

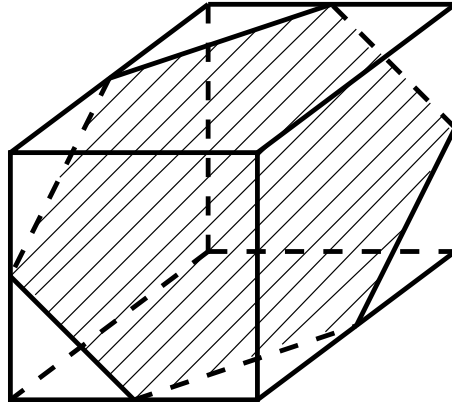
CHAPTER 1

Introduction

The study of the geometry of convex bodies based on information about sections and projections of these bodies has important applications to many areas of mathematics and science. In this book we discuss a new Fourier analytic approach that has recently been developed. The idea is to express certain geometric properties of bodies in Fourier analytic terms and then use methods of harmonic analysis to solve geometric problems. We start with a few examples.

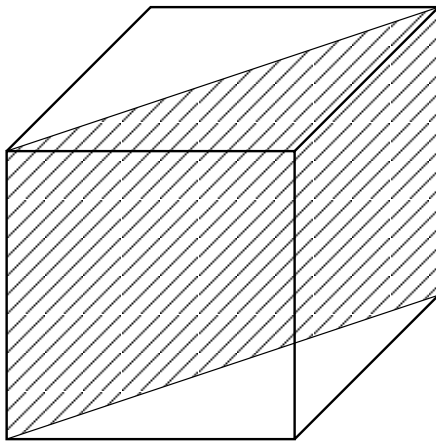
The Fourier analytic approach to sections of convex bodies is based on certain formulas expressing the volume of sections in terms of the Fourier transform of powers of the Minkowski functional of a body. The first formula of this kind was known to Laplace [La], who wrote that the $(n-1)$ -dimensional volume of the central hyperplane section of the unit cube Q_n in \mathbb{R}^n perpendicular to the main diagonal is equal to

$$(1.1) \quad \text{Vol}_{n-1}(Q_n \cap (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})^\perp) = \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{\sin(r/\sqrt{n})}{r/\sqrt{n}} \right)^n dr.$$



Section orthogonal to $(1, 1, \dots, 1)$.

The latter integral is exactly what appears in the central limit theorem for the sums of uniformly distributed random variables, and it is easy to see that the volume of the diagonal section tends to $6/\sqrt{\pi}$, as $n \rightarrow \infty$. This result is quite surprising, because it contradicts our intuition that the section perpendicular to the main diagonal has the maximal volume among all central hyperplane sections of the cube. In fact, in every dimension the section perpendicular to the vector $(1, 1, 0, \dots, 0)$ has volume $\sqrt{2} > 6/\sqrt{\pi}$.

Section orthogonal to $(1, 1, 0, \dots, 0)$.

The problem of finding the maximal volume of central hyperplane sections of the cube remained open until the 1980's, when first Hensley [He1] proved that these volumes are bounded from above by a constant not depending on the dimension, and then Ball [Ba1] found that this constant is exactly $\sqrt{2}$. Both Hensley and Ball used a general Fourier transform formula for the volume of hyperplane sections of the cube first mentioned in the literature by Polya [Po1]:

$$(1.2) \quad \text{Vol}_{n-1}(Q_n \cap \xi^\perp) = \frac{1}{\pi} \int_{-\infty}^{\infty} \prod_{k=1}^n \frac{\sin(r\xi_k)}{r\xi_k} dr,$$

where, for $\xi \in S^{n-1}$, $\xi^\perp = \{x \in \mathbb{R}^n : (x, \xi) = 0\}$ is the central hyperplane perpendicular to the vector ξ . A proof of Ball's result is presented in Chapter 7, Theorem 7.1.

Meyer and Pajor [MeP] found a similar formula for the volume of central hyperplane sections of ℓ_q -balls:

$$(1.3) \quad \text{Vol}_{n-1}(B_q^n \cap \xi^\perp) = \frac{q}{\pi(n-1)\Gamma((n-1)/q)} \int_0^\infty \prod_{k=1}^n \gamma_q(t\xi_k) dt,$$

where γ_q is the Fourier transform of the function $\exp(-|\cdot|^q)$ on \mathbb{R} , and B_q^n is the unit ball of the space ℓ_q^n :

$$(1.4) \quad B_q^n = \{x \in \mathbb{R}^n : \|x\|_q = \left(\sum_{k=1}^n |x_k|^q \right)^{1/q} \leq 1\}.$$

Our work in the area of convex geometry was initiated by formula (1.3). Having known the expression for the Fourier transform of powers of ℓ_q -norms, Lemma 3.6, one can recognize in the right-hand side of (1.3) the Fourier transform of the function $\|\cdot\|_q^{-n+1}$ at the point ξ , up to a constant not depending on q . This suggests that (1.3) is a particular case of a more general formula, and this is indeed the case, as proved in [K7]:

THEOREM 1.1. *For any origin-symmetric star body K in \mathbb{R}^n and any $\xi \in S^{n-1}$,*

$$(1.5) \quad \text{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{\pi(n-1)} (\|\cdot\|_K^{-n+1})^\wedge(\xi),$$

where $\|\cdot\|_K$ is the Minkowski functional of K . The Fourier transform of $\|\cdot\|_K^{-n+1}$ is considered in the sense of distributions and as such coincides with a homogeneous of degree -1 function on \mathbb{R}^n , which is continuous on $\mathbb{R}^n \setminus \{0\}$.

