CHAPTER 2

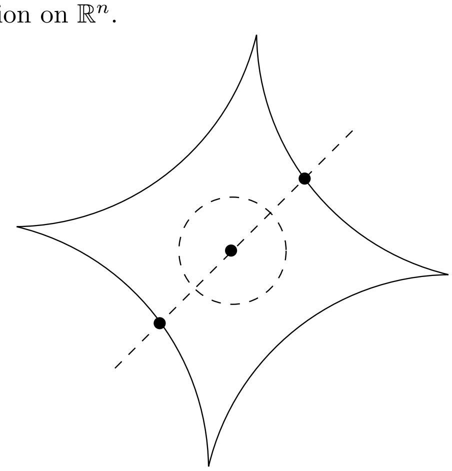
Basic Concepts

2.1. Star bodies

We say that a closed bounded set K in $\mathbb{R}^n$ is a *star body* if for every $x \in K$ each point of the interval $[0, x)$ is an interior point of K (in other words, every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin), and the boundary of K is continuous in the sense that the *Minkowski functional* of K defined by

$$\|x\|_K = \min\{a \geq 0 : \; x \in aK\}$$

is a continuous function on $\mathbb{R}^n$.



Star body.

It is easy to see that the Minkowski functional is a homogeneous function of degree 1 on $\mathbb{R}^n$ and that

$$K = \{x \in \mathbb{R}^n : \; \|x\|_K \leq 1\}.$$

Also, it follows from the definition that the origin is an interior point of every star body, so the Minkowski functional is strictly positive outside of the origin.

The *radial function* of a star body K is defined by

$$\rho_K(x) = \|x\|_K^{-1}, \qquad x \in \mathbb{R}^n.$$

If $x \in S^{n-1}$, then $\rho_K(x)$ is the "radius" of K in the direction of x, i.e. the distance from the origin to the boundary of K in the direction of x.

We define the *radial metric* on the set of all origin-symmetric star bodies in $\mathbb{R}^n$ by

$$\rho(K, L) = \max_{x \in S^{n-1}} |\rho_K(x) - \rho_L(x)|.$$

We use different summations of star bodies. If K and L are star bodies in $\mathbb{R}^n$, then *the Minkowski sum* of K and L is the set

$$K + L = \{x + y : \ x \in K, \ y \in L\},$$

where $x + y$ is the usual sum of vectors in $\mathbb{R}^n$.

If $p \in \mathbb{R} \setminus \{0\}$, the p-sum of K and L is a star body $K +_p L$ defined by

$$\|x\|_{K+_pL} = (\|x\|_K^p + \|x\|_L^p)^{1/p}$$

for every $x \in \mathbb{R}^n$. The case $p = -1$ corresponds to the *radial sum*, where the radius of the body $K +_{-1} L$ in every direction is equal to the sum of radii of K and L in this direction. The 0-sum (or multiplicative sum) $K +_0 L$ of two star bodies K and L is defined by

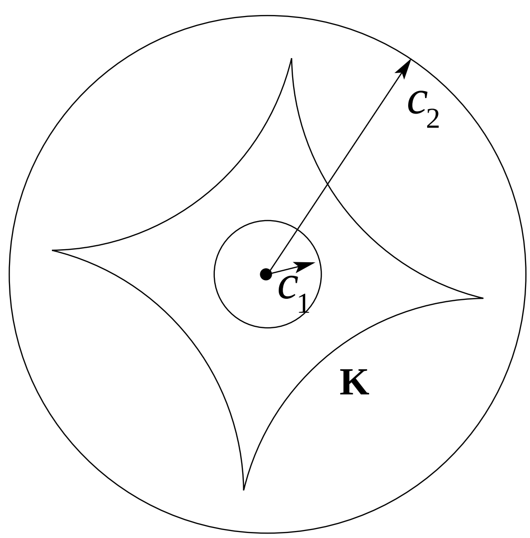
$$\|x\|_{K+_0L} = \sqrt{\|x\|_K \|x\|_L}.$$

A star body is called *k-smooth,* $k \in \mathbb{N} \cup \{0\}$, if the restriction of its Minkowski functional to the sphere S^{n-1} belongs to the space $C^k(S^{n-1})$ of k times continuously differentiable functions on the sphere S^{n-1}. Recall that the norm of the space C^k is defined as the maximum of sup-norms of the function itself and all its derivatives up to the order k. We say that a body is infinitely smooth if it is k-smooth for every $k \in \mathbb{N}$.

The condition that the origin is an interior point of a star body implies that certain negative powers of the Minkowski functional are locally integrable on $\mathbb{R}^n$. This simple fact is very important for us, since we are going to consider powers of the Minkowski functional as distributions, and local integrability implies that the action of these distributions on test functions is by integration; see Section 2.5.

LEMMA 2.1. *Let K be an origin-symmetric star body in $\mathbb{R}^n$. Then, for $0 < p < n$, the function $\|\cdot\|_K^{-p}$ is locally integrable on $\mathbb{R}^n$. Also, if f is a bounded integrable function on $\mathbb{R}^n$, then the function $\|\cdot\|_K^{-p} f(\cdot)$ is integrable on $\mathbb{R}^n$.*

PROOF. By the definition of a star body, K contains a Euclidean ball and is contained in another Euclidean ball with center at the origin. Therefore, there exist constants c_1, c_2 so that, for every $x \in \mathbb{R}^n$, $c_1|x|_2 \le \|x\|_K \le c_2|x|_2$, where $|\cdot|_2$ is the Euclidean norm in $\mathbb{R}^n$.

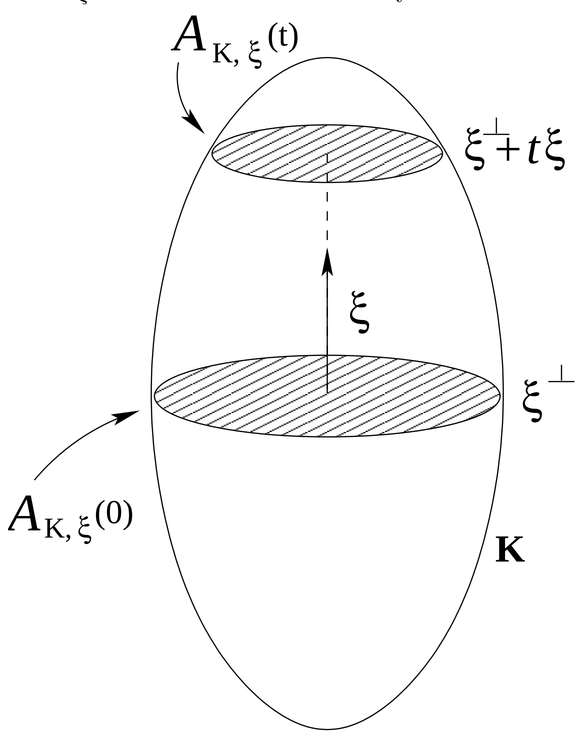


On the other hand, the function $|\cdot|_2^{-p}$ is integrable on any Euclidean ball with center at the origin, which follows from an elementary calculation in polar coordinates. This proves the first statement, and the second easily follows. In fact, on any Euclidean ball with center at the origin the function $\|\cdot\|_K^{-p}$ is locally integrable and f is bounded, while outside of this ball f is integrable and $\|\cdot\|_K^{-p}$ is bounded. $\square$

For $\xi \in S^{n-1}$, we define the *parallel section function* of K in the direction of ξ as a function on $\mathbb{R}$ given by

$$A_{K,\xi}(t) = \mathrm{Vol}_{n-1}(K \cap \{\xi^\perp + t\xi\}),$$

where $\{\xi^\perp + t\xi\}$ is the hyperplane perpendicular to ξ at distance t from the origin. We use the notation A_ξ when it is clear what body is considered.



If χ is the indicator function of the interval $[-1,1]$, then $\chi(\|\cdot\|_K)$ is the indicator function of the body K, and we have

$$A_{K,\xi}(t) = \int_{(x,\xi)=t} \chi(\|x\|_K)\, dx. \tag{2.1}$$

For $t = 0$, writing the integral in the right-hand side in the polar coordinates of the hyperplane $(x,\xi) = 0$, we get the *polar formula for the volume of sections* (note that the dimension of a hyperplane is $n-1$, which explains the power r^{n-2}):

$$A_{K,\xi}(0) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp) = \int_{(x,\xi)=0} \chi(\|x\|_K)\, dx$$
$$= \int_{S^{n-1}\cap\xi^\perp} \left(\int_0^\infty r^{n-2} \chi(r\|\theta\|_K)\, dr \right) d\theta$$

$$= \int_{S^{n-1}\cap\xi^\perp} \left(\int_0^{1/\|\theta\|_K} r^{n-2}\, dr \right) d\theta$$

$$= \frac{1}{n-1} \int_{S^{n-1}\cap\xi^\perp} \|\theta\|_K^{-n+1}\, d\theta$$

$$= \frac{1}{n-1} \int_{S^{n-1}\cap\xi^\perp} \rho_K^{n-1}(\theta)\, d\theta. \tag{2.2}$$

If H is an m-dimensional subspace of $\mathbb{R}^n$, then the m-dimensional volume of the section of K by H can be expressed in terms of the Minkowski functional in a similar way:

$$\mathrm{Vol}_m(K \cap H) = \frac{1}{m} \int_{S^{n-1}\cap H} \|\theta\|_K^{-m}\, d\theta. \tag{2.3}$$

In particular, putting $m = n$,

$$\mathrm{Vol}_n(K) = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(\theta) d\theta. \tag{2.4}$$

2.2. Convex bodies

Our main references for results in convex geometry are the books by Schneider [Sch3] and Gardner [Ga3].

A closed compact set K in $\mathbb{R}^n$ with non-empty interior is called a *convex body* if it contains the line segment connecting any two of its points, i.e. for every $x, y \in K$ and $\lambda \in [0, 1]$, the point $\lambda x + (1 - \lambda)y \in K$.

If a convex body K is origin-symmetric, then its Minkowski functional is a norm on $\mathbb{R}^n$. In fact, as it is for any origin-symmetric star body, this functional is even 1-homogeneous, i.e. for every $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $\|tx\|_K = |t|\|x\|_K$. Also, since K is origin-symmetric and its interior in non-empty, the origin is an interior point of K, which implies that $\|x\|_K = 0$ if and only if $x = 0$. The third defining property of a norm, the triangle inequality

$$\|x + y\|_K \le \|x\|_K + \|y\|_K,$$

follows from convexity of K and is easy to verify.

One of the main tools of convex geometry is the Brunn-Minkowski inequality. This inequality was first proved for three-dimensional convex bodies by Brunn in 1887 and for convex bodies in arbitrary dimension by Minkowski in 1910. Lusternik [Lus] proved the general Brunn-Minkowski inequality for arbitrary non-empty bounded measurable sets. There exist many different proofs of this inequality; see [Sch3, p. 309], [Ga4], [Ba5]. The short proof that we present here is due to Hadwiger and Ohmann [HO].

THEOREM 2.2. *(Brunn-Minkowski Inequality) If K and L are non-empty compact sets in $\mathbb{R}^n$, then for every $\lambda \in [0, 1]$,*

$$\mathrm{Vol}_n(\lambda K + (1-\lambda)L)^{1/n} \ge \lambda \mathrm{Vol}_n(K)^{1/n} + (1-\lambda)\mathrm{Vol}_n(L)^{1/n}. \tag{2.5}$$

PROOF. Since $\mathrm{Vol}_n(\lambda K) = \lambda^n \mathrm{Vol}_n(K)$, it is enough to prove that

$$\mathrm{Vol}_n(K + L)^{1/n} \ge \mathrm{Vol}_n(K)^{1/n} + \mathrm{Vol}_n(L)^{1/n}. \tag{2.6}$$

First, consider the case where

$$K = \prod_{i=1}^{n}[a_i, b_i] \quad \text{and} \quad L = \prod_{i=1}^{n}[c_i, d_i]$$

are rectangular parallelepipeds in $\mathbb{R}^n$ with sides parallel to the coordinate hyperplanes. Denote by $x_i = b_i - a_i,\ y_i = d_i - c_i$ the lengths of the sides of K and L, respectively. Then

$$K + L = \prod_{i=1}^{n}[a_i + c_i, b_i + d_i]$$

and

$$\mathrm{Vol}_n(K+L) = \prod_{i=1}^{n}(x_i + y_i),\ \mathrm{Vol}_n(K) = \prod_{i=1}^{n} x_i,\ \mathrm{Vol}_n(L) = \prod_{i=1}^{n} y_i.$$

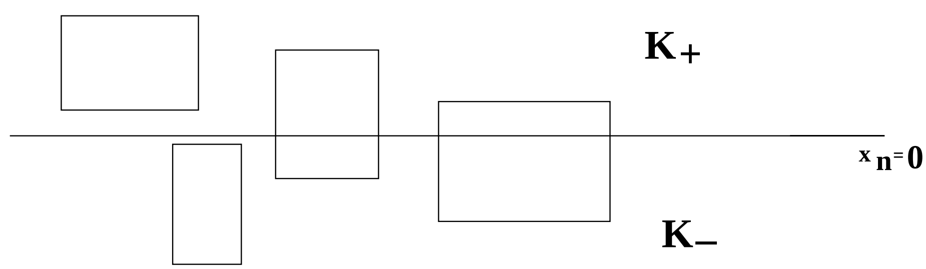
By the arithmetic-geometric mean inequality,

$$\left(\prod_{i=1}^{n}\frac{x_i}{x_i+y_i}\right)^{1/n} + \left(\prod_{i=1}^{n}\frac{y_i}{x_i+y_i}\right)^{1/n} \le \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{x_i+y_i} + \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{x_i+y_i} = 1.$$

This proves inequality (2.6) for two rectangular boxes.

Now suppose that K is the union of m_1 disjoint (up to a set of volume zero) rectangular boxes and L is the union of m_2 disjoint rectangular boxes. We prove inequality (2.6) for such K and L by induction on the number $m_1 + m_2$. Since the result is proved in the case $m_1 + m_2 = 2$, we can assume that $m_1 + m_2 > 2$ and that K contains at least two disjoint boxes.

Note that both sides of inequality (2.6) do not change if we translate any of the sets K or L. Translating K if necessary, we can place it in such a way that at least two boxes in K are separated from each other by a coordinate plane, say $\{x_n = 0\}$. Denote by K_+ and K_- the unions of boxes in $K \cap \{x_n \ge 0\}$ and $K \cap \{x_n \le 0\}$, respectively. Clearly, the number of boxes in each of K_+ and K_- is smaller than m_1.



Now translate L so that

$$\frac{\mathrm{Vol}_n(K_+)}{\mathrm{Vol}_n(K)} = \frac{\mathrm{Vol}_n(L_+)}{\mathrm{Vol}_n(L)} \quad \text{and} \quad \frac{\mathrm{Vol}_n(K_-)}{\mathrm{Vol}_n(K)} = \frac{\mathrm{Vol}_n(L_-)}{\mathrm{Vol}_n(L)},$$

where L_+ and L_- are the parts of L contained in $\{x_n \ge 0\}$ and $\{x_n \le 0\}$ (the translation must provide one of these equalities; the second will hold automatically, since the sum of the left- and right-hand sides is 1).

The sets $K_+ + L_+$ and $K_- + L_-$ are disjoint (up to a set of volume zero) subsets of $K + L$, since they are on different sides of the hyperplane $\{x_n = 0\}$. Besides, the sum of numbers of boxes in K_+ and L_+ is smaller than $m_1 + m_2$, and the same is true for K_- and L_-. By the induction hypothesis, inequality (2.6) holds for each of these two pairs of sets. We get

$$\mathrm{Vol}_n(K+L) \geq \mathrm{Vol}_n(K_+ + L_+) + \mathrm{Vol}_n(K_- + L_-)$$

$$\geq \left(\mathrm{Vol}_n(K_+)^{1/n} + \mathrm{Vol}_n(L_+)^{1/n}\right)^n + \left(\mathrm{Vol}_n(K_-)^{1/n} + \mathrm{Vol}_n(L_-)^{1/n}\right)^n$$

$$= \mathrm{Vol}_n(K_+)\left(1 + \frac{\mathrm{Vol}_n(L)^{1/n}}{\mathrm{Vol}_n(K)^{1/n}}\right)^n + \mathrm{Vol}_n(K_-)\left(1 + \frac{\mathrm{Vol}_n(L)^{1/n}}{\mathrm{Vol}_n(K)^{1/n}}\right)^n$$

$$= \left(\mathrm{Vol}_n(K)^{1/n} + \mathrm{Vol}_n(L)^{1/n}\right)^n.$$

We have proved (2.6) for finite unions of disjoint boxes, and now we can get the general result approximating compact sets by such unions. □

It follows from the Brunn-Minkowski inequality that if K is an origin-symmetric convex body in $\mathbb{R}^n$, then the central hyperplane section has maximal $(n-1)$-dimensional volume among all hyperplane sections orthogonal to a given direction.

Theorem 2.3. *(Brunn's Theorem) Let K be a convex body in $\mathbb{R}^n$. Fix $\xi \in S^{n-1}$ and let $A_{K,\xi}$ be the parallel section function in the direction of ξ. Then the following hold.*
(i) The function $A_{K,\xi}^{1/(n-1)}$ is concave on its support.
(ii) If K is origin-symmetric, then, for each $\xi \in S^{n-1}$ and $t \in \mathbb{R}$,

$$A_{K,\xi}(t) \leq A_{K,\xi}(0) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp),$$

i.e. the central section has maximal $(n-1)$-dimensional volume among all hyperplane sections of K perpendicular to a given direction.
(iii) If K is origin-symmetric and 2-smooth, then the second derivative $A''_{K,\xi}(0) \leq 0$.

Proof. For every $t \in \mathbb{R}$, consider the compact convex set

$$K_t = K \cap \{x \in \mathbb{R}^n : \ (x,\xi) = t\}.$$

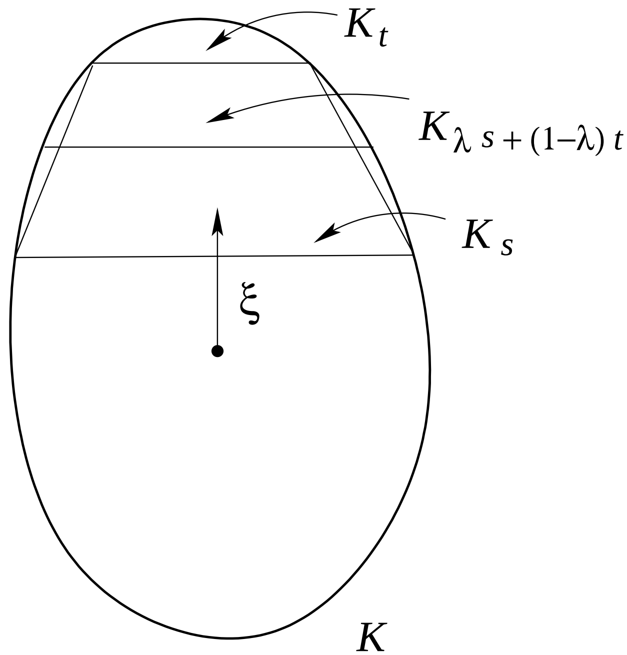
By the convexity of K, for any $\lambda \in [0,1]$ and any $s,t \in \mathbb{R}$ such that the sets K_s and K_t are non-empty (i.e. s and t belong to the support of the function $A_{K,\xi}$), we have

$$\lambda K_s + (1-\lambda)K_t \subset K_{\lambda s + (1-\lambda)t},$$

and therefore

$$\lambda P(K_s) + (1-\lambda)P(K_t) \subset P(K_{\lambda s + (1-\lambda)t}),$$

where P is the orthogonal projection to the hyperplane $(x,\xi) = 0$, and the addition of sets is considered in the hyperplane $(x,\xi) = 0$.



The orthogonal projections have the same $(n-1)$-dimensional volume as the original sets. By the Brunn-Minkowski inequality applied to the compact convex sets $P(K_s)$ and $P(K_t)$ in $\mathbb{R}^{n-1} = \xi^\perp$,

$$\begin{aligned}
&\lambda(\mathrm{Vol}_{n-1}(K_s))^{1/(n-1)} + (1-\lambda)(\mathrm{Vol}_{n-1}(K_t))^{1/(n-1)} \\
&= \lambda(\mathrm{Vol}_{n-1}(P(K_s)))^{1/(n-1)} + (1-\lambda)(\mathrm{Vol}_{n-1}(P(K_t)))^{1/(n-1)} \\
&\le (\mathrm{Vol}_{n-1}(\lambda P(K_s) + (1-\lambda)P(K_t)))^{1/(n-1)} \\
&\le (\mathrm{Vol}_{n-1}(P(K_{\lambda s+(1-\lambda)t})))^{1/(n-1)} = (\mathrm{Vol}_{n-1}(K_{\lambda s+(1-\lambda)t}))^{1/(n-1)}.
\end{aligned}$$

Since $\mathrm{Vol}_{n-1}(K_s) = A_{K,\xi}(s)$, we have proved that

$$\lambda A_{K,\xi}^{1/(n-1)}(s) + (1-\lambda)A_{K,\xi}^{1/(n-1)}(t) \le A_{K,\xi}^{1/(n-1)}(\lambda s + (1-\lambda)t),$$

so the function $A_{K,\xi}^{1/(n-1)}$ is concave on its support. A concave even function has maximum at zero, which proves the second statement. If, in addition, K is 2-smooth, then, as we show below in Lemma 2.4, the function $A_{K,\xi}$ is twice differentiable in a neighborhood of zero, which implies the third statement of the theorem. □

We have just used the fact that the parallel section function of a smooth body is also smooth. We need this fact in a general form. We say that a family of functions is uniformly differentiable if convergence in the limits defining the derivatives is uniform with respect to this family of functions.

LEMMA 2.4. *Let K be an m-smooth origin-symmetric convex body in $\mathbb{R}^n$, where $m \in \mathbb{N} \cup \{0\}$. Then for all $\xi \in S^{n-1}$ the parallel section functions $A_{K,\xi}$ are (uniformly with respect to ξ) m times continuously differentiable in some neighborhood U of zero. Moreover, for every fixed $t \in U$, the derivative $A_{K,\xi}^{(m)}(t)$ is a continuous function of the variable ξ on the sphere.*

PROOF. Fix a direction $\xi \in S^{n-1}$. For every t consider the hyperplane $H_t = \{x : (x,\xi) = t\}$. Since the interior of K is non-empty and K is origin-symmetric, it contains a Euclidean ball with radius r and center at the origin. We assume that $|t| < r$. Let S^{n-2} be the unit Euclidean sphere in H_t with center at the orthogonal projection of the origin to H_t (which is inside of $K \cap H_t$ since $|t| < r$), and let $\rho_{K\cap H_t}$ be the radial function of the body $K \cap H_t$ in $(H_t, S^{n-2}) = \mathbb{R}^{n-1}$ with respect to this center. By the polar formula for the volume of sections (see (2.4)),

$$A_{K,\xi}(t) = \mathrm{Vol}_{n-1}(K \cap H_t) = \frac{1}{n-1}\int_{S^{n-2}} \rho_{K\cap H_t}^{n-1}(\theta)\, d\theta. \tag{2.7}$$

The question is reduced to uniform differentiability (with respect to θ and ξ) of $\rho_{K\cap H_t}(\theta)$ by the variable t.

