigma$-algebra $\mathcal{B}$.

## 2.4. Lebesgue Measure

The $\sigma$-algebra of primary interest to us is the one generated by Borel sets and null sets. Alternatively, as a consequence of Exercise 2.3.3 (5), it is the $\sigma$-algebra of subsets of $\mathbb{R}$ generated by open intervals, and null sets, or the one generated by closed intervals and null sets.

**Definition 2.4.1. (Lebesgue measurable set).** *The $\sigma$-algebra of subsets of $\mathbb{R}$ generated by open intervals and null sets will be denoted by $\mathcal{M}$. Sets in $\mathcal{M}$ will be called Lebesgue measurable, or measurable for short. If $I$ is a closed interval, then $\mathcal{M}(I)$ will denote the Lebesgue measurable subsets of $I$.*

For simplicity we will focus on subsets of $I = [0, 1]$ though we could just as well use any other interval. Notice that it is a consequence of part (4) of Exercise 2.3.3 that $\mathcal{M}(I)$ is a $\sigma$-algebra of subsets of $I$. It is by no means obvious that $\mathcal{M}$ is not the $\sigma$-algebra of all subsets of $\mathbb{R}$. However, in Appendix C we will construct a subset of $I$ which is not in $\mathcal{M}$.

We are now ready to state the main theorem of this chapter.

**Theorem 2.4.2. (Existence of Lebesgue measure).** *There exists a unique function $\mu$, called* Lebesgue measure*, from $\mathcal{M}(I)$ to the non-negative real numbers satisfying:*

- **I. Length:** *If $A = (a, b)$, then $\mu(A) = \operatorname{len}(A) = b - a$, i.e., the measure of an open interval is its length.*
- **II. Translation invariance:** *Suppose $A \in \mathcal{M}(I)$, $c \in \mathbb{R}$ and $A + c \subset I$ where $A + c$ denotes the set $\{x + c \mid x \in A\}$. Then $(A + c) \in \mathcal{M}(I)$ and $\mu(A + c) = \mu(A)$.*
- **III. Countable additivity:** *If $\{A_n\}_{n=1}^{\infty}$ is a countable collection of elements of $\mathcal{M}(I)$, then*

$$\mu(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

*and if the sets are pairwise disjoint, then*

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

**IV. Monotonicity:** *If* $A, B \in \mathcal{M}(I)$ *and* $A \subset B$, *then* $\mu(A) \leq \mu(B)$.

**V. Null sets:** *If a subset* $A \subset I$ *is a null set, then* $A \in \mathcal{M}(I)$ *and* $\mu(A) = 0$. *Conversely, if* $A \in \mathcal{M}(I)$ *and* $\mu(A) = 0$, *then* $A$ *is a null set.*

**VI. Regularity:** *If* $A \in \mathcal{M}(I)$, *then*

$$\mu(A) = \inf\{\mu(U) \mid U \text{ is open and } A \subset U\}.$$

Note that the countable additivity of property III implies the analogous statements about finite additivity. Given a finite collection $\{A_n\}_{n=1}^m$ of sets just let $A_i = \emptyset$ for $i > m$ and the analogous conclusions follow.

We have relegated the proof of this theorem to Appendix A, because it is somewhat technical and it is a diversion from our main task of developing a theory of integration. However, it is worth noting that properties I, III and VI imply the other three and we have included this as an exercise in this section.

Recall that *set difference* $A \setminus B = \{x \in A \mid x \notin B\}$. Since we are focusing on subsets of $I$ complements are with respect to $I$, so $A^c = I \setminus A$.

**Proposition 2.4.3.** *If* $A$ *and* $B$ *are in* $\mathcal{M}(I)$, *then* $A \setminus B$ *is in* $\mathcal{M}(I)$ *and* $\mu(A \cup B) = \mu(A \setminus B) + \mu(B)$. *In particular, if* $I = [0, 1]$, *then* $\mu(I) = 1$, *so* $\mu(A^c) = 1 - \mu(A)$.