This formula has many geometric applications. For example, the Minkowski uniqueness theorem (see [Ga3, Th. 7.2.3]) states that every origin-symmetric star body is uniquely determined by the volumes of its central hyperplane sections. By (1.5), this result immediately follows from the uniqueness theorem for the Fourier transform, because $(\|\cdot\|_K^{-n+1})^\wedge$ is an even homogeneous function on \mathbb{R}^n .

The Fourier analytic approach is quite effective in the study of intersection bodies. Let K and L be origin-symmetric star bodies in \mathbb{R}^n . Following Lutwak [Lu1], we say that K is the *intersection body of L* if the radius of K in every direction is equal to the volume of the central hyperplane section of L perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$,

$$(1.6) \quad \|\xi\|_K^{-1} = \text{Vol}_{n-1}(L \cap \xi^\perp).$$

A more general class of *intersection bodies* can be defined as the closure in the radial metric of the class of intersection bodies of star bodies.

Using formula (1.5), one can get a Fourier analytic characterization of intersection bodies. In fact, applying (1.5) to the right-hand side of (1.6), we get

$$\|\xi\|_K^{-1} = \frac{1}{\pi(n-1)} (\|\cdot\|_L^{-n+1})^\wedge(\xi).$$

Since the Fourier transform of even distributions is self-invertible (up to a constant),

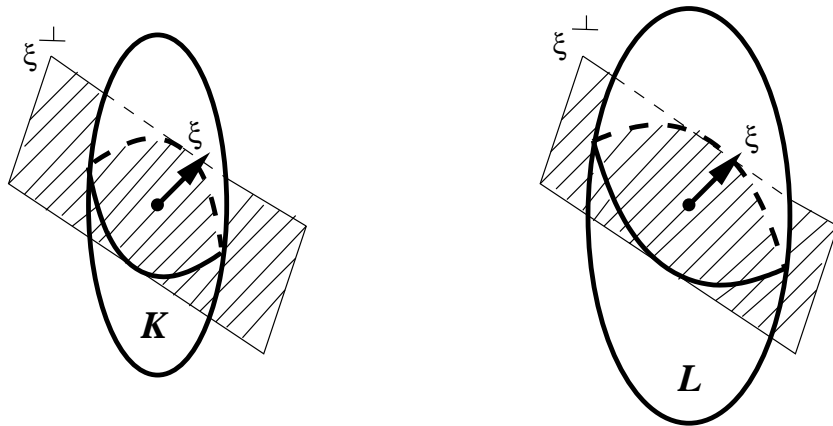
$$(\|\cdot\|_K^{-1})^\wedge(\xi) = \frac{(2\pi)^n}{\pi(n-1)} \|\xi\|_L^{-n+1}.$$

In particular, the Fourier transform of $\|\cdot\|_K^{-1}$ is non-negative. This consideration leads to a Fourier analytic characterization of intersection bodies that was proved in [K10]:

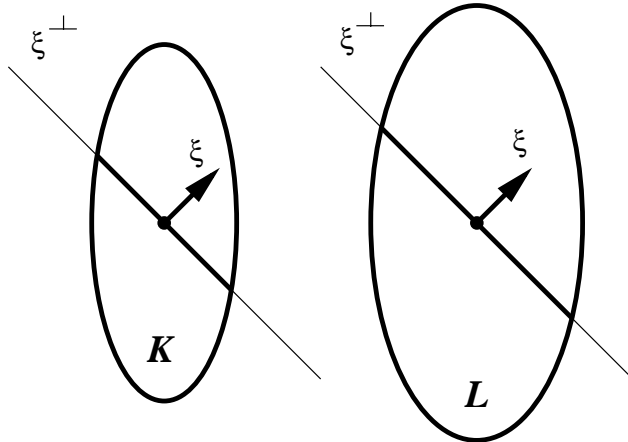
THEOREM 1.2. *An origin-symmetric star body K in \mathbb{R}^n is an intersection body if and only if the Fourier transform of $\|\cdot\|_K^{-1}$ is a positive distribution, i.e. its action on any non-negative test function gives a non-negative result.*

In Section 4.1. we prove Theorem 1.2. This characterization allows us to give many examples of bodies that are and are not intersection bodies. For instance, as proved in [K10], the unit ball of any finite dimensional subspace of L_q with $0 < q \leq 2$ is an intersection body. On the other hand, as proved in [K8], the unit balls of the spaces ℓ_q^n , $2 < q < \infty$, are intersection bodies if the dimension $n \leq 4$, but they are not intersection bodies if $n \geq 5$. We prove these facts in Section 4.3, Theorems 4.11 and 4.13.