Fix θ and consider the two-dimensional plane passing through the origin and spanned by ξ and θ. Let D be the section of K by this plane, and let ρ_D be the radial function of D defined on $[0, 2\pi]$. Since K is C^m, the function ρ_D is C^m. Let us now get an implicit formula expressing $\rho_{K\cap H_t}(\theta)$ in terms of ρ_D. The line connecting the origin with the point on the boundary corresponding to $\rho_{K\cap H_t}(\theta)$ is at the angle $\phi = \arctan(\frac{t}{\rho_{K\cap H_t}(\theta)})$ with respect to the hyperplane $\xi^\perp$. Therefore, the radius at this point is equal to $\rho_D(\arctan(\frac{t}{\rho_{K\cap H_t}(\theta)}))$. From the right triangle,

$$\rho_{K\cap H_t}(\theta) = \sqrt{\rho_D^2(\arctan(\frac{t}{\rho_{K\cap H_t}(\theta)})) - t^2}. \tag{2.8}$$

This is an implicit formula that we are now going to use to compute the derivatives of $y(t) = \rho_{K\cap H_t}(\theta)$. Formula (2.8) now has the form

$$y = \sqrt{\rho_D^2(\arctan(\frac{t}{y})) - t^2}.$$

Differentiating both sides by t and separating the derivative of y, we get

$$y'(t) = \frac{\rho_D(\arctan(t/y))\rho_D'(\arctan(t/y))\frac{1}{y^2+t^2}(y - ty'(t)) - t}{y};$$

therefore

$$y'(t) = \frac{\rho_D(\arctan(\frac{t}{y}))\rho_D'(\arctan(\frac{t}{y}))(1/(y^2+t^2))y - t}{y + t\rho_D(\arctan(\frac{t}{y}))\rho_D'(\arctan(\frac{t}{y}))\frac{1}{y^2+t^2}}.$$

Again, K contains a Euclidean ball with center at the origin, so ρ_D and y are separated from zero for small t with a bound independent on θ and ξ. Also, $\rho_K \in C^m(S^{n-1})$, so ρ_D and all its derivatives up to the order m are bounded from above uniformly with respect to θ and ξ. This shows that the denominator in the expression for y' is positive for small t, so the derivative exists in a neighborhood of zero uniformly with respect to θ and ξ. Computing derivatives of higher orders is similar (note that we never get zero in the denominator if t is small). This shows that y is m times uniformly differentiable in some neighborhood of zero.

It follows from (2.7) that uniform differentiability of $y = \rho_{K\cap H_t}(\theta)$ implies that $A_{K,\xi}(t)$ is uniformly (with respect to ξ) differentiable in a neighborhood of zero. This also implies the last statement of the lemma by induction by m. Given the result for $m-1$, we use the fact that, for every t in some neighborhood of zero, the function $\xi \mapsto A^{(m)}_{K,\xi}(t)$ is the limit in $C(S^{n-1})$ (i.e. uniformly by ξ) of continuous functions

$$\xi \mapsto \frac{A^{(m-1)}_{K,\xi}(t+h) - A^{(m-1)}_{K,\xi}(t)}{h},$$

as $h \to 0$. □

Another application of the Brunn-Minkowski inequality is the following result of Busemann [Bu1] (a proof can be found in [Ga3, Th. 8.1.10] or [MiP, Th. 3.9]; the result in [MiP] is more general):

Theorem 2.5. *(Busemann's Theorem) Suppose that K is an origin-symmetric convex body in $\mathbb{R}^n$. For each $\xi \in S^{n-1}$, let $r(\xi)$ be the $(n-1)$-dimensional volume of the central hyperplane section of K perpendicular to ξ. Then the body whose radius in each direction ξ is $r(\xi)$ is itself convex.*

We say that a compact set K with volume 1 in $\mathbb{R}^n$ is in *isotropic position* if for each $\xi \in S^{n-1}$

$$\int_K (x,\xi)^2 dx = L_K^2, \tag{2.9}$$

where L_K is a constant that is called the *isotropic constant* of K. Since, for any compact set K, the left-hand side of (2.9) is a positive quadratic form of ξ, there exists a linear transformation T of $\mathbb{R}^n$ so that TK is in isotropic position; see [Ba7] for details. Note that the unit ball of any normed space with a symmetric basis is in isotropic position after a dilation making the volume equal to 1 (if we expand $(x,\xi)^2$, the terms with $x_i x_j,\ i \neq j$, integrate to zero because of symmetry, and the terms with x_i^2 generate equal integrals). In particular, if we dilate the unit balls of the spaces $\ell_p^n,\ 0 < p \le \infty$, so that the volumes of the dilated balls are equal to 1, then we get bodies that are in isotropic position.

Hensley [He2] has proved (see Corollary 2.7 below) that there exist absolute constants a_1, a_2 so that for any symmetric convex body K in $\mathbb{R}^n$ in isotropic position and any $\xi_1, \xi_2 \in S^{n-1}$,

$$a_1 \le \frac{\mathrm{Vol}_{n-1}(K \cap \xi_1^\perp)}{\mathrm{Vol}_{n-1}(K \cap \xi_2^\perp)} \le a_2.$$

The proof that we present here is due to Bourgain [Bo3]. For other proofs, see [MiP] or [Ba7].

Lemma 2.6. *Let K be an origin-symmetric convex body in $\mathbb{R}^n,\ n \ge 2$, with $\mathrm{Vol}_n(K) = 1$. There exist absolute (not depending on K, n) constants $0 < c, d < \infty$ so that for any $\xi \in S^{n-1}$ and $t \in \mathbb{R}$,*

$$A_{K,\xi}(t) \le c A_{K,\xi}(0) \exp(-d A_{K,\xi}(0)|t|),$$

where $A_{K,\xi}$ is the parallel section function of K in the direction of ξ.

Proof. Fix ξ and let $\phi(t) = A_{K,\xi}(t)$. We can assume that

$$|t|\phi(0) > 20. \tag{2.10}$$

In fact, by Brunn's theorem, Theorem 2.3, $\phi(0) > \phi(t)$, so if $|t|\phi(0) < 20$, then we can put $d = 1, c = e^{20}$ to get the required estimate.

If $\phi(t) \neq 0$, then $K \cap \{(x,\xi) = t\} \neq \emptyset$, and, since K is convex, there exists a cone with the base $K \cap \xi^{\perp}$ and height $|t|$ which is inside of K, so the volume of this cone is less than or equal to the volume of K :

$$1 = \mathrm{Vol}_n(K) \geq \frac{1}{n}|t|\phi(0). \tag{2.11}$$

Therefore, we can assume that

$$\phi(t) \geq \phi(0)e^{-n} \tag{2.12}$$

because, otherwise, by (2.11),

$$\phi(t) \leq \phi(0)e^{-n} \leq \phi(0)\exp(-|t|\phi(0)),$$

and we get the required estimate again.

For every $t \in \mathbb{R}$ and $\epsilon > 0$, we have (by Theorem 2.3 the function ϕ is decreasing from zero)

$$1 = \mathrm{Vol}_n(K) > \int_0^{\infty} \phi(s)\ ds \geq \int_0^{\epsilon|t|} \phi(s)\ ds \geq \epsilon|t|\phi(\epsilon|t|). \tag{2.13}$$

Again by Brunn's theorem, Theorem 2.3, the function $\phi(\cdot)^{1/(n-1)}$ is concave, so

$$\begin{aligned}\phi(\epsilon|t|) &\geq \left((1-\epsilon)\phi(0)^{1/(n-1)} + \epsilon\phi(|t|)^{1/(n-1)}\right)^{n-1} \\ &= \phi(0)\left(1-\epsilon+\epsilon\left(\frac{\phi(t)}{\phi(0)}\right)^{1/(n-1)}\right)^{n-1} \\ &= \phi(0)\left(1-\epsilon+\epsilon\exp(\ln(\alpha)/(n-1))\right)^{n-1},\end{aligned}$$

where $\alpha = \phi(t)/\phi(0)$. Combining the latter inequality with (2.13) and using an elementary inequality $\exp(x) \geq 1 + x$, we get

$$\begin{aligned}1 \geq \epsilon|t|\phi(\epsilon|t|) &\geq \epsilon|t|\phi(0)\left(1-\epsilon+\epsilon(1+\frac{\ln\alpha}{n-1})\right)^{n-1} \\ &= \epsilon|t|\phi(0)\left(1+\frac{\epsilon\ln\alpha}{n-1}\right)^{n-1}.\end{aligned}$$

Suppose now that $\epsilon < 1/4$. Note that $\alpha \leq 1$, so $\ln\alpha \leq 0$. By (2.12), $\ln\alpha > -n$, so $\epsilon\ln\alpha/(n-1) > -1/2$. The function $\ln(1+x)/x$ is bounded on the interval $[-1/2, 0)$, so there exists an absolute constant A so that

$$\ln\left(1+\frac{\epsilon\ln\alpha}{n-1}\right) \geq A\frac{\epsilon\ln\alpha}{n-1}.$$

Therefore, we have

$$1 \geq \epsilon|t|\phi(0)\left(\frac{\phi(t)}{\phi(0)}\right)^{A\epsilon}.$$

In the latter inequality, we can put

$$\epsilon = \frac{e}{|t|\phi(0)},$$

because, by (2.10), this ϵ is less than 1/4. We get

$$\phi(t) \leq \phi(0)\exp(-\frac{|t|\phi(0)}{Ae}).$$

We have proved the desired inequality with $c = 1$ and $d = 1/Ae$. Now adjust c, d to accommodate the assumptions (2.10) and (2.12). □

We now prove the result of Hensley.

COROLLARY 2.7. *There exist absolute constants a_1, a_2 so that for any origin-symmetric convex body K in isotropic position we have*

$$\frac{a_1}{L_K} \le \mathrm{Vol}_{n-1}(K \cap \xi^\perp) \le \frac{a_2}{L_K}, \quad \forall \xi \in S^{n-1}.$$

PROOF. Note that the condition $\mathrm{Vol}_n(K) = 1$ is included in the definition of isotropic position. By the Fubini theorem,

$$L_K^2 = \int_K (x,\xi)^2 dx = \int_{\mathbb{R}} t^2 A_{K,\xi}(t)\ dt.$$

Using Brunn's theorem, Theorem 2.3, for any $\xi \in S^{n-1}$,

$$1 = \mathrm{Vol}_n(K) = \int_{\mathbb{R}} A_{K,\xi}(t)\ dt$$

$$\le \int_{-2L_K}^{2L_K} A_{K,\xi}(0)\ dt + \int_{|t|>2L_K} \frac{t^2}{4L_K^2} A_{K,\xi}(t)\ dt$$

$$\le 4L_K A_{K,\xi}(0) + \frac{1}{4L_K^2}\int_{\mathbb{R}} t^2 A_{K,\xi}(t)\ dt = 4L_K A_{K,\xi}(0) + \frac{1}{4},$$

so $L_K A_{K,\xi}(0) > 3/16$.

On the other hand, by Lemma 2.6,

$$L_K^2 = \int_{\mathbb{R}} t^2 A_{K,\xi}(t)\ dt$$

$$\le cA_{K,\xi}(0)\int_{\mathbb{R}} t^2 \exp(-d|t|A_{K,\xi}(0))\ dt = \frac{C}{A_{K,\xi}(0)^2},$$

where C is an absolute constant. □

Definitions and results in the rest of this section will be used in the study of projections of convex bodies in Chapter 8.

If K and L are two convex bodies in $\mathbb{R}^n$, the *mixed volume* $V_1(K, L)$ is equal to

$$V_1(K,L) = \frac{1}{n}\lim_{\epsilon\to+0}\frac{\mathrm{Vol}_n(K+\epsilon L) - \mathrm{Vol}_n(K)}{\epsilon};$$

see [Ga3, p. 354] or [Sch3, p. 272]. Using the Brunn-Minkowski inequality, one can easily prove the so-called first Minkowski inequality (see [Ga3, p. 369]):

THEOREM 2.8. *(First Minkowski Inequality) For any convex bodies K, L in $\mathbb{R}^n$,*

$$V_1(K,L) \ge \mathrm{Vol}_n(K)^{(n-1)/n}\mathrm{Vol}_n(L)^{1/n}.$$

PROOF. By the Brunn-Minkowski inequality (2.5),

$$nV_1(K,L) \ge \lim_{\epsilon\to+0}\frac{(\mathrm{Vol}_n(K)^{1/n} + \epsilon\mathrm{Vol}_n(L)^{1/n})^n - \mathrm{Vol}_n(K)}{\epsilon}.$$

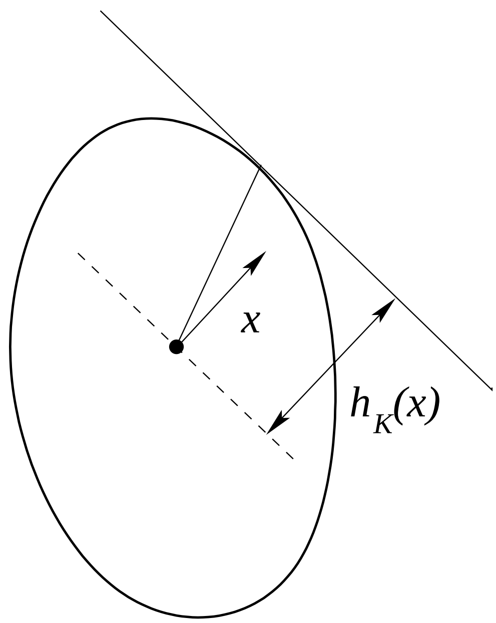
Computing the limit, we get the result. □

The *support function* of a convex body K in $\mathbb{R}^n$ is defined by

$$h_K(x) = \max_{\{\xi \in \mathbb{R}^n : \|\xi\|_K = 1\}} (x, \xi), \quad x \in \mathbb{R}^n.$$

If K is origin-symmetric, then h_K is a norm on $\mathbb{R}^n$. The convex body K^* which is the unit ball of this norm is called the *polar body* of K:

$$h_K(x) = \|x\|_{K^*}, \qquad x \in \mathbb{R}^n.$$



The support function h_K.

The *surface area measure* $S(K, \cdot)$ of a convex body K in $\mathbb{R}^n$ is defined as follows: for every Borel set $E \subset S^{n-1}$, $S(K, E)$ is equal to Lebesgue measure of the part of the boundary of K where normal vectors belong to E (see, for example, [Ga3, p. 351]). The Minkowski existence theorem shows that one gets essentially every symmetric measure on S^{n-1} as the surface area measure of some convex body:

THEOREM 2.9. *(Minkowski Existence Theorem) A finite Borel measure μ on the sphere S^{n-1} is the surface area measure of some convex body K in $\mathbb{R}^n$ if and only if μ is not supported in any great subsphere and the vector-valued integral*

$$\int_{S^{n-1}} u \; d\mu(u) = 0.$$

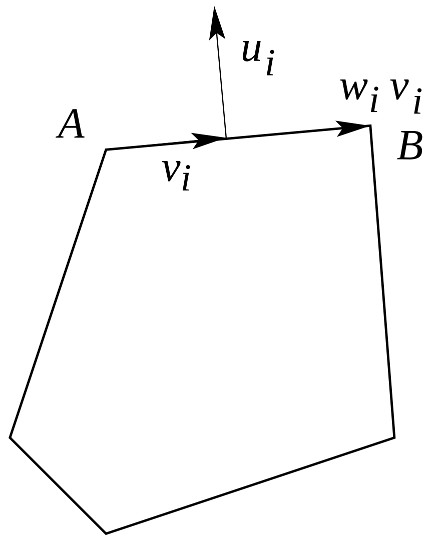
For a proof of Theorem 2.9, see [Sch3, p. 389]. At first glance, the result of this theorem looks surprising, because the condition on the measure is too general. To illustrate this theorem, consider the case where $n = 2$ and μ is the sum of m atoms at the points $u_1, ..., u_m$ with weights $w_1, ..., w_m$. The condition of the theorem is that

$$\sum_{i=1}^{m} w_i u_i = 0.$$

Let v_i be vectors obtained from u_i by rotation by the angle $\pi/2$. Clearly,

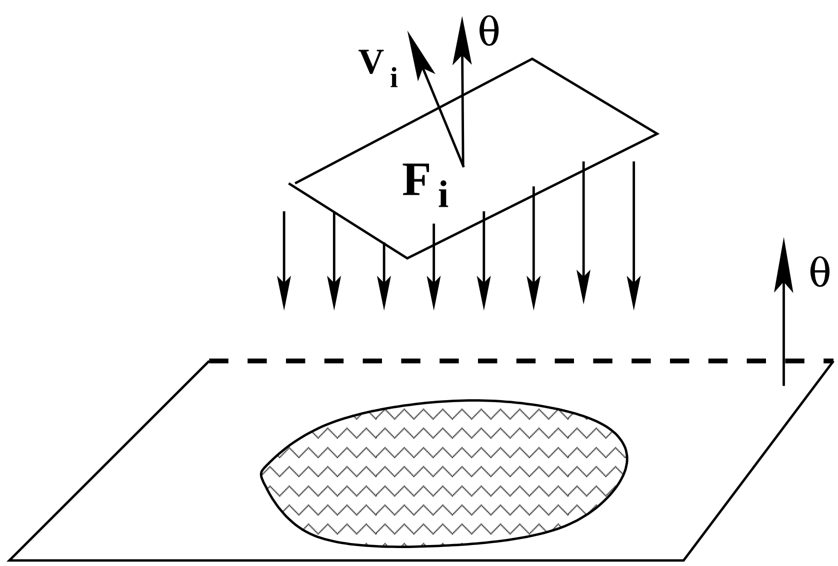
$$\sum_{i=1}^{m} w_i v_i = 0.$$

Adding vectors $w_i v_i$ geometrically (starting every next vector from the endpoint of the previous one), we can get a polygon whose surface area measure is exactly μ.



The well-known *Cauchy projection formula* ([Ga3, p. 361]) expresses the volume of projections of the body K in terms of the surface area measure:

$$\text{Vol}_{n-1}\left(K\Big|\theta^{\perp}\right) = \frac{1}{2}\int\limits_{S^{n-1}} |(\theta, v)|dS(K, v), \qquad \theta \in S^{n-1}. \tag{2.14}$$



To prove this formula, first consider the case where K is a polytope with faces $F_1, ..., F_k$. The surface area measure is then an atomic measure on S^{n-1}, supported in the points $v_1, ..., v_k$ corresponding to unit normal vectors to all faces, with weights $\mathrm{Vol}_{n-1}(F_i) = S(K, \{v_i\})$. Then the projection of each face to the hyperplane $\theta^\perp$ is equal to the cosine of the angle between the normal vector v_i and the direction θ, which is $|(v_i, \theta)|$, times the area of the face, which is equal to $S(K, \{v_i\})$. Taking the sum, we get the integral in the Cauchy formula. The coefficient $1/2$ is necessary because we cover each point of the projection twice. The general case follows by approximation.

The mixed volume can be expressed in terms of the support functions and surface area measures, [Sch3, p. 275]:

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(x) dS(K, x). \tag{2.15}$$

To see this, again consider the case where K is a polytope. When we add ϵL to K, each face F_i of K moves outward in the direction of its normal vector v_i by $\epsilon h_L(v_i)$. Then the volume changes by the area of F_i (which is $S(K, v_i)$) times $\epsilon h_L(v_i)$. Adding these changes, we get the integral in (2.15) times ϵ. Other changes of the volume are at the rate of ϵ^2 or smaller.

We often consider bodies with absolutely continuous surface area measures. A convex body K is said to have the *curvature function*

$$f_K : S^{n-1} \to \mathbb{R}$$

if its surface area measure $S(K, \cdot)$ is absolutely continuous with respect to Lebesgue measure σ_{n-1} on S^{n-1} and if

$$\frac{dS(K, \cdot)}{d\sigma_{n-1}} = f_K \in L_1(S^{n-1}).$$

The curvature function f_K is the reciprocal Gauss curvature, viewed as a function of the unit normal vector (see [Sch3, p. 419]).

In the case where K has the curvature function formula (2.15) becomes

$$V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} h_L(x) f_K(x) dx.$$

If $K = L$, we have $V_1(K, L) = \mathrm{Vol}_n(K)$, so

$$\mathrm{Vol}_n(K) = \frac{1}{n} \int_{S^{n-1}} h_K(x) f_K(x) dx. \tag{2.16}$$

It is well known that one can approximate any convex body in $\mathbb{R}^n$ in the radial metric by a sequence of infinitely smooth convex bodies. This can be proved by a simple convolution argument (see [Sch3, Th. 3.3.1] for the proof):

THEOREM 2.10. *Let K be an origin-symmetric convex body with support function h_K and containing a Euclidean ball with center at the origin and radius $1/C$. For $\epsilon > 0$, let $\phi_\epsilon : [0, \infty) \to [0, \infty)$ be an infinitely differentiable function supported in $[\epsilon/2, \epsilon]$ and such that*

$$\int_{\mathbb{R}^n} \phi_\epsilon(|x|_2)\ dx = 1.$$

Define a function f_ϵ on $\mathbb{R}^n$ by

$$f_\epsilon(x) = \int_{\mathbb{R}^n} h_K(x + |x|_2 z) \phi_\epsilon(|z|_2)\ dz.$$

Then the function f_ϵ is infinitely differentiable on $\mathbb{R}^n \setminus \{0\}$ and is the support function of a convex body K_ϵ so that $f_\epsilon = h_{K_\epsilon}$ and for every $x \in S^{n-1}$,

$$|h_K(x) - h_{K_\epsilon}(x)| < C\epsilon.$$

Since the Minkowski functional of a convex body is the support function of the polar body, the latter theorem gives an approximation of any convex body by infinitely smooth convex bodies in the radial metric.

Also, considering bodies K_ϵ with Minkowski functionals

$$\|\cdot\|_{K_\epsilon} = \|\cdot\|_K + \epsilon|\cdot|_2,$$

one can approximate a convex body K in the radial metric by convex bodies with strictly positive curvature. More results on approximation with special properties can be found in [GrZ, Section 5].

An argument similar to that of Theorem 2.10 shows that one can approximate bodies in the norm of the space $C^k(S^{n-1})$ in the following sense: any k-smooth star body D can be approximated by a sequence of infinitely smooth star bodies D_m so that the radial functions ρ_{D_m} converge to ρ_D in the metric of the space $C^k(S^{n-1})$.

2.3. Radon transforms

The Radon transform and its connection with the Fourier transform play an important role in this text. We refer the reader to the books by Helgason [H1], [H2] for a systematic study of the Radon transforms.