**Proof.** Note that $A \setminus B = A \cap B^c$ which is in $\mathcal{M}(I)$. Also, $A \setminus B$ and $B$ are disjoint and their union is $A \cup B$. So additivity implies that $\mu(A \setminus B) + \mu(B) = \mu(A \cup B)$. Since $A^c = I \setminus A$ this implies $\mu(A \setminus I) + \mu(A) = \mu(A^c \cup A) = \mu(I) = 1$. □

We have already discussed properties I-IV and null sets, but property VI is new and it is worth discussing. It is extremely useful because it allows us to approximate arbitrary measurable sets by sets we understand better. In fact, it gives us a way to approximate any measurable set $A$ "from the outside" by a countable union of pairwise disjoint open intervals and "from the inside" by a closed set. More precisely, we have the following:

**Proposition 2.4.4. (Regularity).** *If $A \in \mathcal{M}(I)$ and $\varepsilon > 0$, then there is a closed set $C \subset A$ such that*

$$\mu(C) > \mu(A) - \varepsilon$$

*and a countable union of pairwise disjoint open intervals $U = \bigcup U_n$ such that*

$$A \subset U \quad \text{and} \quad \mu(U) < \mu(A) + \varepsilon.$$

**Proof.** Given $\varepsilon > 0$ the existence of an open set $U$ with $A \subset U$ and $\mu(U) < \mu(A) + \varepsilon$ is exactly a restatement of property VI. Any open set $U$ is a countable union of pairwise disjoint open intervals by Theorem A.6.3.

To see the existence of $C$ let $V$ be an open set containing $A^c$ with $\mu(V) < \mu(A^c) + \varepsilon$. Then $C = V^c$ is closed and a subset of $A$. Also, $\mu(C) = 1 - \mu(V) > 1 - \mu(A^c) - \varepsilon = \mu(A) - \varepsilon$. □

If we have a countable increasing family of measurable sets, then the measure of the union can be expressed as a limit.

**Proposition 2.4.5.** *If $A_1 \subset A_2 \subset \cdots \subset A_n \ldots$ is an increasing sequence of measurable subsets of $I$, then*

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \mu(A_n).$$

*If $B_1 \supset B_2 \supset \cdots \supset B_n \ldots$ is a decreasing sequence of measurable subsets of $I$, then*

$$\mu(\bigcap_{n=1}^{\infty} B_n) = \lim_{n\to\infty} \mu(B_n).$$

**Proof.** Let $F_1 = A_1$ and $F_n = A_n \setminus A_{n-1}$ for $n > 1$. Then $\{F_n\}_{n=1}^{\infty}$ are pairwise disjoint measurable sets, $A_n = \bigcup_{i=1}^{n} F_i$ and

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} F_i.$$

Hence, by countable additivity we have

$$\begin{aligned}\mu\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \mu\Big(\bigcup_{i=1}^{\infty} F_i\Big) &= \sum_{i=1}^{\infty} \mu(F_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(F_i) = \lim_{n\to\infty} \mu\Big(\bigcup_{i=1}^{n} F_i\Big) \\ &= \lim_{n\to\infty} \mu(A_n).\end{aligned}$$

For the decreasing sequence we define $E_n = B_n^c$. Then $\{E_n\}_{n=1}^{\infty}$ is an increasing sequence of measurable functions and

$$\Big(\bigcap_{n=1}^{\infty} B_n\Big)^c = \bigcup_{n=1}^{\infty} E_n.$$

Hence,

$$\begin{aligned}\mu\Big(\bigcap_{n=1}^{\infty} B_n\Big) &= 1 - \mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) \\ &= 1 - \lim_{n\to\infty} \mu(E_n) \\ &= \lim_{n\to\infty} (1 - \mu(E_n)) \\ &= \lim_{n\to\infty} \mu(B_n).\end{aligned}$$

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**Exercise 2.4.6.**

(1) Prove for $a, b \in I$ that $\mu([a,b]) = \mu((a,b]) = \mu([a,b)) = b-a$.