Intersection bodies played an important role in the solution to the Busemann-Petty problem. Posed in 1956 in [BP], the Busemann-Petty problem asks the following question. Let K and L be origin-symmetric convex bodies in \mathbb{R}^n so that the $(n-1)$ -dimensional volume of every central hyperplane section of K is smaller than the same for L . Does it follow that the n -dimensional volume of K is smaller than the n -dimensional volume of L ?



The Cavalieri principle and the Minkowski uniqueness theorem mentioned above provide strong intuition in the direction of the affirmative answer. Indeed, how can the answer to the Busemann-Petty problem be negative if every origin-symmetric convex body is uniquely determined by the volumes of its central hyperplane sections? Another piece of evidence in favor of the affirmative answer is that it is true in dimension $n = 2$, because the assumption of the problem implies that $K \subset L$.



Also, during the first twenty years the results related to the problem all went in the positive direction. Busemann [Bu2] proved that the answer is affirmative if K is an ellipsoid. Hadwiger [Ha2] gave an affirmative answer in the case where K and L are solids of revolution in \mathbb{R}^3 .

It came as a big surprise when, in 1975, Larman and Rogers [LR] used probabilistic methods to prove that the answer is negative if the dimension $n \geq 12$. Ball [Ba2] used his $\sqrt{2}$ bound mentioned above to construct counterexamples for $n \geq 10$. Ball's examples are the most "visible" so far, if one can say that about objects in \mathbb{R}^{10} : the body with smaller sections is the unit cube and the body with larger sections is the Euclidean ball whose radius is chosen in such a way that the volume of each central hyperplane section is equal to $\sqrt{2}$. Giannopoulos [Gi] and Bourgain [Bo1] constructed counterexamples for $n \geq 7$. Papadimitrakis [Pa] proved that the answer is negative for $n \geq 5$, and Gardner [Ga1] and Zhang [Zh2] proved the same by constructing the first examples of non-intersection bodies in \mathbb{R}^5 . They

used Lutwak's connection with intersection bodies (see Theorem 1.3). In another surprising development, Gardner [Ga2] proved that every origin-symmetric convex body in \mathbb{R}^3 is an intersection body, and, hence, the answer to the Busemann-Petty problem in dimension 3 is affirmative. For several years the problem was considered as completely solved, since the paper [Zh2] published in 1994 claimed that the unit cube in \mathbb{R}^4 was not an intersection body, which would imply a negative answer in dimension 4. However, it was proved in [K8] in 1997 that the cube in \mathbb{R}^4 is an intersection body, which reopened the four-dimensional case and suggested that the answer for $n = 4$ must be affirmative instead of negative. After this, Zhang [Zh4] proved that the answer to the Busemann-Petty problem in dimension 4 is indeed affirmative, and a unified Fourier analytic solution to the problem in all dimensions was given in [GKS].

The last steps of the solution were made using two different approaches. Both of them used Lutwak's concept of an intersection body. However, the study of intersection bodies, which requires inversion of the spherical Radon transform in a special situation, was done differently. The first, more geometric, approach was developed by Gardner [Ga1], [Ga2] and Zhang [Zh1], [Zh4]. The second, Fourier analytic, approach of [K10] and [GKS] provided a unified solution to the problem in all dimensions at the same time. This approach clearly exposed the reason behind the transition between dimensions 4 and 5 and traced this transition to the fact that convexity controls the derivatives of the second order, but not the derivatives of higher orders. The main ideas of the geometric approach can be found in [Ga3], so we concentrate on the Fourier analytic part.

The Fourier analytic proof in [GKS] is based on three main ingredients. The first is a connection between intersection bodies and the Busemann-Petty problem found by Lutwak [Lu1] and slightly modified by Gardner [Ga1] and Zhang [Zh1], [Zh2]. First, if K is an intersection body, then the answer to the question of the Busemann-Petty problem is affirmative for this K and any origin-symmetric star body L . On the other hand, if L is an origin-symmetric convex body that is not an intersection body, then one can perturb L to construct a counterexample to the Busemann-Petty problem. Using these connections, Lutwak reduced the Busemann-Petty problem to the study of intersection bodies:

THEOREM 1.3. *A solution to the Busemann-Petty problem in \mathbb{R}^n is affirmative if and only if every origin-symmetric convex body in \mathbb{R}^n is an intersection body.*

We prove Lutwak's connections twice. First we present the original geometric proofs in Section 5.1, and then we prove more general facts in Section 5.2 using Fourier methods.