Let ϕ be an integrable function on $\mathbb{R}^n$, which is also integrable on every hyperplane. The *Radon transform* of the function ϕ is defined as a function of $(\xi; t),\ \xi \in S^{n-1},\ t \in \mathbb{R}$:

$$\mathcal{R}\phi(\xi; t) = \int_{(x,\xi)=t} \phi(x)\ dx.$$

We frequently use a well-known connection between the Radon and Fourier transforms:

LEMMA 2.11. *For a fixed ξ, the Fourier transform of the function $g(t) = \mathcal{R}\phi(\xi; t),\ t \in \mathbb{R}$, is equal to the function $z \mapsto \hat{\phi}(z\xi),\ z \in \mathbb{R}$.*

PROOF. Making a change of variables $(x, \xi) = t$, we get

$$\hat{\phi}(z\xi) = \int_{\mathbb{R}^n} \phi(x) e^{-iz(x,\xi)}\ dx = \int_{\mathbb{R}} e^{-izt} \left(\int_{(x,\xi)=t} \phi(x)\ dx \right) dt = \hat{g}(z).$$

□

The *spherical Radon transform* $R : C(S^{n-1}) \mapsto C(S^{n-1})$ is a linear operator defined by

$$Rf(\xi) = \int_{S^{n-1} \cap \xi^\perp} f(x)\ dx, \quad \xi \in S^{n-1},$$

for every function $f \in C(S^{n-1})$. Using (2.2), one can express the volume of central hyperplane sections in terms of the spherical Radon transform: for every origin-symmetric star body K in $\mathbb{R}^n$ and every $\xi \in S^{n-1}$,

$$A_{K,\xi}(0) = \mathrm{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{n-1} R(\|\cdot\|_K^{-n+1})(\xi). \tag{2.17}$$

Let $G(n,m)$ be the Grassman manifold of m-dimensional linear subspaces of $\mathbb{R}^n$, and let dH be the rotationally invariant probability measure on $G(n,m)$. The m-dimensional spherical Radon transform is an operator $R_m : C(S^{n-1}) \to C(G(n,m))$ defined by

$$R_m f(H) = \int_{S^{n-1}\cap H} f(x)\ dx.$$

The dual transform R_m^* (see [H2, p. 144] or [SW, pp. 146-150]) is an operator from $C(G(n,m))$ to $C(S^{n-1})$ such that, for every $F \in C(G(n,m))$,

$$R_m^* F(\xi) = \int_{\xi\in H} F(H)\ dH, \qquad \xi \in S^{n-1}, \tag{2.18}$$

with the latter integral being equal to

$$\int_{\xi\in H} F(H)\ dH = \frac{|S^{m-1}|}{|S^{n-1}|}\int_{G_\xi(n-1,m-1)} F(\mathrm{span}(H,\xi))\ dH, \tag{2.19}$$

where $G_\xi(n-1,m-1)$ stands for the Grassmanian of $(m-1)$-dimensional linear subspaces of the $(n-1)$-dimensional space $\xi^\perp$, and the rotationally invariant probability measure on this Grassmanian is still denoted by dH. Also $|S^{m-1}| = 2\pi^{m/2}/\Gamma(m/2)$ is the surface area of the unit sphere S^{m-1}.

The duality means that, for any functions $f \in C(S^{n-1})$ and $F \in C(G(n,m))$,

$$\begin{aligned}(R_m f, F) &= \int_{G(n,m)} \left(\int_{S^{n-1}\cap H} f(\xi)\ d\xi\right) F(H)\ dH \\ &= \int_{S^{n-1}} f(\xi)\left(\int_{\xi\in H} F(H)\ dH\right) d\xi = (f, R_m^* F).\end{aligned} \tag{2.20}$$

We outline the proof of (2.20), communicated to us by S. Alesker. Let X be the set of pairs (ξ, H), where $\xi \in S^{n-1}$ and H is an m-dimensional subspace of $\mathbb{R}^n$ so that $\xi \in H$. The orthogonal group $O(n)$ acts on X transitively, so X admits a unique Haar probability measure μ. For any continuous function g on X, we have

$$\begin{aligned}\int_X g\ d\mu &= c_1 \int_{G(n,m)} \left(\int_{\xi\in H} g(\xi,H)\ d\xi\right) dH \\ &= c_2 \int_{S^{n-1}} \left(\int_{\xi\in H} g(\xi,H)\ dH\right) d\xi,\end{aligned} \tag{2.21}$$

where c_1 and c_2 are some constants. In fact, all three integrals define continuous $O(n)$-invariant functionals on $C(X)$; therefore, all of them correspond to constant multiples of the Haar measure. To prove (2.20), consider any two functions $f \in C(S^{n-1})$ and $F \in C(G(n,m))$. Then put $g(\xi,H) = f(\xi)F(H)$ and compute the integrals in equation (2.21).

Put $F \equiv 1$ in equation (2.20). We get that, for any function $f \in C(S^{n-1})$,

$$\int_{G(n,m)} \left(\int_{S^{n-1}\cap H} f(\xi)\ d\xi\right) dH = \frac{|S^{m-1}|}{|S^{n-1}|}\int_{S^{n-1}} f(\xi)\ d\xi. \tag{2.22}$$

Note that, while the measure on the Grassmanian is probabilistic, the uniform measures on the spheres are not normalized in formulas (2.20) and (2.22), which explains the constant in the right-hand side of (2.22) (this constant can be verified by putting $f \equiv 1$).

The following lemma was proved in [K14, Lemma 1].

LEMMA 2.12. *Let $1 \le m < n$, and let $f, g \in C(S^{n-1})$. Then*

$$\int_{S^{n-1}} f(\xi) \left(\int_{S^{n-1}\cap\xi^\perp} g(x)\ dx \right) d\xi$$

$$= c(n,m) \int_{G(n,m)} \left(\int_{S^{n-1}\cap H} f(\xi) d\xi \right) \left(\int_{S^{n-1}\cap H^\perp} g(x) dx \right)\ dH,$$

where

$$c(n,m) = (|S^{n-1}||S^{n-2}|)/(|S^{m-1}||S^{n-m-1}|).$$

PROOF. Apply equality (2.20) to the functions f and

$$F(H) = \int_{S^{n-1}\cap H^\perp} g(x)\ dx.$$

We get

$$\int_{G(n,m)} \left(\int_{S^{n-1}\cap H} f(\xi) d\xi \right) \left(\int_{S^{n-1}\cap H^\perp} g(x) dx \right)\ dH$$

$$= c \int_{S^{n-1}} f(\xi) \left(\int_{\xi \in H} \left(\int_{S^{n-1}\cap H^\perp} g(x)\ dx \right)\ dH \right)\ d\xi, \tag{2.23}$$

with a constant to be computed later. Now apply (2.19) and change the variables $H \to H^\perp$:

$$\int_{\xi\in H} \left(\int_{S^{n-1}\cap H^\perp} g(x)\ dx \right) dH = c_1 \int_{G_\xi(n-1,n-m)} \left(\int_{S^{n-1}\cap H} g(x)\ dx \right) dH,$$

where $G_\xi(n-1, n-m)$ is the Grassmanian of $(n-m)$-dimensional subspaces of $\xi^\perp$ with its rotationally invariant probability measure. With (2.22) applied to the unit sphere $S^{n-1} \cap \xi^\perp$ of the hyperplane $\xi^\perp$, the latter is equal to

$$c_2 \int_{S^{n-1}\cap\xi^\perp} g(x)\ dx,$$

which proves the lemma up to a constant. To compute the constant, put $f \equiv 1$ and $g \equiv 1$. □

Applying Lemma 2.12 with $m = n-1$, we see that the spherical Radon transform $R = R_{n-1}$ is self-dual (see [Gr, Lemma 1.3.3] for a simple proof not involving Grassmanians).

LEMMA 2.13. *For any functions $f, g \in C(S^{n-1})$,*

$$\int_{S^{n-1}} Rf(\xi)\ g(\xi)\ d\xi = \int_{S^{n-1}} f(\xi)\ Rg(\xi)\ d\xi. \tag{2.24}$$

Finally, we extend the spherical Radon transform to measures. Let μ be a finite Borel measure on S^{n-1}. We define the spherical Radon transform of μ as a functional $R\mu$ on the space $C(S^{n-1})$ acting by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x) d\mu(x).$$

By F. Riesz's characterization of continuous linear functionals on the space $C(S^{n-1})$ (see [DS, p. 262]) $R\mu$ is also a finite Borel measure on S^{n-1}. If μ has continuous density g, then, by Lemma 2.13, the Radon transform of μ has density Rg.

2.4. The Gamma-function

For $z \in \mathbb{C},\ \Re z > 0$, the Γ-function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt. \tag{2.25}$$

A simple change of variables shows that, for any $p, q > 0$,

$$\int_0^\infty x^{p-1} e^{-x^q} dx = \frac{\Gamma(p/q)}{q}. \tag{2.26}$$

Integrating (2.25) by parts, we get that for $z \in \mathbb{C},\ \Re z > 0$,

$$\Gamma(z+1) = z\Gamma(z). \tag{2.27}$$

Since $\Gamma(1) = 1$, (2.27) implies $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$. Computing the integral

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2} dx\ dy = \left(\int_0^\infty e^{-x^2} dx \right)^2$$

in polar coordinates and using (2.26), we get

$$\Gamma(1/2) = \sqrt{\pi}. \tag{2.28}$$

Lemma 2.14. *The function* $\ln \Gamma(\cdot)$ *is convex on* $(0, \infty)$.

Proof. By the Cauchy-Schwartz inequality, for $x_1, x_2 > 0$,

$$\Gamma\left(\frac{x_1+x_2}{2}\right) = \int_0^\infty t^{\frac{x_1+x_2}{2}-1} e^{-t} dt = \int_0^\infty t^{(x_1-1)/2} t^{(x_2-1)/2} e^{-t} dt$$
$$\le \left(\int_0^\infty t^{x_1-1} e^{-t} dt \right)^{1/2} \left(\int_0^\infty t^{x_2-1} e^{-t} dt \right)^{1/2} = \sqrt{\Gamma(x_1)\Gamma(x_2)},$$

which implies the result. □

Differentiating under the integral in (2.25) by z, one can see that the Γ-function is analytic in $\{z \in \mathbb{C} :\ \Re z > 0\}$. We can use formula (2.27) to extend the Γ-function to an analytic function in $\mathbb{C} \setminus (-\mathbb{N} \cup \{0\})$ with simple poles at non-positive integers. The residue of the Γ-function at $z = -k$ is equal to $(-1)^k/k!$.

The Γ-function is positive on the intervals $(0, \infty)$ and $(-2k, -2k+1),\ k \in \mathbb{N}$, and it is negative on each of the intervals $(-2k+1, -2k+2),\ k \in \mathbb{N}$.

The analyticity of the Γ-function allows us to prove different formulas on an interval and then extend them to larger sets using the uniqueness of analytic continuation.

Lemma 2.15. *If* x *and* $x + 1/2$ *are complex numbers that do not belong to the set* $-\mathbb{N} \cup \{0\}$, *then*

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma(x + \frac{1}{2}). \tag{2.29}$$

Proof. We can prove (2.29) on any interval and then use analytic continuation. For $a, b > 0$, consider the double integral

$$\Gamma(a)\Gamma(b) = \int_0^\infty \int_0^\infty x^{a-1} y^{b-1} e^{-x-y} dx\ dy.$$

Substituting $x = u^2,\ y = v^2$, we write the latter integral as

$$4\int_0^\infty \int_0^\infty u^{2a-1}v^{2b-1}e^{-u^2-v^2}du\ dv.$$

Passing to polar coordinates and using (2.26), we get

$$2\Gamma(a+b)\int_0^{\pi/2}(\cos\theta)^{2a-1}(\sin\theta)^{2b-1}d\theta.$$

We get a formula for the Beta-function B defined by

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = 2\int_0^{\pi/2}(\cos\theta)^{2a-1}(\sin\theta)^{2b-1}d\theta. \tag{2.30}$$

Now making a substitution $t = \cos\theta$,

$$B(a,b) = \int_{-1}^1 |t|^{2a-1}(1-t^2)^{b-1}dt. \tag{2.31}$$

Making another substitution $x = t^2$,

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

Putting $a = b$, we get

$$\begin{aligned} B(a,a) &= \int_0^1 \left(\frac{1}{4} - (\frac{1}{2}-x)^2\right)^{a-1} dx \\ &= 2\int_0^{1/2} \left(\frac{1}{4} - (\frac{1}{2}-x)^2\right)^{a-1} dx. \end{aligned}$$

After the additional substitution $1/2 - x = (1/2)\sqrt{u}$, we write the latter integral as

$$\frac{1}{2^{2a-1}}\int_0^1 u^{-1/2}(1-u)^{a-1}du = \frac{1}{2^{2a-1}}B(1/2,a).$$

Finally,

$$B(a,a) = \frac{1}{2^{2a-1}}B(1/2,a),$$

and the result follows from the definition of the Beta-function and (2.28). □

Next we compute the Euler integral:

LEMMA 2.16. *For* $0 < p < 1$,

$$\int_0^\infty \frac{x^{p-1}}{1+x^2}dx = \frac{\pi}{2\sin\frac{\pi p}{2}}.$$

PROOF. Consider the function $f(z) = z^{p-1}/(1+z^2),\ z \in \mathbb{C}$. This function has simple poles at $z = i$ and $z = -i$. Take $0 < t < T$ and consider a contour starting at the point $(t,0)$, going along the x-axis to the point $(T,0)$, then around the circle with center at the origin and radius T, then along the line $ue^{2\pi i}, u \in [T,t]$, and, finally, around the circle with center at the origin and radius t. Integrals over the circles vanish, as $t \to 0$ and $T \to \infty$. We get the result by computing the residues of f at $z = \pm i$ and performing contour integration. □

LEMMA 2.17. *If* $x \in \mathbb{C}$ *is not a real integer, then*

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}. \tag{2.32}$$

PROOF. Put $a = 1 - b,\ b \in (0, 1/2)$, in (2.30). We get

$$\Gamma(b)\Gamma(1-b) = 2\int_0^{\pi/2} (\tan\theta)^{2b-1} d\theta.$$

Making the change of variables $x = \tan\theta$ and using Lemma 2.16, we get (2.32) for $x \in (0, 1/2)$. Then use analytic continuation. □

Our next formula will be used in Lemma 2.23.

LEMMA 2.18. *If $q \in \mathbb{C}$ is not a non-negative even integer and not a negative odd integer, then*

$$\frac{2^{q+1}\sqrt{\pi}\Gamma((q+1)/2)}{\Gamma(-q/2)} = -2\Gamma(1+q)\sin(\pi q/2). \tag{2.33}$$

PROOF. Write $\Gamma(1+q)$ using (2.29) with $x = (q+1)/2$, and then use (2.32) with $x = -q/2$. □

For $p > 0$, let B_p^n be the unit ball of the space ℓ_p^n; see (1.4).

LEMMA 2.19.

$$\mathrm{Vol}_n(B_p^n) = \frac{2^n\left(\Gamma(1+\frac{1}{p})\right)^n}{\Gamma(1+\frac{n}{p})}.$$

PROOF. Using (2.26) and (2.27), we compute the integral

$$\int_{\mathbb{R}^n} e^{-|x_1|^p - \ldots - |x_n|^p} dx = \left(\frac{2\Gamma(1/p)}{p}\right)^n = 2^n\left(\Gamma(1+\frac{1}{p})\right)^n.$$

On the other hand, compute the same integral in polar coordinates and use (2.26) and the polar formula for the volume (2.4):

$$= \int_{S^{n-1}}\left(\int_0^\infty r^{n-1}e^{-r^p\|\theta\|_p^p}dr\right)d\theta = \frac{\Gamma(n/p)}{p}\int_{S^{n-1}}\|\theta\|_p^{-n}d\theta$$

$$= n\mathrm{Vol}_n(B_p^n)\frac{\Gamma(n/p)}{p} = \mathrm{Vol}_n(B_p^n)\Gamma(1+\frac{n}{p}).$$

Comparing these two expressions for the same integral, we get the result. □

COROLLARY 2.20. *The volume and surface area of the unit Euclidean ball are equal to*

$$\mathrm{Vol}(B_2^n) = \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})} \quad \text{and} \quad |S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

PROOF. To get the formula for the surface area, write the integral

$$\mathrm{Vol}(B_2^n) = \int_{\mathbb{R}^n}\chi(|x|_2)dx$$

in polar coordinates, where, as before, χ is the indicator function of the interval $[-1, 1]$. □

2.5. The Fourier transform of distributions

The Fourier transform of distributions is the main tool used in this book. In this section we collect definitions, notation and results from the theory of distributions that we need later. We refer the reader to the books by Rudin [Ru] and Gelfand and Shilov [GS] for systematic studies of distributions. Note that in this text we use a definition of the Fourier transform which is slightly different from those in [Ru] and [GS]. We define the Fourier transform of a function $\phi \in L_1(\mathbb{R}^n)$ by

$$\mathcal{F}\phi(\xi) = \hat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x) e^{-i(x,\xi)} dx, \quad \xi \in \mathbb{R}^n.$$

Rudin [Ru] uses the normalized Lebesgue measure, i.e. Lebesgue measure multiplied by the number $(2\pi)^{-n/2}$, while Gelfand and Shilov use Lebesgue measure without a coefficient, but they have $\exp(i(x,\xi))$ in place of $\exp(-i(x,\xi))$. The reason for our choice is that it is easy to adjust the results from [Ru], and, at the same time, we can directly use computations of the Fourier transform of certain distributions given in [GS], because the functions, for which we compute the Fourier transform in the sense of distributions, are always real-valued and even. For the same reason, we follow [Ru] by defining the action of a complex-valued function f on a test function ϕ as

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x)\ dx,$$

while [GS] has this definition with the complex conjugate of $f(x)$. Finally, we again follow [Ru] in defining the Fourier transform of a distribution f by

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle,$$

while the definition in [GS] is different:

$$\langle \hat{f}, \hat{\phi} \rangle = (2\pi)^n \langle f, \phi \rangle.$$

Let us stress once again that the definitions below are chosen in such a way that all the results from [Ru] can be used up to the coefficient $(2\pi)^{-n/2}$ in the formulas related to the Fourier transform, and all the results from [GS] can be used without any changes, because we always deal with real-valued even distributions (except for one occasion in Lemma 4.16), for which our formulas coincide with those from [GS].

Our choice of the space of test functions is the Schwartz space of rapidly decreasing infinitely differentiable functions. For a multiindex $\alpha = (\alpha_1, ..., \alpha_n)$, where $\alpha_k \in \mathbb{N} \cup \{0\}$, let $|\alpha| = \sum_{k=1}^n \alpha_k$. Let D^α be a differential operator defined for every $f \in C^{|\alpha|}(\mathbb{R}^n)$ by

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

We denote by $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ the space of complex-valued functions $\phi \in C^\infty(\mathbb{R}^n)$ converging to zero at infinity together with all their derivatives faster than any negative power of $|\cdot|_2$, i.e. for every $k \in \mathbb{N}$,

$$p_k(\phi) = \sup_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} (1 + |x|_2)^k |D^\alpha \phi(x)| < \infty.$$

Throughout the text, elements of the space $\mathcal{S}$ will be called test functions.

A locally convex topology on $\mathcal{S}$ is generated by a sequence of seminorms p_k (see [Ru, Th 1.37]), where the local base of open sets is given by intersections of finite collections of sets of the form

$$V(k,m) = \{\phi \in \mathcal{S} : \ p_k(\phi) < \frac{1}{m}\}, \qquad k \in \mathbb{N}, \ m \in \mathbb{N}.$$

As usual, we denote by $\mathcal{S}'$ the space of linear continuous functionals on $\mathcal{S}$, which we call distributions over $\mathcal{S}$. Suppose that f is a locally integrable complex-valued function on $\mathbb{R}^n$ with *power growth at infinity*, i.e. there exists a number $\beta > 0$ so that

$$\lim_{|x|\to\infty} \frac{f(x)}{|x|_2^\beta} = 0.$$

Then f represents a distribution acting by integration: for every $\phi \in \mathcal{S}$,

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x)\phi(x) \ dx.$$

For an open set Ω in $\mathbb{R}^n$, denote by $\mathcal{S}(\Omega)$ the space of test functions with support in Ω. Suppose that $f \in \mathcal{S}'$ and W is the union of all open sets Ω in $\mathbb{R}^n$ on which $f \equiv 0$, i.e. $\langle f, \phi \rangle = 0$ for every $\phi \in \mathcal{S}(\Omega)$. Then the set $\operatorname{supp}(f) = \mathbb{R}^n \setminus W$ is called the *support of the distribution* f. If $\phi \in \mathcal{S}$ is a test function so that $\operatorname{supp}(\phi) \cap \operatorname{supp}(f) = \emptyset$, then $\langle f, \phi \rangle = 0$.

An infinitely differentiable function g on $\mathbb{R}^n$ is called a *multiplicator* over $\mathcal{S}$ if $g\phi \in \mathcal{S}$ for every $\phi \in \mathcal{S}$ and the mapping $\phi \mapsto g\phi$ is continuous on $\mathcal{S}$. Every function $\phi \in \mathcal{S}$ is a multiplicator over $\mathcal{S}$, and so is every polynomial; see [Ru, Th. 7.4].

If ϕ is a test function, then so is its Fourier transform

$$\mathcal{F}\phi(x) = \hat{\phi}(x) = \int_{\mathbb{R}^n} \phi(\xi) e^{-i(x,\xi)} \ d\xi;$$

see [Ru, Th. 7.4]. The Fourier transform is invertible on $\mathcal{S}$, and the inverse operator is given by

$$\tilde{\mathcal{F}}\phi(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} \phi(x) e^{i(x,\xi)} dx,$$

see [Ru, Th 7.7]. It follows that, for every $\phi \in \mathcal{S}$,

$$(\hat{\phi})^\wedge(\xi) = (2\pi)^n \phi(-\xi).$$

Also both operators $\mathcal{F}$ and $\tilde{\mathcal{F}}$ are continuous on the space $\mathcal{S}$, [Ru, Th. 7.7].

An easy application of the Fubini theorem is the following *Parseval formula* for test functions. If ϕ, ψ are test functions, then

$$\int_{\mathbb{R}^n} \hat{\phi}(x) \ \psi(x) \ dx = \int_{\mathbb{R}^n} \phi(\xi) \ \hat{\psi}(\xi) \ d\xi. \tag{2.34}$$

In fact, the left-hand side can be written as

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \phi(\xi) e^{-i(x,\xi)} \ d\xi \right) \psi(x) \ dx,$$

and, by the Fubini theorem, the latter is equal to

$$\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \psi(x) e^{-i(x,\xi)} \ dx \right) \phi(\xi) \ d\xi = \int_{\mathbb{R}^n} \phi(\xi) \ \hat{\psi}(\xi) \ d\xi.$$

Since test functions are dense in the spaces $L_1(\mathbb{R}^n)$ and $L_2(\mathbb{R}^n)$, Parseval's formula can be extended to the cases where both functions ϕ and ψ belong to $L_1(\mathbb{R}^n)$ or both belong to $L_2(\mathbb{R}^n)$ (see [Ru, Th 7.9]).

Parseval's formula motivates the definition of the *Fourier transform of a distribution* f as a distribution $\hat{f}$ acting by

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$$

for every test function ϕ. Since the mapping $\phi \mapsto \hat{\phi}$ is an isomorphism of $\mathcal{S}(\mathbb{R}^n)$, the Fourier transform of a distribution is well defined, and $\hat{f} = \hat{g}$ implies $f = g$. When g is given by a longer expression, we write $(g)^\wedge$ for the Fourier transform of g.

If a test function ϕ is even, we have

$$(\hat{\phi})^\wedge = (2\pi)^n \phi \qquad \text{and} \qquad \langle \hat{f}, \hat{\phi} \rangle = (2\pi)^n \langle f, \phi \rangle$$

for every $f \in \mathcal{S}'$. Throughout this text (except for one occasion in Lemma 4.16) we consider only real-valued even test functions ϕ, for which the Fourier transform $\hat{\phi}$ is also an even real-valued function.

A distribution f is called even homogeneous of degree $p \in \mathbb{R}$ if

$$\langle f(x), \phi(x/\alpha) \rangle = |\alpha|^{n+p} \langle f, \phi \rangle \tag{2.35}$$

for every test function ϕ and every $\alpha \in \mathbb{R},\ \alpha \neq 0$.