(2) Let $X$ be the subset of irrational numbers in $I$. Prove $\mu(X) = 1$. Prove that if $Y \subset I$ is a closed set and $\mu(Y) = 1$, then $Y = I$.

(3) If $A$ and $B$ are measurable subsets of $[0,1]$, prove that

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B).$$

(4) Prove that if $X \subset I$ is measurable, then for any $\varepsilon > 0$ there is an open set $U$ containing $X$ such that $\mu(U \setminus X) < \varepsilon$. This is sometimes referred to as the first of *Littlewood's three principles.*

(5) Suppose $a < b$ and let $\mathcal{M}([a,b])$ denote the Lebesgue measurable subsets of $[a,b]$. Define the function $f : [0,1] \to [a,b]$ by $f(x) = mx + a$ where $m = b - a$. Show that the correspondence $A \mapsto f(A)$ is a bijection from $\mathcal{M}([0,1])$ to $\mathcal{M}([a,b])$. Define the function $\mu_0 : \mathcal{M}([a,b]) \to \mathbb{R}$ by $\mu_0(A) = m\mu(f^{-1}(A))$. Prove that Theorem 2.4.2 remains valid if $I$ is replaced by $[a,b]$ and $\mu$ is replaced by $\mu_0$.

(6) The *symmetric difference* between two sets $A$ and $B$ is defined to be $(A \setminus B) \cup (B \setminus A)$. It is denoted $A \,\Delta\, B$. Suppose $A_n \subset [a,b]$ for $n \in \mathbb{N}$ is measurable and $B$ is also. Prove that if $\lim_{n\to\infty} \mu(A_n \,\Delta\, B) = 0$, then $\lim_{n\to\infty} (\mu(A_n)) = \mu(B)$.

(7) In this exercise we show that properties I, III and VI of Theorem 2.4.2 actually imply the other three properties. Let $\mu$ be a function from $\mathcal{M}(I)$ to the non-negative real numbers satisfying properties I, III and VI.
   (a) Prove that if $A, B \in \mathcal{M}(I)$ and $A \subset B$, then $\mu(A) \leq \mu(B)$, i.e., property IV is satisfied. (This only requires property III.)
   (b) Prove that if $X \subset I$ is a null set, then $X \in \mathcal{M}(I)$ and $\mu(X) = 0$. (This only requires properties I and III.)
   (c) Conversely, prove that if $X \in \mathcal{M}(I)$ and $\mu(X) = 0$, then $X$ is a null set.
   (d) Prove that $\mu$ satisfies property II.

## 2.5. The Lebesgue Density Theorem

The following theorem asserts that if a subset of an interval $I$ is "equally distributed" throughout the interval, then it must be a null set or a set of full measure, i.e., the complement of a null set. For example, it is not possible to have a set $A \subset [0,1]$ which contains half of each subinterval, i.e., it is impossible to have

$$\mu(A \cap [a,b]) = \mu([a,b])/2$$

for all $0 < a < b < 1$. There will always be small intervals with a "high concentration" of points of $A$ and other subintervals with a low concentration. Put another way, it asserts that given any $p < 1$ there is an interval $U$ such that a point in $U$ has “probability” at least $p$ of being in $A$.

**Theorem 2.5.1.** *If $A$ is a Lebesgue measurable set and $\mu(A) > 0$ and if $0 < p < 1$, then there is an open interval $U = (a, b)$ such that $\mu(A \cap U) \geq p\mu(U) = p(b - a)$.*

**Proof.** Let $p \in (0, 1)$ be given. We know from Proposition 2.4.4 that for any $\varepsilon > 0$ there is an open set $V$ which contains $A$ such that $\mu(V) < \mu(A) + \varepsilon$ and that we can express $V$ as $V = \bigcup_{n=1}^{\infty} U_n$ where $\{U_n\}_{n=1}^{\infty}$ is a countable collection of pairwise disjoint open intervals.