The solution in [GKS] also uses the Fourier analytic characterization of intersection bodies from Theorem 1.2 and the following formula proved in [GKS]:

THEOREM 1.4. *Let D be an infinitely smooth origin-symmetric convex body in \mathbb{R}^n , $k \in \mathbb{N} \cup \{0\}$, $k \neq n - 1$, $\xi \in S^{n-1}$. Then if k is an even integer,*

$$(\|\cdot\|_D^{-n+k+1})^\wedge(\xi) = (-1)^{k/2} \pi(n-k-1) A_{D,\xi}^{(k)}(0).$$

If k is an odd integer,

$$\begin{aligned} & (\|\cdot\|_D^{-n+k+1})^\wedge(\xi) \\ &= (-1)^{(k+1)/2} 2(n-k-1)k! \int_0^\infty \frac{A_{D,\xi}(z) - A_{D,\xi}(0) - \dots - A_{D,\xi}^{(k-1)}(0) \frac{z^{k-1}}{(k-1)!}}{z^{k+1}} dz. \end{aligned}$$

Here $A_{D,\xi}^{(k)}(0)$ is the derivative of the order k at zero of the parallel section function $A_{D,\xi}$ of the body D in the direction of ξ defined by

$$A_{D,\xi}(z) = \text{Vol}_{n-1}(D \cap \{\xi^\perp + z\xi\}), \quad z \in \mathbb{R}.$$

In both cases, the Fourier transform of $\|\cdot\|_D^{-n+k+1}$ is considered in the sense of distributions and coincides with a homogeneous of degree $-k-1$ function on \mathbb{R}^n , which is infinitely smooth on $\mathbb{R}^n \setminus \{0\}$.

Let us show how the solution to the Busemann-Petty problem follows from these results. By a simple approximation argument, it is enough to consider the case where the bodies K and L are infinitely smooth. Suppose that the dimension $n = 4$. Putting $n = 4$, $k = 2$ in Theorem 1.4, we get that, for every $\xi \in S^{n-1}$,

$$(\|\cdot\|_K^{-1})^\wedge(\xi) = -\pi A_{K,\xi}''(0).$$

Now we use the fact that the body K is origin-symmetric convex. By a well-known theorem of Brunn, Theorem 2.3, the central section has maximal volume among all hyperplane sections of K perpendicular to a given direction. This means that the function $A_{K,\xi}$ has maximum at zero, so the second derivative of this function at zero is less than or equal to zero. Hence, the Fourier transform of $\|\cdot\|_K^{-1}$ is non-negative for every origin-symmetric convex body K in \mathbb{R}^4 . By the Fourier characterization of intersection bodies, Theorem 1.2, every origin-symmetric convex body in \mathbb{R}^4 is an intersection body, and the affirmative solution to the Busemann-Petty problem follows from Lutwak's connection, Theorem 1.3.

On the other hand, suppose that the dimension $n = 5$. Put $k = 3$ in Theorem 1.4. We use the second formula of this theorem, and we are essentially looking at the third derivative of the function $A_{K,\xi}$. But convexity does not control the third derivatives! With this in mind, it is quite easy to construct an example of an infinitely smooth convex body for which the Fourier transform of $\|\cdot\|_K^{-1}$ is sign-changing (we construct such an example in Corollary 4.4). Again by the Fourier analytic characterization of intersection bodies, this gives an example of an infinitely smooth convex body in \mathbb{R}^5 that is not an intersection body, and a negative solution to the Busemann-Petty problem in dimension 5 follows from Lutwak's connection, Theorem 1.3.

This solution clearly shows that the transition between the dimensions 4 and 5 in the Busemann-Petty problem occurs due to the fact that convexity controls the second derivatives, but not the derivatives of higher orders.

We finish the preview by showing a Fourier analytic proof of the first Lutwak connection. This proof is short and gives a good idea of how the Fourier approach works. We are going to prove a more general result later in Theorem 5.5.

THEOREM 1.5. *Let K be an infinitely smooth origin-symmetric intersection body in \mathbb{R}^n , and let L be any origin-symmetric star body in \mathbb{R}^n so that, for every $\xi \in S^{n-1}$,*

$$(1.7) \quad \text{Vol}_{n-1}(K \cap \xi^\perp) \leq \text{Vol}_{n-1}(L \cap \xi^\perp).$$

Then $\text{Vol}_n(K) \leq \text{Vol}_n(L)$.