LEMMA 2.21. *The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$.*

PROOF. A simple computation shows that, for any test function ϕ and any $\alpha \neq 0$,

$$\widehat{\phi(x/\alpha)}(\xi) = |\alpha|^n \hat{\phi}(\alpha\xi).$$

By (2.35),

$$\begin{aligned} \langle \hat{f}(x), \phi(x/\alpha) \rangle &= \langle f, \widehat{\phi(x/\alpha)} \rangle = \langle f(\xi), |\alpha|^n \hat{\phi}(\alpha\xi) \rangle \\ &= |\alpha|^{-p} \langle f, \hat{\phi} \rangle = |\alpha|^{n+(-n-p)} \langle \hat{f}, \phi \rangle, \end{aligned}$$

which means that $\hat{f}$ is an even homogeneous distribution of degree $-n-p$. □

For any multiindex α, the derivative of the order α of a distribution f is defined by

$$\langle D^\alpha f, \phi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \phi \rangle.$$

The Fourier transform is related to differentiation as follows:

$$(D^\alpha f)^\wedge = i^{|\alpha|} x_1^{\alpha_1} ... x_n^{\alpha_n} \hat{f}, \tag{2.36}$$

and

$$D^\alpha \hat{f} = (-i)^{|\alpha|} (x_1^{\alpha_1} ... x_n^{\alpha_n} f(x))^\wedge;$$

see [Ru, Th 7.4]. The product of a polynomial and a distribution is defined by

$$\langle x_1^{\alpha_1} ... x_n^{\alpha_n} f, \phi \rangle = \langle f, x_1^{\alpha_1} ... x_n^{\alpha_n} \phi \rangle,$$

which is possible because every polynomial is a multiplicator. The connection between the Fourier transform and differentiation follows from similar properties of test functions, which can be proved by simple integration by parts.

If a distribution is supported in the origin, then it is a finite linear combination of derivatives of the δ-function defined by $\langle \delta, \phi \rangle = \phi(0)$, [Ru, Th. 6.25]. Since the Fourier transform of a linear combination of derivatives of the δ-function is a polynomial, any two distributions, whose Fourier transforms are equal outside of the origin, can differ only by a polynomial; see [Ru, Section 7.16].

For any open set $Q \subset \mathbb{C}$ we say that a family of distributions $f_q,\ q \in Q$, is analytic on Q if for every test function ϕ, the function

$$q \mapsto \langle f_q, \phi \rangle$$

is analytic on Q. This concept is important for us, because we usually prove certain formulas for $q \in (-1, 0)$ and then use analytic continuation to prove these formulas for other values of q. We often use the following fact which can be found, in a similar form, in [GS, Ch. 1, Sect. 3]:

LEMMA 2.22. *For any star body K in $\mathbb{R}^n$ the family of distributions $\|\cdot\|_K^q$ is analytic in the domain $q \in \mathbb{C},\ \Re(q) > -n$. The family of Fourier transforms $(\|\cdot\|_K^q)^\wedge$ is also analytic in the same domain.*

PROOF. By Lemma 2.1, the function $\|\cdot\|_K^q$ is locally integrable on $\mathbb{R}^n$. Therefore, for any test function ϕ,

$$h(q) = \langle \|\cdot\|_K^q, \phi \rangle = \int_{\mathbb{R}^n} \|x\|_K^q \phi(x)\ dx.$$

Differentiating under the integral, it is easy to see that the derivative $h'(q)$ exists everywhere in the domain $\Re(q) > -n$ and is equal to

$$h'(q) = \int_{\mathbb{R}^n} \|x\|_K^q \ln(\|x\|_K) \phi(x)\ dx.$$

Note that an easy way of proving the possibility of differentiation by q under the integral is to do it in the opposite direction – integrate by q the equality for the derivatives and use the Fubini theorem.

The second statement follows from the first and the definition of the Fourier transform, because $\langle (\|\cdot\|_K)^\wedge, \phi \rangle = \langle \|\cdot\|_K^q, \hat{\phi} \rangle$. □

A crucial role in what follows belongs to the distributions $t \mapsto |t|^q,\ t \in \mathbb{R}$, where $q \in \mathbb{C}$. If $\Re(q) \le -1$, this function is not locally integrable on $\mathbb{R}$, so to define the corresponding distribution, we need a special procedure that is called *regularization*. We want this procedure to produce an analytic family of distributions on $\mathbb{C}$ without negative integers.

For $t \in \mathbb{R}$, let $t_+ = \max\{0, t\}$. We first define the distributions t_+^q. If $\Re q > -1$, then the function t_+^q is locally integrable and its action on a test function ϕ is defined by

$$\langle t_+^q, \phi(t) \rangle = \int_0^\infty t^q \phi(t)\, dt.$$

We write the right-hand side of the latter equality in a different form:

$$\int_0^\infty t^q \phi(t)\, dt = \int_0^1 t^q (\phi(t) - \phi(0))\, dt + \int_1^\infty t^q \phi(t) dt + \frac{\phi(0)}{1+q}.$$

Now the integrals in the right-hand side converge absolutely for every q with $\Re q \in (-2, -1)$, and we define the distributions t_+^q with $\Re q \in (-2, -1)$ by

$$\langle t_+^q, \phi(t) \rangle = \int_0^1 t^q (\phi(t) - \phi(0))\, dt + \int_1^\infty t^q \phi(t) dt + \frac{\phi(0)}{1+q}. \tag{2.37}$$

In the case $\Re q > -1$ the latter equality coincides with the original definition, so we can say that formula (2.37) defines the distributions t_+^q for all $q \in \mathbb{C}$, $\Re q > -2$, $q \neq -1$. Moreover, by the same argument as in Lemma 2.22, the family of distributions $q \mapsto t_+^q$ is analytic in this domain.

We now continue this procedure: for $q \in \mathbb{C}$ and $m \in \mathbb{N}$ such that $-m-1 < \Re q$, $q \neq -1, -2, \ldots, -m$, and for every $\phi \in \mathcal{S}$, we put

$$\langle t_+^q, \phi(t)\rangle = \int_0^1 t^q \left(\phi(t) - \phi(0) - t\phi'(0) - \cdots - \frac{t^{m-1}}{(m-1)!}\phi^{(m-1)}(0)\right) dt$$

$$+ \int_1^\infty t^q \phi(t) dt + \sum_{k=1}^m \frac{\phi^{(k-1)}(0)}{(k-1)!(q+k)}. \tag{2.38}$$

If $-m-1 < \Re(q) < -m$, we have

$$\langle t_+^q, \phi(t)\rangle = \int_0^\infty t^q \left(\phi(t) - \phi(0) - t\phi'(0) - \cdots - \frac{t^{m-1}}{(m-1)!}\phi^{(m-1)}(0)\right) dt,$$

since in this case

$$\int_1^\infty t^{q+k-1} dt = -\frac{1}{q+k},$$

for $k = 1, \ldots, m$. It can be checked again by direct differentiation that the family

$$\{t_+^q : q \in \mathbb{C} \setminus \{-1, -2, \ldots\}\}$$

forms an analytic distribution, that is, for any $\phi \in \mathcal{S}$, the function $q \mapsto \langle t_+^q, \phi(t)\rangle$ is analytic on $\mathbb{C} \setminus \{-1, -2, \ldots\}$.

It is clear that $\langle t_+^q, \phi(t)\rangle$ has, for each $k \in \mathbb{N}$, a simple pole at $q = -k$ with residue $\phi^{(k-1)}(0)/(k-1)!$. The function $q \mapsto \Gamma(q+1)$ also has a simple pole at $q = -k$ with residue $(-1)^{k-1}/(k-1)!$; see Section 2.4. We conclude that we can extend

$$\left\{\frac{t_+^q}{\Gamma(q+1)} : q \in \mathbb{C} \setminus \{-1, -2, \ldots\}\right\}$$

to an analytic distribution on $\mathbb{C}$, still denoted by $\{t_+^q/\Gamma(q+1) : q \in \mathbb{C}\}$, and for $q = -k$ and $\phi \in \mathcal{S}$,

$$\left\langle \frac{t_+^q}{\Gamma(q+1)}, \phi(t)\right\rangle = (-1)^{k-1}\phi^{(k-1)}(0).$$

Outside of any neighborhood of 0, the functional $t_+^q/\Gamma(q+1)$ acts like a continuous function, so we can actually apply $t_+^q/\Gamma(q+1)$ to any continuous function that is infinitely differentiable in a neighborhood of 0, drawing the same conclusions as for functions in $\mathcal{S}$. We are going to do it later in the definition of fractional derivatives.

For $q \in \mathbb{C} \setminus -2\mathbb{N}+1$, we define the distribution $|t|^q$ by

$$\langle |t|^q, \phi\rangle = \langle t_+^q, \phi\rangle + \langle t_+^q, \phi(-t)\rangle.$$

For any $\phi \in \mathcal{S}$, the function $q \mapsto \langle |t|^q, \phi\rangle$ is analytic in $\mathbb{C} \setminus -2\mathbb{N}+1$. Note that negative even integers are included, because the singularities in (2.38) corresponding to even k disappear after adding the same expression with $\phi(-t)$.

We now compute the Fourier transform of $|t|^q$. We present this computation with all the details, some of which will be omitted in similar calculations later.

LEMMA 2.23. *If $q \in \mathbb{C}$ is not a non-negative even integer and not a negative odd integer, then the Fourier transform of the distribution $|t|^q$, $t \in \mathbb{R}$, is equal to*

$$(|t|^q)^\wedge(\xi) = -2\Gamma(1+q)\sin(\pi q/2)\,|\xi|^{-q-1}, \quad \xi \in \mathbb{R}.$$

PROOF. First, suppose that $q \in (-1, 0)$. Making the substitution $u = zt$ in the integral and using (2.26), we get

$$|t|^q = \frac{2^{q/2+1}}{\Gamma(-q/2)} \int_0^\infty z^{-1-q} e^{-z^2t^2/2}\, dz.$$

We often use representations like this to compute the Fourier transform. The reason is that the Fourier transform of the Gaussian density is well known (and easily computable):

$$(e^{-t^2/2})^\wedge(z) = \sqrt{2\pi}e^{-z^2/2}.$$

Now for any even test function ϕ :

$$\langle (|t|^q)^\wedge, \phi\rangle = \langle |t|^q, \hat{\phi}\rangle = \int_{\mathbb{R}} |t|^q \hat{\phi}(t)\, dt$$

$$= \frac{2^{q/2+1}}{\Gamma(-q/2)} \int_{\mathbb{R}} \left(\int_0^\infty z^{-1-q} e^{-z^2t^2/2}\, dz \right) \hat{\phi}(t)\, dt$$

$$= \frac{2^{q/2+1}}{\Gamma(-q/2)} \int_0^\infty z^{-1-q} \left(\int_{\mathbb{R}} e^{-(tz)^2/2} \hat{\phi}(t)\, dt \right) dz.$$

Applying Parseval's equality (2.34) to the integral over $\mathbb{R}$, the fact that

$$(f(tx))^\wedge(\xi) = t^{-1}\hat{f}(\xi/t) \tag{2.39}$$

and formula (2.26) (make the substitution $u = \xi/z$), we continue the calculation:

$$= \frac{2^{q/2+1}}{\Gamma(-q/2)} \int_0^\infty z^{-1-q} \left(\int_{\mathbb{R}} \sqrt{2\pi} z^{-1} e^{-\xi^2/2z^2} \phi(\xi)\, d\xi \right) dz$$

$$= \frac{\sqrt{2\pi}2^{q/2+1}}{\Gamma(-q/2)} \int_{\mathbb{R}} \left(\int_0^\infty z^{-2-q} e^{-\xi^2/2z^2} dz \right) \phi(\xi)\, d\xi$$

$$= \frac{2^{q+1}\sqrt{\pi}\Gamma((q+1)/2)}{\Gamma(-q/2)} \int_{\mathbb{R}} |\xi|^{-1-q} \phi(\xi)\, d\xi.$$

We see that for every even test function ϕ,

$$\langle (|t|^q)^\wedge, \phi\rangle = \frac{2^{q+1}\sqrt{\pi}\Gamma((q+1)/2)}{\Gamma(-q/2)} \langle |\xi|^{-1-q}, \phi\rangle. \tag{2.40}$$

Now we can use formula (2.33) to simplify the constant, which proves the lemma for every $q \in (-1, 0)$. The distributions $|t|^q$ are defined in such a way that both sides of (2.40) are analytic functions of q everywhere in $\mathbb{C}$ without negative odd integers and non-negative even integers. By uniqueness of analytic continuation, we can extend the formula from the interval $(-1, 0)$ to all such q. □

2.6. Fractional derivatives

Fractional derivatives play an important role in the Fourier analytic solution of the Busemann-Petty problem in [GKS]. However, we do not use fractional derivatives in full generality; we need fractional derivatives at only one point, the origin. In general, the fractional derivative of the order q of a test function $\phi \in \mathcal{S}(\mathbb{R})$ is defined as the convolution of ϕ with $t_+^{-1-q}/\Gamma(-q)$:

$$\phi^{(q)}(x) = \langle \frac{t_+^{-1-q}}{\Gamma(-q)}, \phi(x-t)\rangle.$$

We need this definition only with $x = 0$. This allows us to replace the test function ϕ by any function differentiable up to a certain order in a neighborhood of zero. Let us formulate this precisely, though the procedure is essentially the same as in the definition of the distributions t_+^q; see (2.38).

Let $m \in \mathbb{N} \cup \{0\}$ and suppose h is a continuous integrable function on $\mathbb{R}$ that is m times continuously differentiable in some neighborhood of zero.

For $q \in \mathbb{C},\ -1 < \Re(q) < m,\ q \neq 0, 1, ..., m-1$, the *fractional derivative* of the order q of the function h at zero is defined as the action of the distribution $t_+^{-1-q}/\Gamma(-q)$ on the function h, as follows:

$$h^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^1 t^{-1-q} \left(h(t) - h(0) - ... - h^{(m-1)}(0)\frac{t^{m-1}}{(m-1)!} \right)\ dt$$

$$(2.41) \qquad + \frac{1}{\Gamma(-q)} \int_1^\infty t^{-1-q} h(t) dt + \frac{1}{\Gamma(-q)} \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!(k-q)}.$$

It is easy to see that for a fixed q the definition does not depend on the choice of $m > \Re(q)$, as long as f is m times continuously differentiable. Note that without dividing by $\Gamma(-q)$ the expression for the fractional derivative represents an analytic function in the domain $\{q \in \mathbb{C} :\ \Re(q) > -1\}$ not including integers and has simple poles at integers. The function $\Gamma(-q)$ is analytic in the same domain and also has simple poles at non-negative integers, so after the division we get an analytic function in the whole domain $\{q \in \mathbb{C} :\ \ m > \Re(q) > -1\}$, which also defines fractional derivatives of integer orders. Moreover, computing the limit as $q \to k$, where k is a non-negative integer, we see that the fractional derivatives of integer orders coincide with usual derivatives up to a sign (when we compute the limit, the first two summands in the right-hand side of (2.41) converge to zero, since $\Gamma(-q) \to \infty$, and the limit in the third summand can be computed using the property $\Gamma(x+1) = x\Gamma(x)$ of the Γ-function):

$$h^{(k)}(0) = (-1)^k \frac{d^k}{dt^k} h(t)|_{t=0}.$$

If h is an even function, its derivatives of odd orders at the origin are equal to zero and, for $m-2 < \Re(q) < m$, expression (2.41) becomes

$$(2.42) \qquad h^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-q-1} \left(h(t) - \sum_{j=0}^{(m-2)/2} \frac{t^{2j}}{(2j)!} h^{(2j)}(0) \right) dt.$$

We also note that if $-1 < q < 0$, then

$$h^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^\infty t^{-1-q} h(t)\ dt. \tag{2.43}$$

2.7. Positive definite distributions

We say that a distribution f is *positive definite* if its Fourier transform is a *positive distribution*, i.e. we have $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function ϕ. L. Schwartz's generalization of Bochner's theorem (see, for example, [GV, p. 152]) states that a distribution is positive definite if and only if it is the Fourier transform of a tempered measure on $\mathbb{R}^n$. Recall that a (non-negative, not necessarily finite) measure μ is called tempered if

$$\int_{\mathbb{R}^n} (1 + |x|_2)^{-\beta}\ d\mu(x) < \infty$$

for some $\beta > 0$. We do not prove Schwartz's theorem here and instead refer the reader to [GV, p. 152]. However, we are going to prove a similar fact for distributions that are positive outside of the origin. This proof follows the steps of the proof of Schwartz's theorem.

We mostly deal with positive definite distributions that are at the same time homogeneous. The corresponding tempered measures are also homogeneous as distributions and can be written in polar coordinates, as follows:

LEMMA 2.24. *Let μ be a tempered measure on $\mathbb{R}^n$ which is, at the same time, an even homogeneous distribution of degree $-n+p$, $p \in (0, n)$. Then there exists a finite Borel measure μ_0 on the sphere S^{n-1} so that, for every even test function ϕ,*

$$\langle \mu, \phi \rangle = \int_{S^{n-1}} \left(\int_0^\infty t^{p-1} \phi(t\xi)\ dt \right) d\mu_0(\xi). \tag{2.44}$$

PROOF. Let us first show that μ cannot have an atom at the origin. In fact, suppose that $\mu = \mu_1 + a\delta$, where $\mu_1(\{0\}) = 0$ and δ is the unit mass at the origin. Since μ is homogeneous of degree $-n+p$, for every non-negative test function ϕ with $\phi(0) > 0$ and every $t > 0$, we have

$$\langle \mu, \phi(x/t) \rangle = t^p \langle \mu, \phi \rangle \to 0$$

as $t \to 0$. On the other hand,

$$\langle \mu, \phi(x/t) \rangle = \langle \mu_1, \phi(x/t) \rangle + a\phi(0) \geq a\phi(0),$$

so $a = 0$.

For every Borel subset $A \subset S^{n-1}$ and every interval $(a, b] \in [0, \infty)$ let

$$A \times (a, b] = \{x \in \mathbb{R}^n : \ x = t\theta,\ t \in (a, b],\ \theta \in A\},$$

and let $\chi_{A \times (a,b]}$ be the indicator of this set.

By the definition of a homogeneous distribution, we have

$$\int_{\mathbb{R}^n} \phi(x/t)\ d\mu(x) = \langle \mu, \phi(x/t) \rangle$$

$$= t^p \langle \mu, \phi \rangle = t^p \int_{\mathbb{R}^n} \phi(x)\ d\mu(x)$$

for every test function ϕ and $t > 0$. We approximate the function $\chi(A \times [0,1])$ by test functions and use the dominated convergence theorem to get that, for every open set $A \subset S^{n-1}$ and $t > 0$,

$$\mu(A \times [0,t]) = t^p \mu(A \times [0,1]).$$

Now, for every Borel subset $A \subset S^{n-1}$ and every $0 \le a < b$ we have

$$\mu(A \times (a,b]) = (b^p - a^p)\mu(A \times [0,1]).$$

Define a measure μ_0 on S^{n-1} by $\mu_0(A) = p\mu(A \times [0,1])$ for every Borel set $A \subset S^{n-1}$. Clearly,

$$\int_{S^{n-1}} \left(\int_{\mathbb{R}} |t|^{p-1} \chi_{A\times(a,b]}(t\theta)\ dt \right) d\mu_0(\theta) = \frac{b^p - a^p}{p} \mu_0(A) = \mu(\chi_{A\times(a,b]}).$$

Therefore, we get equality (2.44) with $\phi = \chi_{A\times(a,b]}$ and the result follows by approximation, since A, a, b are arbitrary. □

We often consider measures that are homogeneous distributions of degree $-n-p$, where $p > 0$. In this case the integral in (2.44) might diverge, so we consider only test functions ϕ supported outside of the origin.

Let D be an open set in $\mathbb{R}^n$, and let $K(D)$ be the space of test functions with compact support in D. We say that two distributions f, g are equal on D if $\langle f, \phi \rangle = \langle g, \phi \rangle$ for every $\phi \in K(D)$.

A distribution f is said to be positive (negative) in D if $\langle f, \phi \rangle \ge 0$ (≤ 0) for every non-negative test function $\phi \in K(D)$.

A sequence of non-negative test functions ω_k is called a *δ-sequence* if for each k the function ω_k is supported in the Euclidean ball with center at the origin and radius $1/k$ and if $\int_{\mathbb{R}^n} \omega_k = 1$. It is easy to see that for any continuous function h with compact support A in $\mathbb{R}^n$, the convolutions $h * \omega_k$ are test functions that converge to h in the uniform metric, as $k \to \infty$. For any open set D containing A, these convolutions are supported in D starting from some k. An example of a δ-sequence is

$$\omega_k(x) = c_k \exp\left(-\frac{1}{1 - k^2 |x|_2^2} \right) \quad \text{if} \quad |x|_2 < \frac{1}{k},$$

and $\omega_k(x) = 0$ if $|x|_2 \ge 1/k$, where c_k is a constant so that $\int_{\mathbb{R}^n} \omega_k = 1$.

LEMMA 2.25. *Let $0 < a < b < \infty$. Suppose that F is a positive distribution in the domain $(a,b) \times S^{n-1}$. Then for every interval $[c,d] \subset (a,b)$ there exists a finite Borel measure μ on $[c,d] \times S^{n-1}$ so that $F = \mu$ on $[c,d] \times S^{n-1}$.*

If $p > 0$ and $F \in \mathcal{S}(\mathbb{R}^n)$ is a positive outside of the origin even homogeneous distribution of degree $-n-p$, then there exists a finite Borel measure μ_0 on S^{n-1} so that for every even test function ϕ with compact support outside of the origin,

$$\langle F, \phi \rangle = \int_{S^{n-1}} \left(\int_0^\infty t^{-1-p} \phi(t\xi) dt \right) d\mu_0(\xi).$$

PROOF. Let $\varepsilon > 0$ be such that $[c-\varepsilon, d+\varepsilon] \subset (a,b)$. There exists a test function ψ supported in $(a,b) \times S^{n-1}$ so that $\psi = 1$ everywhere on $[c-\varepsilon, d+\varepsilon] \times S^{n-1}$.

First, we show that F is continuous on $K([c-\varepsilon, d+\varepsilon] \times S^{n-1})$ in the topology of uniform convergence. Consider any sequence of test functions ϕ_k supported in $[c-\varepsilon, d+\varepsilon] \times S^{n-1}$ so that $\phi_k \to 0$ uniformly. This means that for every $\delta > 0$, starting from some k, $-\delta\psi < \phi_k < \delta\psi$ on $[c-\varepsilon, d+\varepsilon] \times S^{n-1}$. Since F is a positive distribution, starting from the same k, we have

$$-\delta\langle F, \psi\rangle \le \langle F, \phi_k\rangle \le \delta\langle F, \psi\rangle,$$

so the sequence $\langle F, \phi_k\rangle$ converges to zero, which establishes our claim.

Now we prove that F can be extended to a continuous functional on the space $C([c,d] \times S^{n-1})$. Let h be any continuous function on $[c,d] \times S^{n-1}$. By the remark before this lemma, we can construct a sequence ϕ_k of test functions supported in $[c-\varepsilon, d+\varepsilon] \times S^{n-1}$ that converges to h uniformly. This sequence is a Cauchy sequence in the uniform metric, so for every $\delta > 0$ we have

$$-\delta\psi \le \phi_j - \phi_k \le \delta\psi$$

for large enough j and k. Since F is a positive distribution, we have

$$-\delta\langle F, \psi\rangle \le \langle F, \phi_j\rangle - \langle F, \phi_k\rangle \le \delta\langle F, \psi\rangle,$$

so $\langle F, \phi_k\rangle$ is a Cauchy sequence. This sequence converges to a number that we call $F_e(h)$. By continuity of F on test functions with the topology of uniform convergence, the number $F_e(h)$ does not depend on the choice of a sequence ϕ_k, and F_e is a continuous linear functional on $C([c,d] \times S^{n-1})$. By F. Riesz's characterization of continuous linear functionals on the spaces $C(K)$ (see [DS, p.262]) this functional is a finite Borel measure on $[c,d] \times S^{n-1}$. Clearly, the distribution F coincides with μ on test functions supported in $[c,d] \times S^{n-1}$.

