
Preface

I was 21 years when I wrote this song
I'm 22 now, but I won't be for long
Time hurries on
And the leaves that are green turn to brown
– Paul Simon, *Leaves that Are Green*

1968 was a tumultuous year. America was convulsed by the Vietnam War, nowhere more than on college campuses. The assassinations of Martin Luther King and of Robert Kennedy tore at the nation's heart. The Democratic convention in Chicago was marked by violent riots. America, for many, had become Amerika, the villain. “Do your own thing” was the admonition that resonated so powerfully. Resist authority. Nonconformity was the supreme virtue. For this fledgling mathematician it was a critical juncture. I had left graduate school without a degree. Would my talents find a focus in this chaotic world? My mind swirled with mathematical ideas, but I seemed unable to turn these ideas into a cohesive whole.

Then I met Paul Erdős. Everyone called him *Uncle Paul*.

While others spoke constantly of it, nonconformity was always Uncle Paul's modus operandi. He had no job; he worked constantly. He had no home; the world was his home. Possessions were a nuisance; money a bore. Paul lived on a web of trust, traveling ceaselessly

from center to center spreading his mathematical pollen. “Prove and Conjecture!” was his constant refrain.

Were we, in those halcyon days, *students* of Uncle Paul. I think the word inadequate and inaccurate. Better to say that we were *disciples* of Paul Erdős. We (and the list is long indeed) had energy and talent. Paul, through his actions and his theorems and his conjectures and every fibre of his being, showed us the Temple of Mathematics. The Pages of The Book were there, we had only to open them. Does there exist for all sufficiently large n a triangle free graph on n vertices which does not contain an independent set of size $\sqrt{n \ln n}$? We had no doubts—the answer was either yes or no. The answer was in The Book. Pure thought—our thought—would allow its reading.

I would sit with Uncle Paul and discuss an open problem. Paul would have a blank pad of paper on his lap. “Suppose,” he would say in his strong Hungarian accent,¹ “we set

$$p = \sqrt{\frac{\ln n}{n}}.$$

He would write the formula for p on the blank page and nothing else. Then his mind sped on, showing how this particular value of p led to the solution. How, I wondered, did Uncle Paul know which value of p to take?

The final form of mathematics, the form that students see in textbooks, was described by Bertrand Russell:

Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry.

¹The documentary film “ N is a Number” by George Csicsery [Cs193], available on the web, shows Uncle Paul in action.

Doing mathematics is anything but austere. As an undergraduate teacher of mine, Gian-Carlo Rota, put it:

A mathematician's work is mostly a tangle of guesswork, analogy, wishful thinking and frustration, and proof, far from being the core of discovery, is more often than not a way of making sure that our minds are not playing tricks.

That said, the “guesswork” can be finely honed. Uncle Paul's selection of the right p did not come at random. Brilliance, of course, is more than helpful. But we mortals can also sometimes succeed.

Paul Erdős lived² in Asymptopia. Primes less than n , graphs with v vertices, random walks of t steps—Erdős was fascinated by the limiting behavior as the variables approached, but never reached, infinity. Asymptotics is very much an art. In his masterwork, *The Periodic Table*, Primo Levi speaks of the personalities of the various elements. A chemist will feel when atoms want or do not want to bind. In asymptotics the various functions $n \ln n$, n^2 , $\frac{\ln n}{n}$, $\sqrt{\ln n}$, $\frac{1}{n \ln n}$ all have distinct personalities. Erdős knew these functions as personal friends. This author had the great privilege and joy of learning directly from Paul Erdős. It is my hope that these insights may be passed on, that the reader may similarly feel which function has the right temperament for a given task.

My decision to write this work evolved over many years, and it was my students who opened my eyes. I would teach courses in discrete mathematics, probability, Ramsey theory, graph theory, the probabilistic method, number theory, and other areas. I would carefully give, for example, Erdős's classic result (Theorem 7.1) on Ramsey numbers: If

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1,$$

then $R(k, k) > n$. I spent much less time on the asymptotic implication (7.6), that $R(k, k) \geq (1 + o(1)) \frac{k}{e\sqrt{2}} \sqrt{2}^k$. My students showed me

²Erdős's breadth was extraordinary. This refers to only one aspect of his oeuvre.

that Asymptopia deserved its own emphasis. A facility with asymptotic calculations could be taught, could be learned, and was a highly pragmatic element of their mathematical education.

Laura Florescu began her graduate studies at the Courant Institute in the fall of 2012. Almost immediately we began our study of mathematical results, old and new, and began work on open questions. This pursuit happily continues to this day. Early on, with this project still in its nascent phase, Laura graciously offered her assistance. We have discussed ideas for the various chapters and sections together. Some of the ideas, such as giving a proof of the Law of the Iterated Logarithm, originated entirely from her and all of the ideas were jointly discussed. She has written early drafts of many sections. (However, all errors in the final copy, what you are reading now, are my responsibility.) I hope that this project has been as much a learning experience for Laura as it has been for me. With her talents and energy, Laura has a bright future ahead of her. Thank you, Laura.

My editor, Ina Mette, deserves special recognition. We have known each other for many years and I had always wanted to write a book under her editorship. Conversations about this current work took place over a long period of time. Through lunches at Lure in New York, at Central Kávéház in Budapest, through numerous emails and phone calls, the outlines of this current work came into focus. Ina has always been insightful in her suggestions and fully supportive of my oftentimes ill-defined ruminations. Thank you, Ina.

1968 was special for me personally as well as professionally. It was the year I married my wife Mary Ann, whom I wish to thank once again for her assistance, encouragement, and understanding. Without her, this enterprise would have had little meaning.

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