PROOF. By Theorem 1.1, the Fourier transforms of $\|\cdot\|_K^{-n+1}$ and $\|\cdot\|_L^{-n+1}$ are continuous functions on $\mathbb{R}^n \setminus \{0\}$, and condition (1.7) can be written in the form

$$(1.8) \quad (\|\cdot\|_K^{-n+1})^\wedge(\xi) \leq (\|\cdot\|_L^{-n+1})^\wedge(\xi), \quad \forall \xi \in S^{n-1}.$$

Since K is infinitely smooth, by Corollary 3.17, the Fourier transform of $\|\cdot\|_K^{-1}$ is a continuous function on the sphere. Also K is an intersection body, so, by Theorem 1.2, this function is non-negative. We have

$$\int_{S^{n-1}} (\|\cdot\|_K^{-1})^\wedge(\xi) (\|\cdot\|_K^{-n+1})^\wedge(\xi) d\xi \leq \int_{S^{n-1}} (\|\cdot\|_K^{-1})^\wedge(\xi) (\|\cdot\|_L^{-n+1})^\wedge(\xi) d\xi.$$

Now we can use a version of Parseval's formula on the sphere, Lemma 3.22 to remove the Fourier transforms:

$$\int_{S^{n-1}} \|x\|_K^{-1} \|x\|_K^{-n+1} dx \leq \int_{S^{n-1}} \|x\|_K^{-1} \|x\|_L^{-n+1} dx.$$

By the polar formula for the volume (2.4), the left-hand side of the latter inequality is equal to n times the n -dimensional volume of K . Using Hölder's inequality to estimate the right-hand side, we get

$$\begin{aligned} n \text{Vol}_n(K) &\leq \left(\int_{S^{n-1}} \|x\|_K^{-n} dx \right)^{1/n} \left(\int_{S^{n-1}} \|x\|_L^{-n} dx \right)^{(n-1)/n} \\ &= n (\text{Vol}_n(K))^{1/n} (\text{Vol}_n(L))^{(n-1)/n}, \end{aligned}$$

which implies that the n -dimensional volume of K is smaller than that of L . \square

The book is organized in the following way. In Chapter 2, we gather definitions, notation and technical tools needed in the rest of the text. We include the proofs that are difficult to find and do not prove the facts that can easily be found in one of our main references. We refer the reader to the books by Schneider [Sch3] and Gardner [Ga3] for results in convex geometry and to the books by Gelfand and Shilov [GS], Gelfand and Vilenkin [GV] and Rudin [Ru] for information about distributions.

In Chapter 3, we prove several formulas expressing the volume of sections in terms of the Fourier transform, including Theorems 1.1 and 1.4. In Section 3.4, we prove the spherical Parseval formula that was used in the proof of Theorem 1.5 and will be used many more times in the rest of the book.

Intersection bodies are studied in Chapter 4. We start with the Fourier analytic characterization of intersection bodies, Theorem 1.2, in Section 4.1. In the same section, we include a simple geometric characterization of intersection bodies. In particular, if n is an even integer, then a smooth origin-symmetric convex body in \mathbb{R}^n is an intersection body if and only if for every $\xi \in S^{n-1}$,

$$(-1)^{n/2} A_{K,\xi}^{(n-2)}(0) \geq 0.$$

This fact conveniently shows once again that in dimension 4 every such K is an intersection body, because the second derivative is negative. For odd n the formula is slightly more complicated, but also works well for constructing non-intersection bodies in dimension 5.

The concept of a k -intersection body is considered in Section 4.2. These bodies are useful for generalizations of the Busemann-Petty problem and also provide an interesting link between geometry and functional analysis. Let $1 \leq k < n$, and let D and L be origin-symmetric star bodies in \mathbb{R}^n . We say that D is a k -intersection body of L if for every $(n-k)$ -dimensional subspace H of \mathbb{R}^n

$$\text{Vol}_k(D \cap H^\perp) = \text{Vol}_{n-k}(L \cap H).$$

A more general class of k -intersection bodies can be characterized by the condition that $\|\cdot\|_D^{-k}$ is a positive definite distribution, Theorem 4.8. An important consequence is that every origin-symmetric convex body in \mathbb{R}^n is a k -intersection body for each $k = n - 3, n - 2, n - 1$ such that $k > 0$ (see Corollary 4.9). Note that the original intersection bodies are 1-intersection bodies in this more general setting.

In Section 4.3 we give several examples of bodies that are and that are not k -intersection bodies. We prove that the unit ball of any finite dimensional subspace of L_q , $q \leq 2$, is a k -intersection body for every k . We then prove that the unit balls of the spaces ℓ_q^n , $2 < q \leq \infty$, are k -intersection bodies if and only if $k = n - 3, n - 2, n - 1$ so that $k > 0$, and, therefore, these bodies represent the “extreme” case. The techniques for calculating the Fourier transform used in this section may be of independent interest. In Section 4.4 we present other “extreme” examples by proving the so-called second derivative test for k -intersection bodies, Theorem 4.19. Roughly speaking, if there exists a direction so that the second directional derivatives of the norm in \mathbb{R}^n in this direction vanish in a special way, then the corresponding body is a k -intersection body only for $k \geq n - 3$. The examples include q -sums of normed spaces with $q > 2$ and Orlicz spaces with Orlicz functions M satisfying $M'(0) = M''(0) = 0$.