The proof of the second statement is similar to that of Lemma 2.24. By the first part of Lemma 2.25, the distribution F is equal to some finite Borel measure μ on $[1,2] \times S^{n-1}$. By homogeneity, using

$$\langle \mu, \phi(x/t)\rangle = |t|^{-p}\langle \mu, \phi\rangle,$$

we can extend this measure to $[2,4] \times S^{n-1}$ and so on, so that we get a Borel measure on $\mathbb{R}^n \setminus \{0\}$ that coincides with F on test functions with compact support outside of the origin. Then, approximating by test functions, we get that for any $0 < a < b < \infty$, any $k > 0$ and any Borel subset A of the sphere,

$$\mu([ka, kb) \times A) = k^{-p}\mu([a,b) \times A),$$

so

$$\mu([1,\infty) \times A) = \sum_{j=1}^{\infty} \mu([2^{j-1}, 2^j) \times A) = \mu([1,2) \times A) \sum_{j=1}^{\infty} 2^{-(j-1)p} < \infty,$$

and also

$$\mu([a,b) \times A) = (a^{-p} - b^{-p})\mu([1,\infty) \times A).$$

We can now define a measure μ_0 on the sphere by

$$\mu_0(A) = p\mu([1,\infty) \times A).$$

We get

$$\mu([a,b) \times A) = \int_{S^{n-1}} \left(\int_0^{\infty} t^{-1-p} \chi_{[a,b)\times A}(t\theta)dt \right) d\mu_0(\theta).$$

The result follows, since a, b, A are arbitrary. □

We are going to use Lemmas 2.24 and 2.25 in the following form. Note that the constant $\Gamma(-p/2)$ in the second part does not play any role here and is needed only for future applications.

COROLLARY 2.26. *(i) Let K be an origin-symmetric star body in $\mathbb{R}^n$ and let $p \in (0,n)$. The function $\|\cdot\|_K^{-p}$ represents a positive definite distribution on $\mathbb{R}^n$ if and only if there exists a finite Borel measure μ_0 on S^{n-1} so that for every even test function ϕ,*

$$\int_{\mathbb{R}^n} \|x\|_K^{-p}\phi(x)\ dx = \int_{S^{n-1}} \left(\int_0^\infty t^{p-1}\hat{\phi}(t\xi)dt\right) d\mu_0(\xi). \tag{2.45}$$

(ii) Suppose that $p>0$ is not an even integer. The distribution $\Gamma(-p/2)(\|\cdot\|_K^p)^\wedge$ is positive on $\mathbb{R}^n\setminus\{0\}$ if and only if there exists a finite Borel measure μ_0 on S^{n-1} so that, for every even test function ϕ whose Fourier transform is supported in $\mathbb{R}^n\setminus\{0\}$,

$$\int_{\mathbb{R}^n} \|x\|_K^{p}\phi(x)dx = \frac{1}{\Gamma(-p/2)}\int_{S^{n-1}} \left(\int_0^\infty t^{-1-p}\hat{\phi}(t\xi)dt\right) d\mu_0(\xi). \tag{2.46}$$

PROOF. The "if" parts in both statements are straightforward. In fact, if we have (2.45), then for any non-negative even test function ϕ,

$$\langle(\|\cdot\|^{-p})^\wedge,\phi\rangle = \langle\|\cdot\|^{-p},\hat{\phi}\rangle = \int_{\mathbb{R}^n}\|x\|^{-p}\hat{\phi}(x)\ dx$$

$$= (2\pi)^n\int_{S^{n-1}}\left(\int_0^\infty t^{p-1}\phi(t\xi)dt\right)d\mu_0(\xi)\ge 0.$$

The second statement is similar (note that we start with a non-negative test function ϕ supported in $\mathbb{R}^n\setminus\{0\}$ and then apply (2.46) to the function $\hat{\phi}$ in place of ϕ).

Now we prove the necessity. For the first part, by Schwartz's theorem, the positive definite distribution $\|\cdot\|_K^{-p}$ is the Fourier transform of a tempered measure μ which, by Lemma 2.21, is an even homogeneous distribution of degree $-n+p$. By Lemma 2.24, there exists a finite Borel measure μ_0 on S^{n-1} so that for any even test function ϕ,

$$\int_{\mathbb{R}^n}\|x\|_K^{-p}\phi(x)dx = \langle\hat{\mu},\phi\rangle$$

$$= \langle\mu,\hat{\phi}\rangle = \int_{S^{n-1}}\left(\int_0^\infty t^{p-1}\hat{\phi}(t\xi)dt\right)d\mu_0(\xi).$$

For the second part, the function $\Gamma(-p/2)\|\cdot\|_K^p$ is the Fourier transform of the distribution

$$F = \frac{\Gamma(-p/2)}{(2\pi)^n}(\|\cdot\|_K^p)^\wedge,$$

which is positive in $\mathbb{R}^n\setminus\{0\}$ and, again by Lemma 2.21, even homogeneous of degree $-n-p$. For any even test function ϕ whose Fourier transform is supported outside of the origin, we have

$$\int_{\mathbb{R}^n}\|x\|_K^p\phi(x)dx = \frac{1}{\Gamma(-p/2)}\langle F,\hat{\phi}\rangle,$$

and the result follows from Lemma 2.25. $\square$

2.8. Stable random variables and the function γ_q

In this section we study a special function that will be used many times later. Denote by γ_q the Fourier transform of the function $z \mapsto \exp(-|z|^q),\ z \in \mathbb{R}$. The functions γ_q can be computed precisely only for $q = 2$, where

$$\gamma_2(t) = (e^{-z^2})^\wedge(t) = \sqrt{\pi}\ \exp(-t^2/4),$$

and for $q = 1$, where

$$\gamma_1(t) = \frac{2}{1+t^2}.$$

For other values of q, we have to study the properties of the functions γ_q indirectly. In particular, these functions were studied by Polya [Po2].

If $0 < q \leq 2$, then γ_q is a positive function which is, up to a constant 2π, equal to the density of the standard symmetric q-stable random variable; see this section below. We use Bernstein's theorem to prove the positivity of γ_q for these values of q.

A non-negative function f on $[0,\infty)$ is called *completely monotonic* if it is infinitely differentiable on $(0,\infty)$ and, for all $k \in \mathbb{N}$ and $x \in (0,\infty)$,

$$(-1)^k f^{(k)}(x) \geq 0.$$

The celebrated theorem of Bernstein (see [F, Ch. 18, Section 4]) states that every completely monotonic continuous at zero function is the Laplace transform of a finite measure on $[0,\infty)$, i.e. there exists a finite Borel measure μ on $[0,\infty)$ so that for every $x \geq 0$,

$$f(x) = \int_0^\infty e^{-tx}\ d\mu(t). \tag{2.47}$$

Lemma 2.27. *For $0 < q \leq 2$, the function γ_q is positive everywhere on $\mathbb{R}$.*

Proof. It is well known and easy to compute that

$$(e^{-tz^2})^\wedge(\xi) = \sqrt{\pi}\ t^{-1/2} e^{-\xi^2/4t}. \tag{2.48}$$

This proves the lemma for $q = 2$. We also use this equality to prove the lemma for other values of q.

Let $0 < q < 2$. It is easily seen that, for every $s \in (0,1)$, the function e^{-z^s} is completely monotonic. Put $s = q/2 < 1$. Then by Bernstein's theorem, there exists a finite measure $\mu_{q/2}$ on $[0,\infty)$ so that for every $z \in [0,\infty)$

$$e^{-z^{q/2}} = \int_0^\infty e^{-tz}\ d\mu_{q/2}(t). \tag{2.49}$$

Hence, for every $z \in \mathbb{R}$,

$$e^{-|z|^q} = \int_0^\infty e^{-tz^2}\ d\mu_{q/2}(t). \tag{2.50}$$

Now we use (2.48) to compute the Fourier transform in both sides of the latter equation with respect to the variable z. Note that the function in the left-hand side of (2.50) is integrable, so one can use the Fubini theorem. We get that for every $\xi \in \mathbb{R}$,

$$\gamma_q(\xi) = \sqrt{\pi} \int_0^\infty t^{-1/2} e^{-\xi^2/4t}\ d\mu_{q/2}(t) > 0. \tag{2.51}$$

□

If q is not an even integer, the function $\gamma_q(t)$ decreases at infinity like $|t|^{-q-1}$; see [PS, Chapter 4, Problem 154].

LEMMA 2.28. *For any* $q > 0$

$$\lim_{t\to\infty} t^{1+q}\gamma_q(t) = 2\Gamma(q+1)\sin(\pi q/2).$$

PROOF. For $x > 0$, integrating by parts, we get

$$x^{q+1}\gamma_q(x) = 2x^{q+1}\int_0^\infty e^{-t^q}\cos(tx)\,dt = 2x^{q+1}\int_0^\infty \frac{\sin(tx)}{x}e^{-t^q}qt^{q-1}\,dt.$$

Making the change of variables $z = x^q t^q$ and putting $\delta = x^{-q}$ (note that $\delta \to 0$, as $x \to \infty$), we transform the latter integral into

$$2\,\Im\left(\int_0^\infty e^{-z\delta}e^{iz^{1/q}}\,dz\right).$$

Now use analyticity to change the line of integration from $[0,\infty)$ to $z = re^{i\theta}$ for some small $\theta < \pi q/2$. We get

$$2\,\Im\left(e^{i\theta}\int_0^\infty e^{-r\delta\cos\theta}e^{-ir\delta\sin\theta}e^{ir^{1/q}\cos(\theta/q)}e^{-r^{1/q}\sin(\theta/q)}\,dr\right).$$

Since $\sin(\theta/q) > 0$, we can use the dominated convergence theorem to pass to the limit, as $x \to \infty$ (or $\delta \to 0$.) We get

$$\lim_{x\to\infty} x^{1+q}\gamma_q(x) = 2\,\Im\left(e^{i\theta}\int_0^\infty e^{ir^{1/q}e^{i\theta/q}}\,dr\right).$$

To compute the latter integral, again use analyticity to change the line of integration from $z = re^{i\theta}$ to $z = re^{i\pi q/2}$. We get that the limit is equal to (use (2.26) and (2.27))

$$2\,\Im\left(e^{i\pi q/2}\int_0^\infty e^{-r^{1/q}}\,dr\right) = 2\Gamma(q+1)\sin(\pi q/2).$$

□

The result also holds if q is an even integer, when the limit is equal to zero. This means that, for even q, the function γ_q decreases even faster than t^{-1-q} (in fact, the rate is exponential).

We end this section by computing the moments of certain random variables, and, as a consequence, the moments of the function γ_q. If (Ω, P) is a probability space, then *random variables* on Ω are measurable real-valued functions on Ω. If f is a random variable, its *probability distribution* is a Borel measure μ on $\mathbb{R}$ such that for any Borel set $A \subset \mathbb{R}$,

$$\mu(A) = P\{\omega \in \Omega :\ f(\omega) \in A\}.$$

The *expectation* of a random variable is defined by

$$\mathbb{E}f = \int_\Omega f(\omega)\,dP(\omega) = \int_{\mathbb{R}} x\,d\mu(x).$$

The *characteristic functional* ϕ of a random variable is the Fourier transform of its probability distribution: for every $t \in R$,

$$\phi(t) = \mathbb{E}(e^{-itf}) = \int_\Omega e^{-itf(\omega)}dP(\omega) = \int_{\mathbb{R}} e^{-itx}d\mu(x).$$

We say that a random variable g is *normalized Gaussian* if for $t \in \mathbb{R}$, $\mathbb{E}(e^{-itg}) = e^{-\frac{t^2}{2}}$, a random variable η is *symmetric normalized q-stable* where $0 < q < 2$ if $\mathbb{E}(e^{-it\eta}) = e^{-|t|^q}$, and a positive random variable α is *normalized positive q-stable* for $0 < q < 1$ if $\mathbb{E}(e^{-t\alpha}) = e^{-t^q}$ when $t > 0$. The existence of symmetric q-stable random variables follows from Lemma 2.27, and the function $\gamma_q/(2\pi)$ is the density of the probability distribution of a symmetric normalized q-stable random variable. The existence of positive q-stable random variables follows from Bernstein's theorem, as in formula (2.49).

If g is normalized Gaussian, then the density of its probability distribution is $(1/\sqrt{2\pi})e^{-t^2/2}$, and it follows from (2.26) that

$$\mathbb{E}(|g|^z) = \frac{1}{\sqrt{\pi}} 2^{\frac{z}{2}} \Gamma(\frac{z+1}{2}), \qquad \Re z > -1. \tag{2.52}$$

The same argument as in Lemma 2.28 shows that the probability distribution of a positive normalized q-stable random variable α has density that decreases at infinity like t^{-1-q}, so the moments of this random variable exist up to the order q. To compute these moments, assume first that $-1 < \Re z < 0$. Then, by the definition of the Γ-function, we have

$$\begin{aligned} \mathbb{E}(\alpha^z) &= \frac{1}{\Gamma(-z)} \int_0^\infty t^{-1-z} \mathbb{E}(e^{-t\alpha})\, dt \\ &= \frac{1}{\Gamma(-z)} \int_0^\infty t^{-1-z} e^{-t^q} dt = \frac{\Gamma(-z/q)}{q\Gamma(-z)}. \end{aligned} \tag{2.53}$$

Now we use analytic continuation to extend this formula to $-1 < \Re z < q$ (for $z = 0$, compute the limit).

Next, if η is normalized symmetric q-stable, then it has an identical distribution with $g\sqrt{2\alpha}$ where g, α are independent, g is normalized Gaussian and α is normalized positive $q/2$-stable. In fact, for every $t \in \mathbb{R}$,

$$\mathbb{E}(e^{-itg\sqrt{2\alpha}}) = \mathbb{E}(\exp(-t^2(\sqrt{2\alpha})^2/2)) = \mathbb{E}(e^{-t^2\alpha}) = e^{-|t|^q}. \tag{2.54}$$

By Lemma 2.28, the moments of η exist up to the order q. We can use (2.52) and (2.53) to compute these moments: by independence of g and α,

$$\mathbb{E}(|\eta|^z) = 2^{z/2}\mathbb{E}(g^z)\ \mathbb{E}(\alpha^{z/2}) = \frac{2^{z+1}\Gamma(\frac{-z}{q})\Gamma(\frac{z+1}{2})}{q\sqrt{\pi}\Gamma(-\frac{z}{2})}, \qquad -1 < \Re z < q. \tag{2.55}$$

In the special case of $q = 1$ formula (2.55) can be simplified either by using properties of the Γ-function or by a direct calculation, using Lemma 2.16 and the fact that the density of distribution of the symmetric 1-stable random variable ζ is $1/(\pi(1 + x^2))$:

$$\mathbb{E}(|\zeta|^z) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{|x|^z}{1 + x^2} dx = (\cos(\frac{z\pi}{2}))^{-1}, \qquad -1 < \Re z < 1. \tag{2.56}$$

The latter formula allows us to rewrite (2.55) in a different form in the case $0 < q < 1$. If $0 < q < 1$ and η is normalized symmetric q-stable, then η has the same distribution as $\alpha\zeta$ where α, ζ are independent, α is normalized positive q-stable and ζ is normalized symmetric 1-stable (this argument is similar to (2.54)). Hence we can rewrite (2.55) in the form

$$\mathbb{E}(|\eta|^z) = \mathbb{E}(\alpha^z)\mathbb{E}(|\zeta|^z) = (\cos(\frac{z\pi}{2}))^{-1}\frac{\Gamma(\frac{-z}{q})}{q\Gamma(-z)}, \qquad -1 < \Re z < q. \tag{2.57}$$

Let us go back to our function γ_q. It follows from Lemma 2.28 that, for $-1 < z < q$, the moments

$$s_q(z) = \int_{\mathbb{R}} |t|^z \gamma_q(t)\ dt$$

converge absolutely. These moments can be computed precisely for all $q > 0$.

LEMMA 2.29. *For any $q > 0$ and $-1 < z < q$, where z is not an even integer, we have*

$$s_q(z) = \frac{2^{z+2}\sqrt{\pi}\Gamma(-z/q)\Gamma((z+1)/2)}{q\Gamma(-z/2)}.$$

In particular, if $q > 2$, the moments $s_q(z)$ are positive for $z \in (-1, 0)$ and $z \in (0, 2)$, and they are negative for $z \in (2, \min(q, 4))$.

PROOF. First assume that $0 < q \leq 2$ and $-1 < z < q$. Then the function γ_q is equal to 2π times the density of the normalized symmetric q-stable variable. Hence,

$$s_q(z) = 2\pi\ \mathbb{E}(|\eta|^z),$$

where η is a normalized symmetric q-stable random variable, and the result follows from (2.55).

Now suppose that $z \in (0, 2)$ is fixed and consider $s_q(z)$ as a function of q. This function of q is analytic on $z < \Re q$, which follows from direct differentiation by q. By analytic continuation, the formula for $s_q(z)$ holds for any $q > 2$ and $z \in (0, 2)$. Now fix $q > 2$ and note that $s_q(z)$ is an analytic function of z in $-1 < \Re z < q$. Again by analytic continuation the formula holds for all $q > 2$ and $-1 < z < q$, where z is not an even integer.

The sign of the number $s_q(z)$ can easily be determined from the signs of the corresponding values of the Γ-function; see Section 2.4. □

We now give an alternative proof of Lemma 2.29, which is an almost direct application of Parseval's formula.

PROOF. Assume that $-1 < z < 0$. We apply Parseval's formula, (2.34) with a non-integrable function $|t|^z$. To justify Parseval's formula in this case, one can use the definition of the Γ-function and (2.26) to write

$$t^z = \frac{2^{z+1/2}}{\Gamma(-z/2)}\int_0^\infty r^{-1-z}e^{-r^2t^2/2}\ dr,$$

which reduces the formula to the case of L_1-functions; see Lemma 3.21 for a similar argument.

By Lemma 2.23 with the coefficient appearing in (2.40), before simplifications:

$$\begin{aligned} s_q(z) &= \int_{\mathbb{R}} |t|^z\gamma_q(t)\ dt = \int_{\mathbb{R}} (|\cdot|^z)^\wedge(s)e^{-|s|^q}\ ds \\ &= \frac{2^{z+1}\sqrt{\pi}\Gamma((z+1)/2)}{\Gamma(-z/2)}\int_{\mathbb{R}} |s|^{-1-z}e^{-|s|^q}\ ds \\ &= \frac{2^{z+2}\sqrt{\pi}\Gamma(-z/q)\Gamma((z+1)/2)}{q\Gamma(-z/2)}. \end{aligned}$$

Now use that $s_q(z)$ is an analytic function of z in the domain $\{z \in \mathbb{C},\ -1 < \Re z < q\}$ to extend the formula to other values of z. $\square$

The result of Lemma 2.29 also shows that, for $q > 2$, the function γ_q is sign-changing, because its moments can be both positive and negative.

CHAPTER 3

Volume and the Fourier Transform

The Fourier analytic approach to sections of convex bodies is based on certain formulas expressing volume in terms of the Fourier transform. In this chapter we present several formulas of this kind.

3.1. The first examples: hyperplane sections of ℓ_q-balls

The results of this section will lead us to a general formula relating the volume of hyperplane sections of an arbitrary star body to the Fourier transform. As happened historically, we start with a formula for the volume of hyperplane sections of the cube. We denote by

$$B_\infty^n = \{x \in \mathbb{R}^n : \ \|x\|_\infty = \max_{k=1,\dots,n} |x_k| \le 1\}$$

the cube with side 2 in $\mathbb{R}^n$, which is the unit ball of the space ℓ_∞^n.

THEOREM 3.1. *For any* $\xi \in S^{n-1}$ *and* $t \in \mathbb{R}$,

$$A_{B_\infty^n,\xi}(t) = \mathrm{Vol}_{n-1}(B_\infty^n \cap \{\xi^\perp + t\xi\}) = \frac{2^n}{\pi}\int_0^\infty \cos(tr)\prod_{k=1}^n \frac{\sin(r\xi_k)}{r\xi_k}\ dr. \tag{3.1}$$

We assume here that $\sin(r\xi_k)/r\xi_k = 1$ *if* $\xi_k = 0$.

PROOF. The case where ξ has only one non-zero coordinate is trivial, so we assume that ξ has at least two non-zero coordinates. Then the function under the integral in (3.1) is integrable on $\mathbb{R}$.

Let χ be the indicator function of the interval $[-1, 1]$. For every $x \in \mathbb{R}^n$,

$$\chi(\|x\|_\infty) = \prod_{k=1}^n \chi(|x_k|). \tag{3.2}$$

For any $r \in \mathbb{R}$, first using (2.1) and then the Fubini theorem and (3.2), we get

$$\widehat{A_\xi}(r) = \int_{\mathbb{R}} A_\xi(t) e^{-itr}\ dt = \int_{\mathbb{R}} e^{-itr}\left(\int_{(x,\xi)=t} \chi(\|x\|_\infty)\ dx\right)\ dt$$
$$= \int_{\mathbb{R}^n} \chi(\|x\|_\infty) e^{-ir(x,\xi)}\ dx = \prod_{k=1}^n \int_{-1}^1 e^{-irx_k\xi_k}\ dx_k = 2^n \prod_{k=1}^n \frac{\sin(r\xi_k)}{r\xi_k}.$$

Since the function A_ξ is even and $\widehat{A_\xi}$ is integrable on $\mathbb{R}$, we can invert the Fourier transform (see [Ru, Th 7.7]):

$$2\pi A_\xi(t) = (\widehat{A_\xi})^\wedge(t) = 2^n \int_{\mathbb{R}} e^{-itr}\prod_{k=1}^n \frac{\sin(r\xi_k)}{r\xi_k}\ dr.$$

$\square$

A similar formula for ℓ_q-balls with $1 \le q \le 2$ was found by Meyer and Pajor, [MeP]. This formula was generalized to all $0 < q < \infty$ in [K7]. Let

$$B_q^n = \{x \in \mathbb{R}^n : \ \|x\|_q = \left(\sum_{k=1}^n |x_k|^q\right)^{1/q} \le 1\}$$

be the unit ball of the space $\ell_q^n, \ 0 < q < \infty$.

THEOREM 3.2. *For every $\xi \in S^{n-1}$ and every $q > 0$,*

$$\mathrm{Vol}_{n-1}(B_q^n \cap \xi^\perp) = \frac{q}{\pi(n-1)\Gamma((n-1)/q)} \int_0^\infty \prod_{k=1}^n \gamma_q(t\xi_k) \ dt,$$

where γ_q is the Fourier transform of the function $\exp(-|\cdot|^q)$ *on* $\mathbb{R}$.

We shall prove this formula in the next section, as a part of a more general result (use the formula of Theorem 3.8 and apply Lemma 3.6).

At this point, we would like to show that the formulas of Theorems 3.1 and 3.2 have something in common. To do this, let us compute the Fourier transform of the powers of ℓ_q^n-norms, $0 < q \le \infty$. We start with the case $q = \infty$, and we need the following simple fact.

LEMMA 3.3. *If $0 < p < n$, then the function*

$$g(\xi) = \int_0^\infty t^{-p-1} \prod_{k=1}^n \left|\frac{\sin(t\xi_k)}{\xi_k}\right| \ dt$$

is locally integrable on $\mathbb{R}^n$.