Then

$$\mu(A) \leq \mu(V) = \sum_{n=1}^{\infty} \operatorname{len}(U_n) < \mu(A) + \varepsilon.$$

Choosing $\varepsilon = (1 - p)\mu(A)$ we get

$$\begin{aligned}\sum_{n=1}^{\infty} \operatorname{len}(U_n) &< \mu(A) + (1 - p)\mu(A) \\ &< \mu(A) + (1 - p) \sum_{n=1}^{\infty} \operatorname{len}(U_n),\end{aligned}$$

so

$$p \sum_{n=1}^{\infty} \operatorname{len}(U_n) < \mu(A) \leq \sum_{n=1}^{\infty} \mu(A \cap U_n), \tag{2.5.1}$$

where the last inequality follows from subadditivity. Since these infinite series have finite sums, there is at least one $n_0$ such that $p\mu(U_{n_0}) \leq \mu(A \cap U_{n_0})$. This is because if it were the case that $p\mu(U_n) > \mu(A \cap U_n)$ for all $n$, then it would follow that

$$p \sum_{n=1}^{\infty} \operatorname{len}(U_n) > \sum_{n=1}^{\infty} \mu(A \cap U_n),$$

contradicting equation (2.5.1). The interval $U_{n_0}$ is the $U$ we want. □

We have shown that given $p \in (0, 1)$ as close to 1 as we like, there is an open interval in which the “relative density” of $A$ is at least $p$. It is often useful to have these intervals all centered at a particular point called a density point.

**Definition 2.5.2. (Density point).** *If $A$ is a Lebesgue measurable set and $x \in A$, then $x$ is called a* Lebesgue density point *if*

$$\lim_{\varepsilon \to 0} \frac{\mu(A \cap [x-\varepsilon, x+\varepsilon])}{\mu([x-\varepsilon, x+\varepsilon])} = 1.$$

There is a much stronger result than Theorem 2.5.1 above, which we now state, but do not prove. A proof can be found in Section 9.2 of [**T**].

**Theorem 2.5.3. (Lebesgue density theorem).** *If $A$ is a Lebesgue measurable set, then there is a subset $E \subset A$ with $\mu(E) = 0$ such that every point of $A \setminus E$ is a Lebesgue density point.*

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**Exercise 2.5.4.**

(1) Prove that if $A \subset I = [0,1]$ has measure $\mu(A) < 1$ and $\varepsilon > 0$, then there is an interval $[a,b] \subset I$ such that $\mu(A \cap [a,b]) < \varepsilon(b-a)$.

(2) Let $A$ be a measurable set with $\mu(A) > 0$ and let

$$\Delta = \{x_1 - x_2 \mid x_1, x_2 \in A\}$$

be the set of differences of elements of $A$. Then for some $\varepsilon > 0$ the set $\Delta$ contains the interval $(-\varepsilon, \varepsilon)$.

## 2.6. Lebesgue Measurable Sets – Summary

In this section we provide a summary outline of the key properties of the collection $\mathcal{M}$ of Lebesgue measurable sets which have been developed in this chapter. If $I$ is a closed interval, then $\mathcal{M}(I)$ denotes the subsets of $I$ which are in $\mathcal{M}$.

(1) The collection of Lebesgue measurable sets $\mathcal{M}$ is a *$\sigma$-algebra*, which means:
   - If $A \in \mathcal{M}$, then $A^c \in M$.
   - If $A_n \in \mathcal{M}$ for $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in M$.
   - If $A_n \in \mathcal{M}$ for $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} A_n \in M$.

(2) All open sets and all closed sets are in $\mathcal{M}$. Any null set is in $\mathcal{M}$.

(3) If $I = [0, 1]$ and $A \in \mathcal{M}(I)$, then there is a non-negative real number $\mu(A)$ called its Lebesgue measure which satisfies:

- The Lebesgue measure of an interval is its length.
- Lebesgue measure is translation invariant.
- If $A \in \mathcal{M}(I)$, then $\mu(A^c) = 1 - \mu(A)$.
- A set $A \in \mathcal{M}(I)$ is a null set if and only if $\mu(A) = 0$.
- *Countable subadditivity:* If $A_n \in \mathcal{M}(I)$ for $n \in \mathbb{N}$, then
$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) \leq \sum_{n=1}^{\infty} \mu(A_n).$$
- *Countable additivity:* If $A_n \in \mathcal{M}(I)$ for $n \in \mathbb{N}$ are pairwise disjoint sets, then
$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mu(A_n).$$
- *Regularity:* If $A \in \mathcal{M}(I)$, then
$$\mu(A) = \inf\{\mu(U) \mid U \text{ is open and } \ A \subset U\}.$$
- *Increasing sequences*: If $A_n \in \mathcal{M}(I)$ for $n \in \mathbb{N}$ satisfy $A_n \subset A_{n+1}$, then
$$\mu\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \mu(A_n).$$
- *Decreasing sequences*: If $A_n \in \mathcal{M}(I)$ for $n \in \mathbb{N}$ satisfy $A_n \supset A_{n+1}$, then
$$\mu\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \mu(A_n).$$

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**Exercise 2.6.1. (The Cantor middle third set).** Recursively define a nested sequence $\{J_n\}_{n=0}^{\infty}$ of closed subsets of $I = [0, 1]$. Each $J_n$ consists of a finite union of closed intervals. We define $J_0$ to be $I$ and let $J_n$ be the union of the closed intervals obtained by deleting the open middle third interval from each of the intervals in $J_{n-1}$.

Thus

$$
\begin{aligned}
J_0 &= [0, 1],\\
J_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right],\\
J_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \text{ etc.}
\end{aligned}
$$

We define the *Cantor middle third set* $C$ by

$$C = \bigcap_{n=0}^{\infty} J_n.$$

(1) When the open middle thirds of the intervals in $J_{n-1}$ are removed we are left with two sets of closed intervals: the left thirds of the intervals in $J_{n-1}$ and the right thirds of these intervals. We denote the union of the left thirds by $L_n$ and the right thirds by $R_n$, and we note that $J_n = L_n \cup R_n$. Prove that $L_n$ and $R_n$ each consist of $2^{n-1}$ intervals of length $1/3^n$ and hence $J_n$ contains $2^n$ intervals of length $1/3^n$.

(2) (*Topological properties*)

(a) Prove $C$ is compact.

(b) A closed subset of $\mathbb{R}$ is called *nowhere dense* if it contains no non-empty open interval. Prove that $C$ is nowhere dense.

(c) A closed subset $A$ of $\mathbb{R}$ is called *perfect* if for every $\varepsilon > 0$ and every $x \in A$ there is $y \in A$ with $x \neq y$ and $|x - y| < \varepsilon$. Prove that $C$ is perfect.

(3) Let $\mathcal{D}$ be the uncountable set of all infinite sequences

$$d_1 d_2 d_3 \dots d_n \dots,$$

where each $d_n$ is either 0 or 1 (see part (4) of Exercise A.5.12) and define a function $\psi : C \to \mathcal{D}$ by $\psi(x) = d_1 d_2 d_3 \dots d_n \dots$, where each $d_n = 0$ if $x \in L_n$ and $d_n = 1$ if $x \in R_n$. Prove that $\psi$ is surjective and hence by Corollary A.5.7 the set $C$ is uncountable. *Hint:* You will need to use Theorem A.7.3. Prove that $\psi$ is also injective and hence a bijection.

(4) Prove that $C$ is Lebesgue measurable and that $\mu(C) = 0$. *Hint:* Consider $C^c$, the complement of $C$ in $I$. Show it

is measurable and calculate $\mu(C^c)$. *Alternative hint:* Show directly that $C$ is a null set by finding for each $\varepsilon > 0$ a collection of open intervals $\{U_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} \operatorname{len}(U_n) < \varepsilon \quad \text{and} \quad C \subset \bigcup_{n=1}^{\infty} U_n.$$

(5) Prove that $C$ is the subset of elements of $[0, 1]$ which can be represented in base three using only the digits 0 and 2. More precisely, prove that $x \in C$ if and only if it can be expressed in the form

$$x = \sum_{n=1}^{\infty} \frac{c_n}{3^n}$$

where each $c_n$ is either 0 or 2.