Chapter 5 is devoted to the Busemann-Petty problem. In Section 5.1, we present in detail the unified analytic solution to the problem from [GKS]. In Section 5.2, we consider a generalization of the Busemann-Petty problem. Since the solution to the problem is negative in most dimensions, it is natural to ask for a condition on the behavior of parallel section functions at the origin, which allows us to compare the n -dimensional volumes in all dimensions. We give such a condition in terms of the $(n - 4)$ -th derivatives of the parallel section function at zero and show that the order $n - 4$ is optimal. This result, Theorem 5.7, generalizes the solution to the original Busemann-Petty problem. The proof is purely Fourier analytic.

In Section 5.3, we give a short proof of the affirmative part of the solution to the Busemann-Petty problem (and also the affirmative part of Theorem 5.7) using spherical harmonics.

A very recent result of Zvavitch [Zv2] gives a generalization of the Busemann-Petty problem to arbitrary measures in place of the volume. We present this result in Section 5.4. Let f be a non-negative locally integrable function on \mathbb{R}^n , which is also locally integrable on each central hyperplane ξ^\perp in \mathbb{R}^n . Let μ be the measure on \mathbb{R}^n with density f . For every closed bounded set $B \in \xi^\perp$ define

$$\mu(B) = \int_B f(x) dx.$$

Given two convex origin-symmetric bodies K and L in \mathbb{R}^n such that, for every $\xi \in S^{n-1}$,

$$\mu(K \cap \xi^\perp) \leq \mu(L \cap \xi^\perp),$$

does it follow that

$$\mu(K) \leq \mu(L)?$$

Surprisingly, for a very large class of functions f , the answer is the same as for the original problem – affirmative if $n \leq 4$ and negative if $n \geq 5$. The proof uses a generalization of (1.5), the spherical Parseval formula from Section 3.4, a clever elementary argument replacing Hölder’s inequality in the end of the proof and results on intersection bodies from Chapter 4.

In Section 5.5 we discuss several open problems, in particular, the lower dimensional Busemann-Petty problem asking the same question with k -dimensional sections, $2 \leq k < n - 2$, in place of hyperplane sections. Bourgain and Zhang [BZ] proved that the answer is negative if $k > 3$. We present a proof of this result from [K14]. The problem remains open for two- and three-dimensional sections, $k = 2, 3$.

In Chapter 6, we show a connection between intersection bodies and the theory of L_p -spaces. Let us discuss this connection in a little more detail. It is a well-known fact going back to P. Levy (see Lemma 6.4) that an n -dimensional space $(\mathbb{R}^n, \|\cdot\|)$ embeds in L_p , $p > 0$, if and only if there exists a finite Borel measure μ on the sphere S^{n-1} in \mathbb{R}^n so that, for every $x \in \mathbb{R}^n$,

$$(1.9) \quad \|x\|^p = \int_{S^{n-1}} |(x, \xi)|^p d\mu(\xi).$$

On the other hand, one can express the volume of hyperplane sections of an origin-symmetric star body in the form

$$\text{Vol}_{n-1}(L \cap \xi^\perp) = \frac{1}{2} \lim_{\epsilon \rightarrow 0} \epsilon \int_L |(x, \xi)|^{-1+\epsilon} d\xi.$$

In fact, use the Fubini theorem to write the right-hand side as

$$\lim_{\epsilon \rightarrow 0} \epsilon \int_0^\infty t^{-1+\epsilon} A_{L, \xi}(t) dt,$$

and it is easy to see that the latter limit equals $A_{L, \xi}(0) = \text{Vol}_{n-1}(L \cap \xi^\perp)$. Thus, a body K is the intersection body of L if and only if, for every $x \in \mathbb{R}^n$,

$$\|x\|_K^{-1} = \frac{1}{2} \lim_{\epsilon \rightarrow 0} \epsilon \int_{\mathbb{R}^n} |(x, \xi)|^{-1+\epsilon} \chi_L(\xi) d\xi,$$

where χ_L is the indicator function of L .

Comparing the latter equality with (1.9), we see that intersection bodies look very much like the unit balls of spaces embedding in L_{-1} , which provides motivation for introducing the concept of embedding in L_p with $p < 0$. To do this, we would like to extend the representation (1.9) to $p \leq -1$. However, if $p \leq -1$, the integral in the right-hand side of (1.9) diverges and we have to regularize it. The standard way of regularization is by using distributions. Applying both sides of (1.9) to a test function ϕ , we get

$$\langle \|x\|^p, \phi(x) \rangle = \int_{S^{n-1}} \langle |(x, \xi)|^p, \phi(x) \rangle d\mu(\xi).$$

Passing to Fourier transforms and using Lemmas 2.23 and 2.11, we get

$$\begin{aligned} \langle |(x, \xi)|^p, \phi(x) \rangle &= \langle |t|^p, \int_{(x, \xi)=t} \phi(x) dx \rangle \\ &= c_p \langle |z|^{-1-p}, \hat{\phi}(z\xi) \rangle. \end{aligned}$$

If $-n < p < 0$, the distributions $\|\cdot\|^p$ on \mathbb{R}^n and $|\cdot|^{-1-p}$ on \mathbb{R} are locally integrable functions and, therefore, act by integration. We arrive at the following definition, first introduced in [K12]:

DEFINITION 1.6. *Let D be an origin-symmetric star body in \mathbb{R}^n , and let $X = (\mathbb{R}^n, \|\cdot\|_D)$. For $-n < p < 0$, we say that X embeds in L_p if there exists a finite Borel measure μ on S^{n-1} so that, for every even Schwartz test function $\phi \in \mathcal{S}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \|x\|_D^p \phi(x) dx = \int_{S^{n-1}} \left(\int_{\mathbb{R}} |z|^{-p-1} \hat{\phi}(z\theta) dt \right) d\mu(\theta).$$

A direct connection between intersection bodies and embeddings in negative L_p was established in [K14] (see Theorem 6.16):

THEOREM 1.7. *Let $1 \leq k < n$. An origin-symmetric star body D in \mathbb{R}^n is a k -intersection body if and only if the space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-k} . Both conditions are also equivalent to $\|\cdot\|_D^{-k}$ being a positive definite distribution.*

The advantage of this connection (and, consequently, of introducing embeddings in negative L_p) is that now one can try to extend to negative values of p different results about usual L_p -spaces. Every such extension gives new information about intersection bodies. Let us give several examples of this approach.

A well-known simple fact is that every two-dimensional normed space embeds in L_1 ([Her], [Fe], [Li]; see Corollary 6.8). How does this fact extend to embeddings in L_{-k} ? It was proved in [K11, Th. 2] (see Corollary 4.9) that for every origin-symmetric convex body K in \mathbb{R}^n and every $p \in [n-3, n)$ such that $p > 0$, the function $\|\cdot\|_K^{-p}$ represents a positive definite distribution. By Theorem 1.7 every n -dimensional Banach space embeds in L_{-n+3} . Putting $n = 2$, we get the property of two-dimensional spaces mentioned above. Putting $n = 4$, we see that every four-dimensional normed space embeds in L_{-1} . By Theorem 1.7, every four-dimensional origin-symmetric convex body is a 1-intersection body, which, by Lutwak's connection, solves in the affirmative the critical four-dimensional case of the Busemann-Petty problem.

Another well-known property of L_p -spaces, proved in [BDK], is that, for any $0 < p < q \leq 2$, the space L_q embeds isometrically in L_p , so L_p -spaces become larger when p decreases from 2 (see Corollary 6.7). This result was extended to negative p in [K12, Th. 2] (see Theorem 6.17): every finite dimensional subspace of L_q , $0 < q \leq 2$, embeds in L_{-p} for every $p \in (0, n)$. Hence, the unit ball of every n -dimensional subspace of L_q , $0 < q \leq 2$, is a k -intersection body for every $k = 1, \dots, n-1$. This gives plenty of examples of intersection bodies, and, in particular, proves a conjecture of M. Meyer [Me] that every polar projection body is an intersection body.

On the negative side, the solution to Schoenberg's problem (see Corollary 6.12) shows that for $q > 2$ and $n \geq 3$, the spaces ℓ_q^n do not embed in L_p with $0 < p \leq 2$. This result was extended to negative p in [K8, Lemma 9] (Theorem 4.13): the spaces ℓ_q^n with $q > 2$ embed in L_{-p} only for $p \in [n-3, n)$. Putting $n = 5$, we see that ℓ_q^5 with $q > 2$ does not embed in L_{-1} , so the unit balls of these spaces are not 1-intersection bodies, which provides counterexamples to the Busemann-Petty problem in the critical dimension 5.

All these examples are isometric. An isomorphic example is given in Section 6.3. The factorization theorem of Maurey and Nikishin, [Ma], [Ni], implies that, for $0 < p < q < 1$, every Banach subspace of L_p is isomorphic to a subspace of L_q . It was proved in [KK2] that, for any $-\infty < p < q < 1$, $p \neq 0$, $q > 0$, every n -dimensional Banach subspace of L_p , $-n < p$, is isomorphic to a subspace of L_q with the Banach-Mazur distance depending only on p and q (see Theorem 6.26). In

terms of intersection bodies this shows that origin-symmetric convex k -intersection bodies are isomorphically equivalent to unit balls of subspaces of L_q , $0 < q < 1$.

Chapter 6 is organized as follows. We start with several results on embedding of normed spaces in L_p with $p > 0$. In particular, we prove a Fourier analytic characterization of subspaces of L_p : if $p > 0$ is not an even integer, then a normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in L_p if and only if the Fourier transform of $\Gamma(-p/2)\|\cdot\|^p$ is a non-negative distribution outside of the origin in \mathbb{R}^n . We also prove a well-known old result (going back to P. Levy in the finite dimensional case and extended to the infinite dimensional case in [BDK]) that a finite dimensional normed space $(\mathbb{R}^n, \|\cdot\|)$ embeds isometrically in L_p with $0 < p \leq 2$ if and only if the function $\exp(-\|\cdot\|^p)$ is positive definite on \mathbb{R}^n .