PROOF. Since the function g is homogeneous of degree $-n+p \in (-n, 0)$, it is enough to show that g is integrable on the unit cube Q_n in $\mathbb{R}^n$. We have

$$\int_{Q_n} g(\xi) \ d\xi = \int_0^\infty t^{-p-1} \left(\prod_{k=1}^n \int_{-1}^1 \left|\frac{\sin t\xi_k}{\xi_k}\right| \ d\xi_k\right) \ dt$$

$$= \int_0^\infty t^{-p-1} \left(\int_{-t}^t \left|\frac{\sin u}{u}\right| \ du\right)^n \ dt.$$

To see that the latter integral converges, break the outer integral into two integrals – over $[0, 1]$ and over $[1, \infty)$. For the first interval, we use the estimate

$$\int_{-t}^t \left|\frac{\sin u}{u}\right| \ du \le 2t,$$

and then we note that $n - p - 1 > -1$, so the integral converges.

For the second interval,

$$\int_{-t}^t \left|\frac{\sin u}{u}\right| \ du \le \int_{-1}^1 \ du + 2\int_1^t \frac{du}{u} = 2 + 2\ln t.$$

The integral over $[1, \infty)$ converges because $-1 - p < -1$. □

The function g is locally integrable and homogeneous of degree $-n+p$. By the argument of Lemma 2.1, for every test function ϕ on $\mathbb{R}^n$, the function $g(\xi)|\phi(\xi)|$ is integrable on $\mathbb{R}^n$, which justifies the use of Fubini's theorem in the next result. We now compute the Fourier transform of powers of the ℓ_∞-norm.

LEMMA 3.4. *If $p \in (0,n)$, then the Fourier transform of the function $\|\cdot\|_\infty^{-p}$ is equal to the locally integrable on $\mathbb{R}^n$ function*

$$\xi \mapsto 2^n p \int_0^\infty t^{-p-1} \prod_{k=1}^n \frac{\sin(t\xi_k)}{\xi_k}\ dt. \tag{3.3}$$

PROOF. For every $x \in \mathbb{R}^n,\ x \neq 0$, making the change of variables $u = z\|x\|_\infty$, we get

$$\|x\|_\infty^{-p} = p \int_0^\infty z^{p-1}\chi(z\|x\|_\infty)\ dz,$$

where χ is the indicator of $[-1,1]$, and the integral converges because $p > 0$. Clearly,

$$\chi(z\|x\|_\infty) = \prod_{k=1}^n \chi(zx_k).$$

Therefore, for every fixed $z \neq 0$,

$$(\chi(z\|\cdot\|_\infty))^\wedge(\xi) = \prod_{k=1}^n \frac{2\sin(\xi_k/z)}{\xi_k}.$$

Since $0 < p < n$, for every test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ the integral

$$\langle (\|\cdot\|_\infty^{-p})^\wedge, \phi\rangle = p \int_{\mathbb{R}^n} \hat{\phi}(x) \left(\int_0^\infty z^{p-1}\chi(z\|x\|_\infty)\ dz \right) dx \tag{3.4}$$

converges absolutely, and we can use the Fubini theorem, the definition of the Fourier transform of distributions and the change of variables $t = 1/z$ to show that the expression in the right-hand side of (3.4) is equal to

$$\begin{aligned}
& p \int_0^\infty z^{p-1} \langle \chi(z\|x\|_\infty), \hat{\phi}(x)\rangle\ dz \\
&= p \int_0^\infty z^{p-1} \langle (\chi(z\|\cdot\|_\infty))^\wedge, \phi\rangle\ dz \\
&= p \int_0^\infty z^{p-1} \left(\int_{\mathbb{R}^n} \prod_{k=1}^n \frac{2\sin(\xi_k/z)}{\xi_k}\ \phi(\xi)\ d\xi \right)\ dz \\
&= 2^n p \int_{\mathbb{R}^n} \left(\int_0^\infty t^{-p-1} \prod_{k=1}^n \frac{\sin(t\xi_k)}{\xi_k}\ dt \right) \phi(\xi)\ d\xi \\
&= 2^n p \left\langle \int_0^\infty t^{-p-1} \prod_{k=1}^n \frac{\sin(t\xi_k)}{\xi_k}\ dt, \phi \right\rangle.
\end{aligned}$$

□

In the next two lemmas we compute the Fourier transform of powers of ℓ_q-norms, $0 < q < \infty$. The necessary properties of the function γ_q can be found in Section 2.8.

LEMMA 3.5. *Let $q > 0,\ n \in \mathbb{N},\ -n < p < 0$. Then the function*

$$h(\xi) = \int_0^\infty t^{n+p-1} \prod_{k=1}^n |\gamma_q(t\xi_k)|\ dt$$

is locally integrable on $\mathbb{R}^n$. If $0 < p < qn$, then the function h is integrable on compact sets outside of the coordinate planes in $\mathbb{R}^n$.

Proof. First, let $-n < p < 0$. The function h is homogeneous of degree $-n-p$ so it is enough to prove that h is integrable on the unit cube Q_n in $\mathbb{R}^n$. We have

$$\int_{Q_n} h(\xi)\ d\xi = \int_0^\infty t^{p-1} \left(\int_{-t}^{t} |\gamma_q(z)|\ dz \right)^n\ dt.$$

When $t \in [0,1]$, we can estimate the function under the outer integral by

$$2^n t^{n+p-1} (\max_{t\in\mathbb{R}} |\gamma_q(t)|)^n.$$

Since $n+p-1 > -1$, the latter function is integrable on $[0,1]$. When $t \in [1,\infty)$, the function under the outer integral is bounded from above by $t^{p-1}\|\gamma_q\|^n_{L_1(\mathbb{R})}$, which is integrable on $[1,\infty)$ because $p < 0$. Note that $\gamma_q \in L_1(\mathbb{R})$ because it decreases at infinity as (or faster than) $|t|^{-1-q}$; see Lemma 2.28.

Let $0 < p < nq$. Since h is homogeneous of degree $-n-p$, it is enough to prove that h is integrable in the cube $[1,2]^n$. The integral of h over this cube is equal to

$$\int_0^\infty t^{p-1} \left(\int_{t}^{2t} |\gamma_q(z)| dz \right)^n\ dt.$$

For $t \in [0,1]$ the function under the outer integral can be estimated from above by $t^{p-1}\|\gamma_q\|^n_{L_1(\mathbb{R})}$, which is integrable on $[0,1]$, since now $p > 0$. By Lemma 2.28, there exists a constant C so that for every $t > 1$, $|\gamma_q(t)| \le C|t|^{-1-q}$ on $[t,2t]$. This implies that the function under the outer integral is smaller than $C^n t^{p-1} t^{n(-1-q)} t^n = C^n t^{-1+p-qn}$, which is integrable on $[1,\infty)$ because $p < qn$. □

We now compute the Fourier transform of $\|\cdot\|_q^p$.

Lemma 3.6. *Let $q > 0$, $n \in \mathbb{N}$, $-n < p < 0$. Then the Fourier transform of the distribution $\|\cdot\|_q^p$ is equal to a locally integrable function*

$$(\|\cdot\|_q^p)^\wedge(\xi) = \frac{q}{\Gamma(-p/q)} \int_0^\infty t^{n+p-1} \prod_{k=1}^n \gamma_q(t\xi_k)\, dt. \tag{3.5}$$

If $0 < p < nq$ and $p/q \notin \mathbb{N} \cup \{0\}$, the Fourier transform of $\|\cdot\|_q^p$ is equal to the same function outside of the coordinate planes in $\mathbb{R}^n$.

Proof. First, suppose that $-n < p < 0$. Making the substitution $t = y\|x\|_q$ and using (2.26), we get

$$(|x_1|^q + \cdots + |x_n|^q)^{p/q} = \frac{q}{\Gamma(-p/q)} \int_0^\infty y^{-1-p} \exp(-y^q(|x_1|^q + \cdots + |x_n|^q))\, dy,$$

where the integral converges because $p < 0$.

For every fixed $y > 0$, the Fourier transform of the function

$$x \mapsto \exp(-y^q(|x_1|^q + \cdots + |x_n|^q))$$

at any point $\xi \in \mathbb{R}^n$ is equal to

$$y^{-n} \prod_{k=1}^n \gamma_q(\xi_k/y).$$

For any even test function ϕ, using Lemmas 3.5 and 2.1 to justify the application of Fubini's theorem and making the change of variables $t = 1/y$, we get

$$\langle (\|\cdot\|_q^p)^\wedge, \phi\rangle = \int_{\mathbb{R}^n} (|x_1|^q + \cdots + |x_n|^q)^{p/q} \hat{\phi}(x)\ dx$$

$$= \frac{q}{\Gamma(-p/q)} \int_0^\infty y^{-1-p} \Big\langle \exp(-y^q(|x_1|^q + \cdots + |x_n|^q)), \hat{\phi}(x) \Big\rangle dy$$

$$= \frac{q}{\Gamma(-p/q)} \int_0^\infty y^{-n-p-1} \left(\int_{\mathbb{R}^n} \prod_{k=1}^n \gamma_q(\xi_k/y)\phi(\xi)\ d\xi \right) dy$$

$$= \Big\langle \frac{q}{\Gamma(-p/q)} \int_0^\infty t^{n+p-1} \prod_{k=1}^n \gamma_q(t\xi_k)\, dt, \phi(\xi) \Big\rangle. \tag{3.6}$$

This proves the lemma for $-n < p < 0$.

By Lemma 2.22, the left-hand side of (3.6) is an analytic function of p in the domain $\Re p > -n$. Suppose that the test function ϕ in (3.6) is supported outside of the coordinate planes in $\mathbb{R}^n$. Then by Lemma 3.5 and an argument similar to that in Lemma 2.22 (recall that γ_q decreases at infinity like t^{-1-q}), the right-hand side of (3.6) is an analytic function of p in the domain $\Re p \in (-n, qn),\ p/q \notin \mathbb{N} \cup \{0\}$. By the uniqueness of analytic continuation, equality (3.6) holds for all p from the latter domain. Since ϕ is an arbitrary test function supported outside of the coordinate planes, we get equality (3.5) outside of the coordinate planes in $\mathbb{R}^n$. □

Comparing the formulas of Theorems 3.1 and 3.2 with the results of Lemmas 3.4 and 3.6, one can notice that the volume of every central hyperplane section coincides (up to the same constant!) with the Fourier transform of $\|\cdot\|_q^{-n+1}$ at the corresponding point. This suggests that both formulas must be particular cases of a more general result.

3.2. A general formula for the volume of hyperplane sections

In this section we prove that the connection between the volume of sections of ℓ_q-balls and the Fourier transform, established in the previous section, can be extended to arbitrary symmetric star bodies.

LEMMA 3.7. *Let f be an even homogeneous function of degree $-n+1$ on $\mathbb{R}^n$, continuous on the sphere S^{n-1}. Then the Fourier transform of f is an even homogeneous of degree -1, continuous on $\mathbb{R}^n \setminus \{0\}$ function such that, for every $\xi \in S^{n-1}$,*

$$Rf(\xi) = \int_{S^{n-1}\cap\xi^\perp} f(\theta)d\theta = \frac{1}{\pi}\hat{f}(\xi),$$

where the spherical Radon transform R is applied to the restriction of f to the sphere.

PROOF. Since f is even, it is enough to consider even test functions ϕ. We have

$$\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle = \int_{\mathbb{R}^n} f(x)\hat{\phi}(x)\ dx$$

$$= \int_{S^{n-1}} f(\theta) \left(\int_0^\infty \hat{\phi}(t\theta)\ dt \right) d\theta. \tag{3.7}$$

Using Lemma 2.11 and the fact that the test function ϕ is even, we get that, for every fixed $\theta \in S^{n-1}$, the Fourier transform of the function $t \mapsto \hat{\phi}(t\theta)$ is equal to the function

$$z \mapsto 2\pi \int_{(x,\theta)=z} \phi(x)dx,$$

so

$$2\int_0^\infty \hat\phi(t\theta)\ dt = \int_{-\infty}^\infty \hat\phi(t\theta)dt$$
$$= (\hat\phi(t\theta))^\wedge(0) = 2\pi \int_{(x,\theta)=0} \phi(x)dx.$$

Substituting the latter integral in (3.7) and writing it in polar coordinates of the hyperplane $(x,\theta)=0$, we get that the expression in (3.7) is equal to

$$\pi \int_{S^{n-1}} f(\theta) \left(\int_{S^{n-1}\cap\theta^\perp} \left(\int_0^\infty r^{n-2}\phi(r\xi)dr \right)\ d\xi \right)\ d\theta$$

$$= \pi \int_{S^{n-1}} f(\theta)\ R\left(\int_0^\infty r^{n-2}\phi(r\xi)\ dr \right)(\theta)\ d\theta. \tag{3.8}$$

We now use self-duality of the spherical Radon transform (Lemma 2.13) with the functions $f(\theta)$ and $g(\xi) = \int_0^\infty r^{n-2}\phi(r\xi)dr$. We get that the expression in (3.8) is equal to

$$\pi \int_{S^{n-1}} \left(\int_0^\infty r^{n-2}\phi(r\xi)dr \right) Rf(\xi)\ d\xi$$
$$= \pi \int_{S^{n-1}} \left(\int_0^\infty r^{n-2}\phi(r\xi)dr \right) \left(\int_{S^{n-1}\cap\xi^\perp} f(\theta)d\theta \right)\ d\xi$$
$$= \pi \int_{\mathbb{R}^n} |x|_2^{-1} \left(\int_{S^{n-1}\cap(x/|x|_2)^\perp} f(\theta)d\theta \right) \phi(x)\ dx,$$

where, as before, $|\cdot|_2$ stands for the Euclidean norm in $\mathbb{R}^n$, and the last equality can be verified by passing to the polar coordinates $x = r\xi$.

We see now that the distribution $\hat f$ is equal to the locally integrable continuous on $\mathbb{R}^n \setminus \{0\}$ function

$$\pi|x|_2^{-1} \int_{S^{n-1}\cap(x/|x|_2)^\perp} f(\theta)d\theta.$$

In particular, for every unit vector $x \in S^{n-1}$, we get

$$\hat f(x) = \pi Rf(x).$$

□

An immediate consequence is a formula relating the volume of sections to the Fourier transform.

Theorem 3.8. *Let K be an origin-symmetric star body in $\mathbb{R}^n$. The Fourier transform of the function $\|\cdot\|_K^{-n+1}$ is a homogeneous of degree -1 function on $\mathbb{R}^n$, continuous on $\mathbb{R}^n \setminus \{0\}$ and such that, for every $\xi \in S^{n-1}$,*

$$A_{K,\xi}(0) = \mathrm{Vol}_{n-1}(K\cap\xi^\perp) = \frac{1}{\pi(n-1)}(\|\cdot\|_K^{-n+1})^\wedge(\xi).$$

Proof. Put $f = \|\cdot\|_K^{-n+1}$ in Lemma 3.7. By Lemma 3.7 and formula (2.17), $(\|\cdot\|_K^{-n+1})^\wedge$ is a continuous function outside of the origin in $\mathbb{R}^n$ and, for every $\xi \in S^{n-1}$,

$$(\|\cdot\|_K^{-n+1})^\wedge(\xi) = \pi R(\|\cdot\|_K^{-n+1})(\xi) = \pi(n-1)\mathrm{Vol}_{n-1}(K\cap\xi^\perp).$$

□

Theorem 3.8 provides a simple proof of Minkowski's uniqueness theorem for sections:

COROLLARY 3.9. *If K, L are origin-symmetric star bodies in $\mathbb{R}^n$ and for every direction $\xi \in S^{n-1}$,*

$$\mathrm{Vol}_{n-1}(K \cap \xi^\perp) = \mathrm{Vol}_{n-1}(L \cap \xi^\perp),$$

then $K = L$.

PROOF. By Theorem 3.8, we get that the Fourier transforms of homogeneous of degree $-n+1$ distributions $\|\cdot\|_K^{-n+1}$ and $\|\cdot\|_L^{-n+1}$ are equal on the whole $\mathbb{R}^n$. This implies that $\|x\|_K = \|x\|_L$ for every $x \in \mathbb{R}^n$, so $K = L$. □

We now prove a more general uniqueness theorem for the spherical Radon transform in arbitrary dimensions.

COROLLARY 3.10. *Let $1 \le m < n$ and let $f, g \in C(S^{n-1})$ be two even functions so that for any m-dimensional subspace H of $\mathbb{R}^n$,*

$$\int_{S^{n-1}\cap H} f(\theta)d\theta = \int_{S^{n-1}\cap H} g(\theta)d\theta. \tag{3.9}$$

Then $f = g$.

PROOF. Fix any $\xi \in S^{n-1}$ and denote by $G_\xi(n-1, m)$ the Grassmanian of m-dimensional subspaces of $\xi^\perp$. By (2.22), integrating both sides of equation (3.9) over $G_\xi(n-1, m)$, we get

$$\int_{S^{n-1}\cap\xi^\perp} f(\theta)d\theta = \int_{S^{n-1}\cap\xi^\perp} g(\theta)d\theta.$$

Extend f, g to homogeneous functions of degree $-n+1$ on the whole $\mathbb{R}^n$. By Lemma 3.7, the Fourier transforms of the extensions $f(\theta)r^{-n+1}$ and $g(\theta)r^{-n+1}$ are equal. By the uniqueness theorem for the Fourier transform, $f = g$. □

3.3. The parallel section function and the Fourier transform

The formula of Theorem 3.8 can be generalized in several different ways. The main result of this section, Theorem 3.18, shows that this formula is a part of a more general relation between the parallel section function and the Fourier transform. This relation serves as an important part of the solution of the Busemann-Petty problem.

We start with some properties of the Fourier transform of homogeneous functions. If f is an even continuous function on S^{n-1} and $p \in \mathbb{R}$, we denote by

$$f(\theta)r^p = f(x/|x|_2)|x|_2^p$$

an even homogeneous of degree p function of the variable $x = r\theta \in \mathbb{R}^n,\ r \in \mathbb{R},\ \theta \in S^{n-1}$. When we write a function in the form $f(\theta)r^p$, we actually mean a function of the variable x, as in the formula above. If $p \ne -n-k$ for even non-negative integers k, then the action of the distribution $f(\theta)r^p$ on a test function ϕ can be written as

$$\begin{aligned}\langle f(\theta)r^p, \phi\rangle &= \langle f(x/|x|_2)|x|_2^p, \phi(x)\rangle \\ &= \frac{1}{2}\int_{S^{n-1}} f(\theta)\langle |z|^{n+p-1}, \phi(z\theta)\rangle d\theta.\end{aligned}$$

Moreover, if $p > -n$ or if ϕ is supported outside of the origin, then the right-hand side of the latter equation can be written as

$$\frac{1}{2}\int_{S^{n-1}} f(\theta)\left(\int_{\mathbb{R}} |z|^{n+p-1}\phi(z\theta)dz\right)d\theta.$$

We know from Lemma 2.21 that the Fourier transform of a homogeneous function of degree p is a homogeneous distribution of degree $-n-p$. We want to know when this Fourier transform is a continuous function on $\mathbb{R}^n \setminus \{0\}$. We also want to study the convergence of such Fourier transforms.

LEMMA 3.11. *(i) Suppose that $p > -n$ and let f_k, $k \in \mathbb{N}$, and f be even continuous functions on the sphere S^{n-1} so that $f_k \to f$ in $C(S^{n-1})$. Then for every even test function ϕ,*

$$\lim_{k\to\infty} \langle (f_k(\theta)r^p)^\wedge, \phi\rangle = \langle (f(\theta)r^p)^\wedge, \phi\rangle.$$

(ii) Suppose that $\delta > 0$, a sequence of numbers $\{p_k\}_{k=1}^\infty$ is such that $p_k > -n+\delta$ for all k, and $\lim_{k\to\infty} p_k = p$. Let $f \in C(S^{n-1})$. Then for every even test function ϕ,

$$\lim_{k\to\infty} \langle (f(\theta)r^{p_k})^\wedge, \phi\rangle = \langle (f(\theta)r^p)^\wedge, \phi\rangle.$$

PROOF. The functions f_k are uniformly bounded by some constant C. Since $p > -n$, by the same argument as in Lemma 2.1, the functions $f_k(\theta)r^p$ are locally integrable on $\mathbb{R}^n$, and the functions $f_k(\theta)r^p\hat{\phi}(x)$ are integrable on $\mathbb{R}^n$. These functions are majorated by an integrable function $Cr^p|\hat{\phi}(x)|$ and converge pointwise to $f(\theta)r^p\hat{\phi}(x)$. By the Lebesgue dominated convergence theorem,

$$\lim_{k\to\infty} \langle (f_k(\theta)r^p)^\wedge, \phi\rangle = \lim_{k\to\infty} \int_{\mathbb{R}^n} f_k(\theta)r^p\hat{\phi}(x)dx$$
$$= \int_{\mathbb{R}^n} f(\theta)r^p\hat{\phi}(x)dx = \langle (f(\theta)r^p)^\wedge, \phi\rangle,$$

and we are done with the first part.

For the second part, the functions $f(\theta)r^{p_k}$ are locally integrable, since each $p_k > -n$. Hence, the functions $f(\theta)r^{p_k}\hat{\phi}(x)$ are integrable on $\mathbb{R}^n$. There exists a constant $s > -n$ so that $p_k < s$ for each k. Now the functions r^{p_k} are majorated by the function $h(r) = \max(r^s, r^{-n+\delta})$. Therefore, the functions $f(\theta)r^{p_k}\hat{\phi}(x)$ are majorated by an integrable function $f(\theta)h(r)|\hat{\phi}(x)|$, and they converge pointwise to $f(\theta)r^p\hat{\phi}(x)$. We can finish the proof in the same way as in the first part. □

The following formula is well known:

LEMMA 3.12. *For every $q > -1$ and every $x \in S^{n-1}$, we have*

$$\int_{S^{n-1}} |(x,\xi)|^q \, d\xi = \frac{2\Gamma((q+1)/2)\pi^{(n-1)/2}}{\Gamma((n+q)/2)}.$$

PROOF. It is easily seen that for any integrable function Φ on $[-1,1]$ and any $x \in S^{n-1}$,

$$\int_{S^{n-1}} \Phi((x,\xi))\, d\xi = |S^{n-2}| \int_{-1}^{1} \Phi(t)(1-t^2)^{(n-3)/2}\, dt,$$

where $|S^{n-2}|$ is the surface area of the unit Euclidean ball in $\mathbb{R}^{n-1}$ (see for example [Gr, p. 9]), and the measure $d\xi$ on the sphere has density 1 and is not normalized.

Putting $\Phi(t) = |t|^q$ (note that $q > -1$ so the integral converges) and using formula (2.31), we get the result. □

LEMMA 3.13. *If $q > -1$ and b is a continuous function on the sphere S^{n-1}, then the function*

$$h(x) = \int_{S^{n-1}} |(x,\xi)|^q b(\xi)\ d\xi$$

is continuous on S^{n-1}. Also,

$$\|h\|_{C(S^{n-1})} \le \frac{2\Gamma((q+1)/2)\pi^{(n-1)/2}}{\Gamma((n+q)/2)} \|b\|_{C(S^{n-1})}.$$

PROOF. Let x_k be a sequence of points in the sphere converging to $x \in S^{n-1}$. Then one can find a sequence of rotations T_k such that $T_k(x) = x_k$, and

$$\|b(T_k) - b\|_{C(S^{n-1})} \to 0,$$

as $k \to \infty$ (b is uniformly continuous on the compact set S^{n-1}). Then,

$$h(x_k) - h(x) = \int_{S^{n-1}} |(x,\xi)|^q (b(T_k\xi) - b(\xi))\ d\xi,$$

and the result follows from Lemma 3.12. □

Our next lemma is a combination of Lemmas 2.23 and 2.11. The general idea behind this lemma is quite simple. If $f \in L_1(\mathbb{R})$ and $\xi \in S^{n-1}$, consider a measure μ in $\mathbb{R}^n$ supported in the line $t\xi,\ t \in \mathbb{R}$, with density f. The Fourier transform of the measure μ is the function

$$x \mapsto \hat{f}((x,\xi)),\ x \in \mathbb{R}^n.$$

In fact,

$$\hat{\mu}(x) = \int_{\mathbb{R}^n} e^{-i(x,\eta)}\ d\mu(\eta) = \int_{\mathbb{R}} f(t) e^{-it(x,\xi)}\ dt = \hat{f}((x,\xi)). \tag{3.10}$$

Therefore for any test function ϕ,

$$\int_{\mathbb{R}^n} \hat{f}((x,\xi))\phi(x)\ dx = \langle \hat{\mu}, \phi \rangle = \langle \mu, \hat{\phi} \rangle = \int_{\mathbb{R}} f(t)\hat{\phi}(t\xi)\ dt.$$

We would like to have a similar formula in the case where f is not necessarily an L_1-function.