In Section 6.2, we present a solution to Schoenberg's problem on positive definite functions. This problem was posed in 1938 in [Sc1] and was completely solved in the beginning of the 1990's in [Mis2] (for $q = \infty$) and [K1] (for $2 < q < \infty$): given $q > 2$, for which $p > 0$ is the function $\exp(-\|\cdot\|_q^p)$ positive definite, or, equivalently, for which $0 < p \leq 2$ does the space ℓ_q^n embed isometrically in L_p ? As mentioned above, the answer is negative for $n \geq 3$, $q > 2$ and every $0 < p \leq 2$. If $n = 2$, the answer is affirmative if and only if $p \in [0, 1]$.

The concept of embedding in L_p with $p < 0$ and related results, described above, are presented in Section 6.3.

In Chapter 7, we consider the problem of finding the extremal sections of ℓ_q -balls, $0 < q \leq \infty$. In Section 7.1, we prove Ball's theorem on the maximal central hyperplane sections of the cube. The extremal sections of ℓ_q -balls with $0 < q \leq 2$ are considered in Section 7.2.

The Fourier analytic approach to sections was extended in [KRZ] to projections of convex bodies. The results for projections are surprisingly similar to those for sections. This similarity once again exposes the duality between sections and projections, which remains one of the most intriguing mysteries of convex geometry.

In place of the Fourier formula for sections (see formula (1.5)), we have a formula connecting the volume of hyperplane projections with the curvature function of the body, [KRZ, Th. 2]:

$$(1.10) \quad \widehat{f}_L(\theta) = -\pi \text{Vol}_{n-1}(L|_{\theta^\perp}), \quad \forall \theta \in S^{n-1},$$

where $L|_{\theta^\perp}$ is the projection of an origin-symmetric convex body L to the hyperplane θ^\perp , and f_L is the curvature function of L extended from the sphere to the whole \mathbb{R}^n as a homogeneous function of degree $-n - 1$. We prove this formula in Section 8.1.

In Section 8.2, we prove a result of Barthe and Naor [BN] on the extremal hyperplane projections of ℓ_q -balls with $q > 2$: the minimal projection corresponds to the hyperplane orthogonal to the vector $(1, 0, \dots, 0)$, and the maximal projection corresponds to the vector $(1, \dots, 1)$.

In Section 8.3 we consider projection bodies that represent an analog of intersection bodies. An origin-symmetric convex body L in \mathbb{R}^n is called a *projection body* if there exists another convex body K so that the support function of L in every direction is equal to the volume of the hyperplane projection of K to this direction: for every $\theta \in S^{n-1}$,

$$h_L(\theta) = \text{Vol}_{n-1}(K|_{\theta^\perp}).$$

The support function $h_L(\theta) = \max_{x \in L}(\theta, x)$ is equal to the dual norm $\|\theta\|_{L^*}$, where L^* stands for the polar body of L . We prove a Fourier analytic characterization of projection bodies. A convex body L is a projection body if and only if the Fourier transform of the support function h_L is a negative distribution outside of the origin. Consequently, by Theorem 1.4, if h_L is infinitely smooth on the sphere and n is an even integer, then L is a projection body if and only if, for every $\xi \in S^{n-1}$,

$$(-1)^{n/2} A_{L^*, \xi}^{(n)} \geq 0.$$

This result is very similar to the characterization of intersection bodies in Section 4.1 and explains the transition between the dimensions 2 and 3 in Shephard's problem.

In Section 8.4 we present a Fourier analytic solution to Shephard's problem, which is the projection counterpart of the Busemann-Petty problem. The problem reads as follows. Let K, L be convex origin-symmetric bodies in \mathbb{R}^n and suppose that, for every $\theta \in S^{n-1}$,

$$(1.11) \quad \text{Vol}_{n-1}(K|\theta^\perp) \leq \text{Vol}_{n-1}(L|\theta^\perp).$$

Does it follow that

$$(1.12) \quad \text{Vol}_n(K) \leq \text{Vol}_n(L)?$$

The problem was solved independently by Petty [Pe] and Schneider [Sch1], who showed that the answer is affirmative if $n \leq 2$ and negative if $n \geq 3$. Our solution to the Shephard problem is based on formula (1.10) and a spherical version of Parseval's formula. This solution is in the spirit of the solution to the Busemann-Petty problem from Section 5.2.

The main goal of this work is to show connections between Fourier analysis and convex geometry. Our coverage of both subjects does not pretend to be complete; we emphasize the results that illustrate connections between the two areas. However, every chapter contains a section with historical remarks, open problems and other results on the subject that are not included in the main text. Many more results and references on sections and projections of convex bodies can be found in monographs and surveys [Ba5], [Ba6], [Ga3], [GiM], [KKo], [MiP], [MiS], [Sch3]. The book [Gr] contains numerous applications of spherical harmonics to convex geometry. The survey [K18] gives a quick introduction to Fourier methods for sections of convex bodies.

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