LEMMA 3.14. *Let $q > -1$, where q is not an even integer. Then for every even test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ supported outside of the origin and every fixed vector $\xi \in \mathbb{R}^n$, $\xi \ne 0$, we have*

$$\int_{\mathbb{R}^n} |(x,\xi)|^q \hat{\phi}(x)\ dx = -2(2\pi)^{n-1}\Gamma(q+1)\sin(\pi q/2) \int_{\mathbb{R}} |t|^{-1-q}\phi(t\xi)\ dt.$$

PROOF. Because of homogeneity, it is enough to prove the lemma for $\xi \in S^{n-1}$. Since ϕ is an even function, we have $(\hat{\phi})^\wedge = (2\pi)^n\phi$. By Lemma 2.11, the function $t \mapsto (2\pi)^n\phi(t\xi)$ is the Fourier transform of the function $z \mapsto \int_{(x,\xi)=z} \hat{\phi}(x)\, dx$. Using the Fubini theorem and Lemma 2.23, we get

$$\int_{\mathbb{R}^n} |(x,\xi)|^q\ \hat{\phi}(x)\, dx = \int_{\mathbb{R}} |z|^q \left(\int_{(x,\xi)=z} \hat{\phi}(x)\, dx \right) dz$$

$$= \left\langle |z|^q, \int_{(x,\xi)=z} \hat{\phi}(x)\, dx \right\rangle$$
$$= \frac{1}{2\pi}\left\langle (-2\Gamma(1+q)\sin(\pi q/2))|t|^{-1-q}, (2\pi)^n \phi(t\xi) \right\rangle$$
$$= (2\pi)^{n-1}(-2\Gamma(1+q)\sin(\pi q/2)) \int_{\mathbb{R}} |t|^{-1-q}\phi(t\xi)\ dt.$$

$\square$

COROLLARY 3.15. *Suppose that $q > -1$ is not an even integer and b is an even continuous function on the sphere S^{n-1}. Define a function F on $\mathbb{R}^n$ by*

$$F(x) = \int_{S^{n-1}} |(x,\xi)|^q\ b(\xi)\ d\xi.$$

Then the Fourier transform $\widehat{F}$ is a homogeneous of degree $-n-q$ continuous on $\mathbb{R}^n \setminus \{0\}$ function such that, for every $y \in \mathbb{R}^n \setminus \{0\}$,

$$\widehat{F}(y) = -4(2\pi)^{n-1}\Gamma(q+1)\sin(\pi q/2)\ b(\xi) r^{-n-q},$$

where $r = |y|_2$ and $\xi = y/|y|_2$.

PROOF. For any even test function ϕ supported outside of the origin, using Lemma 3.14, we get

$$\langle \widehat{F}, \phi\rangle = \langle F, \hat{\phi}\rangle = \int_{S^{n-1}} b(\xi) \left(\int_{\mathbb{R}^n} |(x,\xi)|^q \hat{\phi}(x)\ dx \right) d\xi$$
$$= -2(2\pi)^{n-1}\Gamma(q+1)\sin(\pi q/2) \int_{S^{n-1}} b(\xi) \left(\int_{\mathbb{R}} |t|^{-1-q}\phi(t\xi)\ dt \right) d\xi$$
$$= -4(2\pi)^{n-1}\Gamma(q+1)\sin(\pi q/2)\ \langle b(\xi) r^{-n-q}, \phi\rangle.$$

$\square$

The next lemma shows that the Fourier transform of a smooth homogeneous function is also smooth. It is possible to prove a stronger result with exact degrees of smoothness of a function and its Fourier transform, but this would make the text more complicated, and, on the other hand, we do not really need it here.

LEMMA 3.16. *Let $k \in \mathbb{N} \cup \{0\}$ and let $f \in C^{2k}(S^{n-1})$ be an even function. Suppose that $q \le 2k$, where q is not an odd integer. Then the following hold.*
(i) The Fourier transform of the distribution $f(\theta)r^{-n+q+1}$ is a homogeneous of degree $-1-q$, continuous on $\mathbb{R}^n \setminus \{0\}$ function. If $q < 2k$, then for every $x \in \mathbb{R}^n$,

$$|x|_2^{2k}(f(\theta)r^{-n+q+1})^\wedge(x)$$

(3.11)
$$= \frac{(-1)^k \pi}{-2\Gamma(2k-q)\sin(\pi(2k-q-1)/2)} \int_{S^{n-1}} |(x,\xi)|^{2k-q-1} \Delta^k (f(\theta) r^{-n+q+1})(\xi)\ d\xi.$$

If $q = 2k$, then

$$|x|_2^{2k}\ (f(\theta)r^{-n+q+1})^\wedge(x)$$
$$= (-1)^k \pi |x|_2^{-1} \int_{S^{n-1}\cap(x/|x|_2)^\perp} \Delta^k (f(\theta) r^{-n+q+1})(\xi)\ d\xi. \tag{3.12}$$

Here $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ is the Laplace operator on $\mathbb{R}^n$.

(ii) If $f \in C^\infty(S^{n-1})$, then there exists an even function $g \in C^\infty(S^{n-1})$ so that for every $x = t\xi \in \mathbb{R}^n,\ t \neq 0,\ \xi \in S^{n-1}$,

$$(f(\theta)r^{-n+q+1})^\wedge(x) = g(\xi)t^{-1-q},$$

so the Fourier transform of $f(\theta)r^{-n+q+1}$ is an infinitely smooth function on $\mathbb{R}^n\setminus\{0\}$.

Proof. For any even test function ϕ such that $\hat{\phi}$ is supported outside of the origin, by the connection between differentiation and the Fourier transform, formula (2.36),

$$\langle |x|_2^{2k}(f(\theta)r^{-n+q+1})^\wedge(x), \phi(x)\rangle = (-1)^k\langle \Delta^k(f(\theta)r^{-n+q+1}), \hat{\phi}\rangle.$$

On the other hand, if $q < 2k$, where q is not an odd integer, then, by Corollary 3.15,

$$\Big\langle \int_{S^{n-1}} |(x,\xi)|^{2k-q-1}\Delta^k(f(\theta)r^{-n+q+1})(\xi)\ d\xi, \phi(x)\Big\rangle$$

$$= \frac{1}{(2\pi)^n}\Big\langle \Big(\int_{S^{n-1}} |(x,\xi)|^{2k-q-1}\Delta^k(f(\theta)r^{-n+q+1})(\xi)\ d\xi\Big)^\wedge(u), \hat{\phi}(u)\Big\rangle$$

$$= \frac{-4(2\pi)^{n-1}\Gamma(2k-q)\sin(\pi(2k-q-1)/2)}{(2\pi)^n}\langle \Delta^k(f(\theta)r^{-n+q+1}), \hat{\phi}\rangle,$$

because $\Delta^k(f(\theta)r^{-n+q+1})$ is a homogeneous function of degree $-n-2k+q+1$, which is continuous on $\mathbb{R}^n \setminus \{0\}$. This shows that the functions in both sides of (3.11), considered as distributions, coincide on test functions ϕ whose Fourier transform is supported outside of the origin. Such distributions can differ only by a polynomial. But this polynomial must be zero because both functions are even homogeneous of degree $2k-q-1$, which is not an even integer.

If $q = 2k$, the function $\Delta^k(f(\theta)r^{-n+q+1})$ is homogeneous of degree $-n+1$. Formula (3.12) follows from Lemma 3.7.

Finally, if f is infinitely smooth, then we can choose k in (3.11) as large as we want. The function of x in the right-hand side of (3.11) is $[2k-q-3]$ times differentiable by the variable x. To see this, first note that for $p > 1$

$$\frac{\partial}{\partial x_i}|(x,\xi)|^p = p\xi_i|(x,\xi)|^{p-1}\mathrm{sgn}((x,\xi)),$$

$$\frac{\partial}{\partial x_i}\left(|(x,\xi)|^p\mathrm{sgn}((x,\xi))\right) = p\xi_i|(x,\xi)|^{p-1},$$

and then use a standard justification of differentiation under the integral (the best way to do it is backwards, integrating the derivative and using the Fubini theorem; see [K19] for similar calculations). □

An immediate consequence of Lemma 3.16 and Lemma 3.13 is the following.

Corollary 3.17. *Suppose that $q \le 2k,\ k \in \mathbb{N}\cup\{0\}$.*
(i) If $f \in C^{2k}(S^{n-1})$, then the Fourier transform $(f(\theta)r^{-n+q+1})^\wedge$ is a homogeneous of degree $-1-q$ continuous on $\mathbb{R}^n \setminus \{0\}$ function.
(ii) If $f_m, f \in C^{2k}(S^{n-1})$ and f_m converges to f in $C^{2k}(S^{n-1})$, then the Fourier transforms $(f_m(\theta)r^{-n+q+1})^\wedge$ converge to $(f(\theta)r^{-n+q+1})^\wedge$ in the space $C(S^{n-1})$.

Let D be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$. By Lemma 2.4, for any $m \in \mathbb{N} \cup \{0\}$, there exists an interval $U_m = (-\delta_m, \delta_m) \subset \mathbb{R}$ so that the parallel section functions $A_{D,\xi}$ are uniformly differentiable up to the order m in this interval, and, for any $t \in U_m$ and $0 \le k \le m$, the function $\xi \mapsto A_{D,\xi}^{(k)}(t)$ is continuous on S^{n-1}.

Applying the definition of fractional derivatives (2.41) to the functions $A_{D,\xi}$ and changing the interval $(0,1)$ to $(0,\delta_m)$, we get that for $\Re q < m,\ q \neq 0, 1, ..., m-1$,

$$A_\xi^{(q)}(0) = \frac{1}{\Gamma(-q)} \int_0^{\delta_m} t^{-1-q} \left(A_\xi(t) - A_\xi(0) - ... - A_\xi^{(m-1)}(0) \frac{t^{m-1}}{(m-1)!} \right) \, dt$$

$$+ \frac{1}{\Gamma(-q)} \int_{\delta_m}^\infty t^{-1-q} A_\xi(t) \, dt + \frac{1}{\Gamma(-q)} \sum_{k=0}^{m-1} \frac{\delta_m^{k-q} A_\xi^{(k)}(0)}{k!(k-q)}.$$

Using a formula for the remainder of the Taylor series and an argument similar to that from Lemma 2.22, we see that the fractional derivative $A_{D,\xi}^{(q)}(0)$ is an analytic function of q everywhere in $\mathbb{C}$ (we have to extend the definition to integers, as usual). Moreover, $A_{D,\xi}^{(q)}(0)$ is a continuous function of ξ on the sphere. Extending this function to a homogeneous function of ξ of degree $-1-q$, one can see by the argument of Lemma 2.22 that, for any test function ϕ on $\mathbb{R}^n$, the function

$$q \mapsto \langle A_{D,\xi}^{(q)}(0), \phi(\xi) \rangle$$

is analytic in the domain $\{q \in \mathbb{C} : -1 < \Re q\}$.

We are now ready to prove the main result of this section expressing the derivatives of the parallel section function $A_{D,\xi}$ in terms of the Fourier transform of powers of the Minkowski functional. In the next theorem, by Corollary 3.17, the Fourier transform of $\|\cdot\|_D^{-n+q+1}$ is a homogeneous of degree $-1-q$ continuous function on $\mathbb{R}^n \setminus \{0\}$, which allows us to write all the formulas pointwise on the sphere.

THEOREM 3.18. *Let D be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, $\xi \in S^{n-1}$. Then, for every $q \in (-1, \infty)$, $q \neq n-1$, the fractional derivative of the order q of the parallel section function at zero can be expressed in the form*

$$A_{D,\xi}^{(q)}(0) = \frac{\cos(\pi q/2)}{\pi(n-q-1)} (\|\cdot\|_D^{-n+q+1})^\wedge(\xi). \tag{3.13}$$

In particular, if $k \ge 0$, $k \neq n-1$, is an even integer, then we get the usual derivative of the order k :

$$(\|\cdot\|_D^{-n+k+1})^\wedge(\xi) = (-1)^{k/2} \pi (n-k-1) A_{D,\xi}^{(k)}(0). \tag{3.14}$$

If $k \ge 1$, $k \neq n-1$, is an odd integer, then

$$(\|\cdot\|_D^{-n+k+1})^\wedge(\xi) = (-1)^{(k+1)/2} 2(n-k-1)k!$$

$$\times \int_0^\infty \frac{A_{D,\xi}(z) - A_{D,\xi}(0) - ... - A_{D,\xi}^{(k-1)}(0) \frac{z^{k-1}}{(k-1)!}}{z^{k+1}} \, dz. \tag{3.15}$$

PROOF. Let us start with $q \in (-1, 0)$. Using (2.41), (2.1) and the Fubini theorem, we get (as before, χ is the indicator of $[-1,1]$)

$$A_{D,\xi}^{(q)}(0) = \frac{1}{2\Gamma(-q)} \int_{-\infty}^\infty |z|^{-q-1} A_{D,\xi}(z) \, dz$$

$$= \frac{1}{2\Gamma(-q)} \int_{-\infty}^{\infty} |z|^{-q-1} \left(\int_{(x,\xi)=z} \chi(\|x\|_D)\, dx \right) dz$$

$$= \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}^n} |(x,\xi)|^{-q-1} \chi(\|x\|_D)\, dx$$

$$= \frac{1}{2\Gamma(-q)} \int_{S^{n-1}} |(\theta,\xi)|^{-q-1} \left(\int_0^{\infty} r^{n-q-2} \chi(r\|\theta\|_D)\, dr \right) d\theta$$

$$= \frac{1}{2(n-q-1)\Gamma(-q)} \int_{S^{n-1}} |(\theta,\xi)|^{-q-1} \|\theta\|_D^{-n+q+1}\, d\theta.$$

We now consider $A^{(q)}_{D,\xi}(0)$ as a function of $\xi \in \mathbb{R}^n \setminus \{0\}$ so that it is homogeneous of degree $-1-q$. For every even test function ϕ, using Lemma 3.14, we get

$$\langle A^{(q)}_{D,\xi}(0), \phi(\xi) \rangle = \frac{1}{2\Gamma(-q)(n-q-1)}$$

$$\times \int_{S^{n-1}} \|\theta\|_D^{-n+q+1} \left(\int_{\mathbb{R}^n} |(\theta,\xi)|^{-q-1} \phi(\xi)\, d\xi \right) d\theta$$

$$= \frac{-1}{4(n-q-1)\Gamma(-q)\Gamma(q+1)\sin\frac{q\pi}{2}}$$

$$\times \int_{S^{n-1}} \|\theta\|_D^{-n+q+1} \left(\int_{\mathbb{R}} |t|^q\ \widehat{\phi}(t\theta)\, dt \right) d\theta$$

$$= \frac{\cos(q\pi/2)}{\pi(n-q-1)} \left\langle (\|\cdot\|_D^{-n+q+1})^\wedge(\xi), \phi(\xi) \right\rangle, \tag{3.16}$$

where the last equation follows from (2.32) and the simple calculation

$$\langle \left(\|\cdot\|_D^{-n+q+1} \right)^\wedge, \phi \rangle = \int_{\mathbb{R}^n} \|x\|_D^{-n+q+1} \hat{\phi}(x)\ dx$$

$$= \int_{S^{n-1}} \|\theta\|_D^{-n+q+1} \left(\int_0^{\infty} r^q\ \hat{\phi}(r\theta)\ dr \right)\ d\theta$$

$$= \frac{1}{2} \int_{S^{n-1}} \|\theta\|_D^{-n+q+1} \left(\int_{\mathbb{R}} |r|^q\ \hat{\phi}(r\theta)\ dr \right)\ d\theta.$$

By the remark before Theorem 3.18 and by Lemma 2.22, both sides of (3.16) are analytic functions of q in the domain $-1 < \Re q,\ q \neq n-1$. By analytic extension, this proves (3.16) for all required values of q. Also, by the same remark before the theorem, $A^{(q)}_{D,\xi}(0)$ is a continuous function of ξ on the sphere, and by Corollary 3.17 so is the Fourier transform of $\|\cdot\|_D^{-n+q+1}$, so we get equality (3.13) as the equality of functions on the sphere. If k is even, we immediately get (3.14).

However, if k is odd, both sides of (3.13) vanish (recall that D is origin-symmetric, so $A_{D,\xi}$ is an even function). Since we have already proved (3.16) for $k-1 < q < k$, we divide both sides of (3.16) by $\cos(\pi q/2)$ and compute the limit as $q \to k$. Here we use expression (2.42) for the fractional derivative. The limit in the left-hand side of (3.16) is equal to the action on the test function $\phi(\xi)$ of the following function of ξ :

$$\lim_{q\to k} \frac{1}{\Gamma(-q)\cos(\pi q/2)}$$

$$\times \int_0^\infty \frac{A_{D,\xi}(z) - A_{D,\xi}(0) - A''_{D,\xi}(0)\frac{z^2}{2} - \cdots - A^{(k-1)}_{D,\xi}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}}\, dz.$$

Now,

$$\begin{aligned} &\lim_{q\to k} \Gamma(-q)\cos(\pi q/2) \\ &= \lim_{q\to k} \Gamma(-q)\sin\left(\frac{(q+1)\pi}{2} - \frac{(k+1)\pi}{2}\right)(-1)^{(k+1)/2} \\ &= \lim_{q\to k} \frac{\Gamma(-q+k+1)}{(-q)(1-q)\cdots(k-q)}\sin((q-k)\pi/2)(-1)^{(k+1)/2} \\ &= -\frac{\pi}{2}(-1)^{(k+1)/2}(-1)^k\frac{1}{k!} = \frac{\pi}{2}(-1)^{(k+1)/2}\frac{1}{k!}. \end{aligned}$$

On the other hand, by Lemma 3.11,

$$\lim_{q\to k} \langle (\|\cdot\|_D^{-n+q+1})^\wedge(\xi), \phi(\xi)\rangle = \langle (\|\cdot\|_D^{-n+k+1})^\wedge(\xi), \phi(\xi)\rangle,$$

and the result in the case of odd integers follows. □

3.4. Parseval's formula on the sphere

In this section we prove a spherical version of Parseval's formula. Let us start with an informal reasoning that gives an idea of how this can be done by projecting the classical formula to the sphere.

Suppose that K and L are star bodies in $\mathbb{R}^n$, $0 < p < n$, and f is an even function on $\mathbb{R}$. Consider an integral

$$\int_{\mathbb{R}^n} \|x\|_K^{-p} f(\|x\|_L)\, dx. \tag{3.17}$$

Writing this integral in polar coordinates and making the change of variables $t = r\|\theta\|_L$, we see that (3.17) is equal to

$$\int_{S^{n-1}} \|\theta\|_K^{-p}\left(\int_0^\infty r^{n-p-1} f(r\|\theta\|_L)\, dr\right)\, d\theta$$

$$= \left(\int_0^\infty t^{n-p-1} f(t)\, dt\right)\left(\int_{S^{n-1}} \|\theta\|_K^{-p}\|\theta\|_L^{-n+p}\, d\theta\right). \tag{3.18}$$

On the other hand, applying the classical Parseval formula (informally), we get that the integral (3.17) is also equal to

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (\|\cdot\|_K^{-p})^\wedge(\xi)(f(\|\cdot\|_L))^\wedge(\xi)\, d\xi.$$

Now recall that the Fourier transform of a homogeneous function of degree $-p$ is a homogeneous function of degree $-n+p$ and write the latter integral in polar coordinates $\xi = r\theta$. We get that (3.17) equals

$$\frac{1}{(2\pi)^n}\int_{S^{n-1}} (\|\cdot\|_K^{-p})^\wedge(\theta)\left(\int_0^\infty r^{p-1}(f(\|\cdot\|_L))^\wedge(r\theta)\, dr\right) d\theta. \tag{3.19}$$

The crucial observation is that

$$\int_0^\infty r^{p-1}(f(\|\cdot\|_L))^\wedge(r\theta)\, dr = c_p(f)(\|\cdot\|_L^{-n+p})^\wedge(\theta), \tag{3.20}$$

where $c_p(f)$ is a constant not depending on θ. This means that directional moments of the Fourier transform of functions of the form $f(\|\cdot\|_L)$ do not depend on the choice of f (up to a constant independent of the direction). Combining (3.20) with (3.19) and (3.18), we get a spherical version of Parseval's formula, which is the main goal of this section:

$$\int_{S^{n-1}} (\|\cdot\|_K^{-p})^\wedge(\theta)(\|\cdot\|_L^{-n+p})^\wedge(\theta)\ d\theta = d_p(f) \int_{S^{n-1}} \|\theta\|_K^{-p}\|\theta\|_L^{-n+p}\ d\theta. \tag{3.21}$$

To make this reasoning formally correct, we have to do several things. First, the Fourier transforms of $\|\cdot\|_K^p$ and $\|\cdot\|_L^{-n+p}$ must be continuous functions on the sphere. This can be achieved by assuming that the bodies K and L are sufficiently smooth. Secondly, the integrals involving the function f must converge. To ensure this, we need to choose a specific function f so that the derivatives of the function $f(\|\cdot\|_L)$ of high enough orders are integrable on $\mathbb{R}^n$. Then, using the connection between differentiation and the Fourier transform, we get that the Fourier transform of $f(\|\cdot\|_L)$ is quickly decreasing at infinity. For example, if

$$\Delta^{(n+2)/2}(f(\|\cdot\|_L)) \in L_1(\mathbb{R}^n), \tag{3.22}$$

then, by (2.36),

$$|(f(\|\cdot\|_L))^\wedge(x)| \le M|x|_2^{-n-2},$$

where M is the L_1-norm of the function (3.22). We are also going to show that, as one can expect, the constant $d_p(f) = (2\pi)^n$, and, thus, it does not depend on f or p.

Before we proceed with choosing a specific even function f, let us also informally deduce (3.20). Writing the Fourier integral in polar coordinates $\xi = r\eta$ and making a change of variables $t = r\|\eta\|_L$, we get

$$\begin{aligned}(f(\|\cdot\|_L))^\wedge(\xi) &= \int_{\mathbb{R}^n} f(\|x\|_L)e^{-i(x,\xi)}\ dx \\ &= \int_{S^{n-1}} \left(\int_0^\infty r^{n-1} f(r\|\eta\|_L)e^{-ir(\eta,\xi)}dr\right)\ d\eta \\ &= \frac{1}{2}\int_{S^{n-1}} \|\eta\|_L^{-n}(|t|^{n-1}f(t))^\wedge((\eta,\xi)/\|\eta\|_L)\ d\eta.\end{aligned}$$

Putting $\xi = r\theta$ and integrating with respect to $r^{p-1}dr$, we see that the left-hand side of (3.20) equals

$$\left(\int_0^\infty r^{p-1}(|t|^{n-1}f(t))^\wedge(r)\ dr\right)\left(\int_{S^{n-1}} \|\eta\|_L^{-n+p}|(\eta,\theta)|^{-p}\ d\eta\right).$$

On the other hand (and also informally),

$$\begin{aligned}(\|\cdot\|_L^{-n+p})^\wedge(\theta) &= \int_{\mathbb{R}^n} \|x\|_L^{-n+p}e^{-i(x,\theta)}dx \\ &= \int_{S^{n-1}} \|\eta\|_L^{-n+p}\left(\int_0^\infty r^{p-1}e^{-ir(\eta,\theta)}dr\right)\ d\eta \\ &= c_{p-1}\int_{S^{n-1}} \|\eta\|_L^{-n+p}|(\eta,\theta)|^{-p}\ d\eta,\end{aligned}$$

which expresses the right-hand side of (3.20) (up to a constant) in exactly the same way as the left-hand side. Of course the integrals in the latter argument do not converge, which will be corrected below by using test functions.

Let us make this reasoning work. We put $f(t) = e^{-t^4}$. For an origin-symmetric star body K in $\mathbb{R}^n$, define a function

$$\mu_K(\xi) = (\exp(-\|\cdot\|_K^4))^\wedge(\xi), \qquad \xi \in \mathbb{R}^n.$$

First, we show that our choice of the function f guarantees integrability of μ_K for any infinitely smooth body K.

LEMMA 3.19. *Let K be an origin-symmetric infinitely smooth star body. Then the function μ_K is continuous and integrable on $\mathbb{R}^n$.*

PROOF. Let n be an even integer. Consider a function

$$F(x) = \Delta^{(n+2)/2} \exp(-\|x\|_K^4) = h(x) \exp(-\|x\|_K^4), \qquad x \in \mathbb{R}^n,$$

where Δ is the Laplace operator. The function h that appears in front of the exponential after differentiation is the sum of homogeneous functions of degrees $-n+2$ and higher, which are continuous on the sphere S^{n-1}, because K is infinitely smooth (after each differentiation the degree of homogeneity decreases by 1; this explains the choice of the power 4 in the formula for the function f). Clearly, h is locally integrable, and, therefore, F is integrable on $\mathbb{R}^n$. Using the connection between the Fourier transform and differentiation (2.36), we get

$$\hat{F}(x) = (-1)^{(n+2)/2} |x|_2^{n+2} \mu_K(x).$$

The Fourier transform of an integrable function F is a bounded function on $\mathbb{R}^n$. Thus, there exists a constant C so that $|\mu_K(x)| \le C|x|_2^{-n-2}$ for every $x \in \mathbb{R}^n$. The function $|\cdot|_2^{-n-2}$ is integrable outside of the unit ball in $\mathbb{R}^n$. Also, since $\exp(-\|\cdot\|_K^4)$ is integrable on $\mathbb{R}^n$, μ_K is a continuous function and is locally integrable, so the result follows. A similar argument (with $\Delta^{(n+1)/2}$) works in the case where n is an odd integer. □

Let us now prove formula (3.20) in our particular case.

LEMMA 3.20. *Let $0 < p < n$ and let K be an infinitely smooth origin-symmetric star body in $\mathbb{R}^n$. Then for every $\theta \in S^{n-1}$*

$$\frac{\Gamma((n-p)/4)}{4} (\|\cdot\|_K^{-n+p})^\wedge(\theta) = \int_0^\infty t^{p-1} \mu_K(t\theta)\ dt.$$

PROOF. For any even test function $\phi \in \mathcal{S}(\mathbb{R}^n)$, consider the integral

$$\int_{\mathbb{R}^n} \mu_K(x) \left(\int_0^\infty t^{n-p-1} \phi(tx)\ dt \right) dx. \tag{3.23}$$

This integral converges absolutely, because by Lemma 3.19, the function μ_K is continuous and integrable, and the inner integral represents a continuous on $\mathbb{R}^n \setminus \{0\}$ and homogeneous of degree $-n+p$ function of x, so we can apply the argument of Lemma 2.1.

First, let us write the integral (3.23) in spherical coordinates (r, θ) (we also make a substitution $z = rt$ in the inner integral). We get that

$$(3.23) = \int_{S^{n-1}} \left(\int_0^\infty r^{p-1} \mu_K(r\theta)\ dr \right) \left(\int_0^\infty z^{n-p-1} \phi(z\theta)\ dz \right) d\theta.$$

Now let us write the same integral in a different way. Note that, by Corollary 3.17, $(\|\cdot\|_K^{-n+p})^\wedge$ is a continuous function on S^{n-1}. This function is homogeneous of degree $-p$ on $\mathbb{R}^n$. Also, $(\exp(-t^4\|\cdot\|_K^4))^\wedge(\xi) = t^{-n}\mu_K(\xi/t)$. We have

$$\begin{aligned}
(3.23) &= \int_0^\infty t^{n-p-1}\left(\int_{\mathbb{R}^n}\phi(tx)\mu_K(x)\ dx\right)\ dt \\
&= \int_0^\infty t^{n-p-1}\left(\int_{\mathbb{R}^n}\phi(\xi)\mu_K(\xi/t)t^{-n}\ d\xi\right)\ dt \\
&= \int_0^\infty t^{n-p-1}\left(\int_{\mathbb{R}^n}\hat{\phi}(y)\exp(-t^4\|y\|_K^4)\ dy\right)\ dt \\
&= \int_{\mathbb{R}^n}\hat{\phi}(y)\left(\int_0^\infty t^{n-p-1}\exp(-t^4\|y\|_K^4)\ dt\right)\ dy \\
&= \frac{\Gamma((n-p)/4)}{4}\int_{\mathbb{R}^n}\|y\|_K^{-n+p}\hat{\phi}(y)\ dy \\
&= \frac{\Gamma((n-p)/4)}{4}\langle\|\cdot\|_K^{-n+p},\hat{\phi}\rangle \\
&= \frac{\Gamma((n-p)/4)}{4}\langle(\|\cdot\|_K^{-n+p})^\wedge,\phi\rangle \\
&= \frac{\Gamma((n-p)/4)}{4}\int_{\mathbb{R}^n}(\|\cdot\|_K^{-n+p})^\wedge(\xi)\phi(\xi)\ d\xi \\
&= \frac{\Gamma((n-p)/4)}{4}\int_{S^{n-1}}(\|\cdot\|_K^{-n+p})^\wedge(\theta)\left(\int_0^\infty r^{n-p-1}\phi(r\theta)\ dr\right)\ d\theta.
\end{aligned}$$

Now if we put $\phi(x) = u(r)v(\theta)$ in both expressions for the integral (3.23), where v is any infinitely differentiable function on the sphere and u is a non-negative even test function on $\mathbb{R}$ with compact support outside of zero, we get

$$\begin{aligned}
&\int_{S^{n-1}}\left(\int_0^\infty r^{p-1}\mu_K(r\theta)\ dr\right)v(\theta)\ d\theta \\
&= \frac{\Gamma((n-p)/4)}{4}\int_{S^{n-1}}(\|\cdot\|_K^{-n+p})^\wedge(\theta)v(\theta)\ d\theta
\end{aligned}$$

for every $v \in C^\infty(S^{n-1})$. The result follows. □

We need to apply Parseval's formula on $\mathbb{R}^n$ in the case where one of the functions does not belong to $L_1(\mathbb{R}^n)$ or $L_2(\mathbb{R}^n)$.

LEMMA 3.21. *Let $0 < p < n$, and let K and D be infinitely smooth origin-symmetric star bodies in $\mathbb{R}^n$. Then*

$$(2\pi)^n\int_{\mathbb{R}^n}\|x\|_K^{-p}\exp(-\|x\|_D^4)\ dx = \int_{\mathbb{R}^n}(\|\cdot\|_K^{-p})^\wedge(\xi)\ \mu_D(\xi)\ d\xi. \tag{3.24}$$

PROOF. By Lemma 2.1, the integrals in both sides of (3.24) converge absolutely. By Lemma 3.20 (with p in place of $n-p$; we extend both sides of the equation to the whole $\mathbb{R}^n$ as homogeneous functions of degree $-n+p$), the right-hand side of (3.24) is equal to

$$\frac{4}{\Gamma(p/4)}\int_{\mathbb{R}^n}\left(\int_0^\infty t^{n-p-1}\mu_K(t\xi)\ dt\right)\mu_D(\xi)\ d\xi.$$

By the Fubini theorem, the latter is equal to

$$\frac{4}{\Gamma(p/4)}\int_0^\infty t^{n-p-1}\left(\int_{\mathbb{R}^n}\mu_K(t\xi)\mu_D(\xi)\ d\xi\right)\ dt.$$

We can apply Parseval's formula for L_1-functions (see (2.34)) to the inner integral, because μ_K and μ_D are even continuous integrable functions on $\mathbb{R}^n$, which are the Fourier transforms of the functions $\exp(-\|\cdot\|_K^4)$ and $\exp(-\|\cdot\|_D^4)$. We continue the calculation:

$$= \frac{4(2\pi)^n}{\Gamma(p/4)}\int_0^\infty t^{-p-1}\left(\int_{\mathbb{R}^n}\exp(-\|x\|_K^4/t^4)\exp(-\|x\|_D^4)\ dx\right)\ dt$$

$$= \frac{4(2\pi)^n}{\Gamma(p/4)}\int_{\mathbb{R}^n}\left(\int_0^\infty t^{-p-1}\exp(-\|x\|_K^4/t^4)\ dt\right)\exp(-\|x\|_D^4)\ dx$$

$$= (2\pi)^n\int_{\mathbb{R}^n}\|x\|_K^{-p}\exp(-\|x\|_D^4)\ dx.$$

□

We are now ready to prove Parseval's formula on the sphere in the infinitely smooth case.

LEMMA 3.22. *Let K and D be infinitely smooth origin-symmetric star bodies in $\mathbb{R}^n$, and let $0 < p < n$. Then*

$$\int_{S^{n-1}}(\|\cdot\|_K^{-p})^\wedge(\theta)(\|\cdot\|_D^{-n+p})^\wedge(\theta)\ d\theta = (2\pi)^n\int_{S^{n-1}}\|\theta\|_K^{-p}\|\theta\|_D^{-n+p}\ d\theta.$$

PROOF. Passing to spherical coordinates, we get

$$\int_{\mathbb{R}^n}\|x\|_K^{-p}\exp(-\|x\|_D^4)\ dx \tag{3.25}$$

$$= \int_{S^{n-1}}\|\theta\|_K^{-p}\left(\int_0^\infty r^{n-p-1}\exp(-r^4\|\theta\|_D^4)\ dr\right)\ d\theta$$

$$= \frac{\Gamma((n-p)/4)}{4}\int_{S^{n-1}}\|\theta\|_K^{-p}\|\theta\|_D^{-n+p}\ d\theta.$$

Now use Lemma 3.21 to get a different expression for the integral in (3.25):

$$\int_{\mathbb{R}^n}\|x\|_K^{-p}\exp(-\|x\|_D^4)\ dx$$

$$= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(\|\cdot\|_K^{-p})^\wedge(\xi)\mu_D(\xi)\ d\xi$$

$$= \frac{1}{(2\pi)^n}\int_{S^{n-1}}(\|\cdot\|_K^{-p})^\wedge(\theta)\left(\int_0^\infty r^{p-1}\mu_D(r\theta)\ dr\right)\ d\theta.$$

The result follows from Lemma 3.20. □

We can use an approximation argument to weaken the smoothness assumptions on the bodies K and D. The coefficient $(2\pi)^n$ disappears because of our definition of the measure μ_0 in Corollary 2.26, which is more convenient in other applications.

COROLLARY 3.23. *Let $k \in \mathbb{N}$, $1 \le k \le n-1$, and let K and D be origin-symmetric star bodies in $\mathbb{R}^n$. Suppose that D is $(k-1)$-smooth if k is odd and that D is k-smooth if k is even. Also suppose that $\|\cdot\|_K^{-k}$ is a positive definite distribution, and let μ_0 be the finite Borel measure on S^{n-1} that corresponds to $\|\cdot\|_K^{-k}$ by Corollary 2.26. Then*

$$\int_{S^{n-1}} (\|\cdot\|_D^{-n+k})^\wedge(\theta)\, d\mu_0(\theta) = \int_{S^{n-1}} \|\theta\|_K^{-k}\|\theta\|_D^{-n+k}\, d\theta. \tag{3.26}$$

PROOF. Since D is $(k-1)$-smooth, we can approximate D by a sequence D_m of infinitely smooth symmetric star bodies in such a way that the functions $\|\cdot\|_{D_m}$ converge to $\|\cdot\|_D$ in the metric of the space $C^{k-1}(S^{n-1})$; see the end of Section 2.2. Then, by Corollary 3.17, the Fourier transforms $(\|\cdot\|_{D_m}^{-n+k})^\wedge$ are continuous functions on the sphere that converge in $C(S^{n-1})$ to another continuous function $(\|\cdot\|_D^{-n+k})^\wedge$, as $m \to \infty$.

Also consider any sequence of infinitely smooth symmetric star bodies $K_j \subset K$, $j \in \mathbb{N}$, approximating K in the radial metric. For every even test function ϕ the sequence of integrable (by Lemma 2.1) functions $\|\cdot\|_{K_j}^{-k}|\hat{\phi}(\cdot)|$ is majorated by an integrable function $\|\cdot\|_K^{-k}|\hat{\phi}(\cdot)|$. By the dominated convergence theorem and the first part of Corollary 2.26,

$$\int_{\mathbb{R}^n} (\|\cdot\|_{K_j}^{-k})^\wedge(y)\phi(y)\, dy = \int_{\mathbb{R}^n} \|x\|_{K_j}^{-k}\hat{\phi}(x)\, dx$$

$$\to \int_{\mathbb{R}^n} \|x\|_K^{-k}\hat{\phi}(x)\, dx = (2\pi)^n \int_{S^{n-1}} \left(\int_0^\infty r^{k-1}\phi(r\theta)\, dr\right) d\mu_0(\theta), \tag{3.27}$$

as $j \to \infty$. Since the bodies D_m are infinitely smooth,

$$\phi_m(x) = u(r)(\|\cdot\|_{D_m}^{-n+k})^\wedge(\theta)$$

is a test function, where u is any non-negative even test function on $\mathbb{R}$ with compact support outside of zero. Substituting ϕ_m in (3.27) and passing to spherical coordinates, we get

$$\lim_{j\to\infty} \int_{S^{n-1}} (\|\cdot\|_{K_j}^{-k})^\wedge(\theta)(\|\cdot\|_{D_m}^{-n+k})^\wedge(\theta)\, d\theta$$

$$= (2\pi)^n \int_{S^{n-1}} (\|x\|_{D_m}^{-n+k})^\wedge(\theta)\, d\mu_0(\theta). \tag{3.28}$$

By Corollary 3.17, the functions $(\|\cdot\|_{D_m}^{-n+k})^\wedge(\theta)$ converge in $C(S^{n-1})$ to the function $(\|\cdot\|_D^{-n+k})^\wedge(\theta)$, as $m \to \infty$, so the integral in the right-hand side of (3.28) converges to the integral in the left-hand side of (3.26). Now we get the result if we apply Lemma 3.22 to the integral in the left-hand side of (3.28) and let $m, j \to \infty$. $\square$

The technique of this section allows us to prove a spherical version of another basic fact from Fourier analysis.

LEMMA 3.24. *Let $1 \le k < n$, let ϕ be an even continuous integrable function on $\mathbb{R}^n$, and let H be an $(n-k)$-dimensional subspace of $\mathbb{R}^n$. Suppose that ϕ is integrable on all translations of H and that the Fourier transform $\hat{\phi}$ is integrable on $H^\perp$. Then*

$$(2\pi)^k \int_H \phi(x)\, dx = \int_{H^\perp} \hat{\phi}(x)\, dx.$$

PROOF. Let $\xi_1, ..., \xi_k$ be an orthonormal basis in $H^\perp$. For every $t \in \mathbb{R}^k$, making the change of variables $u_1 = (x, \xi_1), ..., u_k = (x, \xi_k)$, we get

$$\hat{\phi}(t_1\xi_1 + ... + t_k\xi_k) = \int_{\mathbb{R}^n} \phi(x) \exp(-i(x, t_1\xi_1 + ... + t_k\xi_k))\ dx$$

$$= \int_{\mathbb{R}^n} \phi(x) \exp(-it_1(x, \xi_1) + ... + t_k(x, \xi_k))\ dx$$

$$= \int_{\mathbb{R}^k} \exp(-i(t_1u_1 + ... + t_ku_k)) \left(\int_{H+u_1\xi_1+...+u_k\xi_k} \phi(x)\ dx \right)\ du.$$

This means that the function $f(t) = \hat{\phi}(t_1\xi_1 + ... + t_k\xi_k)$ is the Fourier transform of the function $g(u) = \int_{H+u_1\xi_1+...+u_k\xi_k} \phi(x)\ dx$. The function g is even integrable on $\mathbb{R}^k$ because ϕ is even integrable on $\mathbb{R}^n$. Also, since $\hat{\phi}$ is even integrable on $H^\perp$, the function f is integrable and even on $\mathbb{R}^k$. By the inversion theorem for the Fourier transform, [Ru, Th. 7.7],

$$\int_{\mathbb{R}^k} f(t)\ dt = (\hat{g})^\wedge(0) = (2\pi)^k g(0),$$

which immediately implies the result, because the left-hand side is exactly $\int_{H^\perp} \hat{\phi}$. □

The following spherical version of Lemma 3.24 provides a Fourier transform formula for the volume of lower dimensional sections. This formula extends the result of Theorem 3.8 and will be useful in several places later.

LEMMA 3.25. *Suppose that $1 \le k < n$ is an integer and that L is an origin-symmetric star body in $\mathbb{R}^n$ so that L is $(k-1)$-smooth if k is odd and it is k-smooth if k is even. Then for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$ we have*

$$\mathrm{Vol}_{n-k}(L \cap H) = \frac{1}{(2\pi)^k (n-k)} \int_{S^{n-1}\cap H^\perp} (\|\cdot\|_L^{-n+k})^\wedge(\theta)\ d\theta.$$

PROOF. We first assume that L is infinitely smooth. By Lemma 3.20,

$$\int_{S^{n-1}\cap H^\perp} (\|\cdot\|_L^{-n+k})^\wedge(\theta)\ d\theta$$

$$= (4/\Gamma((n-k)/4)) \int_{S^{n-1}\cap H^\perp} \left(\int_0^\infty t^{k-1} \mu_L(t\theta)\ dt \right) d\theta$$

$$= (4/\Gamma((n-k)/4)) \int_{H^\perp} \mu_L(x)\ dx.$$

Using Lemma 3.24 with $\phi = \exp(-\|\cdot\|_L^4)$ and writing the integral in polar coordinates, we get

$$= (4(2\pi)^k/\Gamma((n-k)/4)) \int_H \exp(-\|x\|_L^4)\ dx$$

$$= (2\pi)^k \int_{S^{n-1}\cap H} \|\theta\|_L^{-n+k}\ d\theta = (2\pi)^k (n-k) \mathrm{Vol}_{n-k}(L \cap H),$$

by (2.3). The assumption that L is infinitely smooth can be removed by approximation in the same way, as was done in Corollary 3.23. □

3.5. Remarks and further results

As mentioned in the Introduction, the case of Theorem 3.1 where ξ is parallel to the main diagonal of the cube was known to Laplace [La]. To the best of the author's knowledge, the general formula of Theorem 3.1 first appeared in the paper [Po1] by Polya. This formula was independently derived and applied to the study of hyperplane sections of the cube by Hensley [He1] and Ball [Ba1]. The proof here is the same as in the paper [NP] by Nazarov and Podkorytov.

The formula of Theorem 3.2 in the case $1 \le q \le 2$ was found by Meyer and Pajor [MeP, Lemma II.6]. The proof uses stable random variables. The validity of this formula for other values of $q \in (0, \infty)$ was established in [K7]. The formulas for the Fourier transform of ℓ_q^n-norms, $0 < q \le \infty$, from Lemmas 3.4 and 3.6 are taken from [K8].

The general Fourier transform formula for the volume of hyperplane sections of star bodies, Theorem 3.8, was proved in [K7]. Together with the results of Lemmas 3.4 and 3.6, this formula implies Theorem 3.1 and proves Theorem 3.2 for all $0 < q < \infty$.

The result of Lemma 3.16 represents a way to invert the spherical Radon transform and cosine transforms with negative exponents. This approach was suggested in [K6]. As the author has recently learned, similar formulas relating the Radon and cosine transforms of homogeneous functions to the Fourier transform were proved by Semyanistyi [Se]. A geometric way of inverting the spherical Radon transform in certain situations was used by Gardner [Ga2] and Zhang [Zh2] in their solutions of the Busemann-Petty problem in dimensions 3 and 4. One can find more results on the connection between the Radon, Fourier and cosine transforms; related works include [Al], [AB], [GW1], [Gri], [Rb1], [Rb2], [SW].

Theorem 3.18 was proved in [GKS]. This result contains Theorem 3.8 as a particular case (with $k = 0$) and plays the key role in the unified solution to the Busemann-Petty problem; see Section 5.1. A way to prove similar formulas without using the Fourier transform or analytic continuation was suggested by Barthe, Fradelizi and Maurey in [BFM].

The result of Theorem 3.18 was extended to sections of lower dimensions in [K14]. Let $H \in G(n, n-k),\ 1 \le k < n$, and let $\xi_1, ..., \xi_k$ be an orthonormal basis in $H^\perp$. For a convex body K in $\mathbb{R}^n$, the $(n-k)$-dimensional parallel section function $A_{K,H}$ is a function on $\mathbb{R}^k$ defined by

$$A_{K,H}(u) = \mathrm{Vol}_{n-k}(K \cap \{H + u_1\xi_1 + ... + u_k\xi_k\}),\ u \in \mathbb{R}^k.$$

The result in [K14] is as follows:

Theorem 3.26. *Let K be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n,\ 1 \le k < n$. Then for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$ and every $m \in \mathbb{N} \cup \{0\},\ m \ne (n-k)/2$,*

$$\Delta^m A_{K,H}(0) = \frac{(-1)^m}{2^k \pi^k (n-2m-k)} \int_{S^{n-1}\cap H^\perp} (\|x\|_K^{-n+2m+k})^\wedge(\theta)\ d\theta.$$

The case $m = 0$ of Theorem 3.26 provides an alternative proof of Lemma 3.25. The proof of Theorem 3.26 follows the lines of the proof of Theorem 3.18.

The spherical Parseval formulas of Lemma 3.22 and Corollary 3.23 were proved in [K13]. These formulas are used in many places, most importantly in the generalizations of the Busemann-Petty problem in Sections 5.2 and 5.4. We presented here the original proof from [K13]. This proof projects the classical Parseval formula to the sphere. An alternative proof using spherical harmonics is shorter (see Remark (ii) at the end of Section 5.3), but it does not give a clear idea of what is going on. The spherical Parseval formula (with $p = 1$) combined with Theorem 3.8 gives an expression for the volume of a body in terms of volumes of central sections: for any $(n-2)$-smooth origin-symmetric convex body D in $\mathbb{R}^n$,

$$\mathrm{Vol}_n(D) = \frac{(2\pi)^n \pi (n-1)}{n} \int_{S^{n-1}} \mathrm{Vol}_{n-1}(D \cap \xi^\perp)(\|\cdot\|_D^{-1})^\wedge(\xi)\, d\xi.$$

The formula of Lemma 3.25 was first proved in [K13]. This formula is used in Section 4.2 to characterize k-intersection bodies.