Overview

Let $\mathcal{K}$ denote the collection of analytic functions on the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ that take the form

$$(K\mu)(z) := \int_{\mathbb{T}} \frac{1}{1 - \overline{\zeta} z} \, \mathrm{d}\mu(\zeta),$$

where μ belongs to M, the space of finite, complex, Borel measures on the unit circle $\mathbb{T} = \partial\mathbb{D}$. In the classical setting, as studied by Cauchy, Sokhotski, Plemelj, Morera, and Privalov, the Cauchy transform took the form of a Cauchy-Stieltjes integral

$$\int_{[0,2\pi]} \frac{1}{1 - e^{-i\theta} z} \, dF(\theta),$$

where F is a function of bounded variation on $[0, 2\pi]$.

In this monograph, we plan to study many aspects of the Cauchy transform: its function-theoretic properties (growth estimates, boundary behavior); the properties of the map $\mu \mapsto K\mu$; the functional analysis on the Banach space $\mathcal{K}$ (norm, dual, predual, basis); the representation of analytic functions as Cauchy transforms; the multipliers (functions ϕ such that $\phi\mathcal{K} \subset \mathcal{K}$); the classical operators on $\mathcal{K}$ (shift operators, composition operators); and the distribution function $y \mapsto m(|K\mu| > y)$ (where m is Lebesgue measure on $\mathbb{T}$). We will also examine more modern work, beginning with a seminal paper of D. Clark and later taken up by A. B. Aleksandrov and A. Poltoratski, that uncovers the important role Cauchy transforms play in perturbations of certain linear operators. To set the stage for what follows, we begin with an overview.

We start off in Chapter 1 with a quick review of measure theory, integration, functional analysis, harmonic analysis, and the classical Hardy spaces. This review will provide a solid foundation and clarify the notation.

The heart of the subject begins in Chapter 2 with the basic function properties of Cauchy transforms with special emphasis on how these properties are encoded in the representing measure μ. For example, a Cauchy transform $f = K\mu$ satisfies the growth estimate

$$|f(z)| \leqslant \frac{\|\mu\|}{1 - |z|}, \quad z \in \mathbb{D},$$

($\|\mu\|$ is the total variation norm of μ) as well as the identity

$$\lim_{r \to 1^-} (1 - r) f(r\zeta) = \mu(\{\zeta\}), \quad \zeta \in \mathbb{T}.$$

This last identity says that Cauchy transforms behave poorly at places on the unit circle where the representing measure μ has a point mass. Despite this seemingly poor boundary behavior, Smirnov's theorem says that Cauchy transforms do have some regularity near the circle in that they belong to certain Hardy spaces H^p. More precisely, whenever $f = K\mu$ and $0 < p < 1$,

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p \, \mathrm{d}m(\zeta) < \infty,$$

where $\mathrm{d}m = d\theta/2\pi$ is normalized Lebesgue measure on the unit circle. Let H^p be the space of analytic functions f for which the above inequality holds and let

$$\|f\|_{H^p} := \left(\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)|^p \, \mathrm{d}m(\zeta) \right)^{1/p}.$$

By standard Hardy space theory, Cauchy transforms have radial boundary values

$$f(\zeta) := \lim_{r \to 1^-} f(r\zeta)$$

for m-almost every $\zeta \in \mathbb{T}$. In fact, the formulas of Fatou and Plemelj say that the analytic function f on $\widehat{\mathbb{C}} \setminus \mathbb{T}$ (where $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$) defined by

$$f(z) = \int \frac{1}{1 - \overline{\zeta} z} \, \mathrm{d}\mu(\zeta), \quad z \in \widehat{\mathbb{C}} \setminus \mathbb{T},$$

satisfies

$$\lim_{r \to 1^-} (f(r\zeta) - f(\zeta/r)) = \frac{\mathrm{d}\mu}{\mathrm{d}m}(\zeta)$$

$$\lim_{r \to 1^-} (f(r\zeta) + f(\zeta/r)) = 2P.V. \int \frac{\mathrm{d}\mu(\xi)}{1 - \overline{\xi}\zeta}$$

for m-a.e. $\zeta \in \mathbb{T}$.

In this chapter we also discuss when $f = K\mu$ can be recovered from its boundary function $\zeta \mapsto f(\zeta)$ via the Cauchy integral formula

$$f(z) = \int_{\mathbb{T}} \frac{f(\zeta)}{1 - \overline{\zeta} z} \, \mathrm{d}m(\zeta), \quad z \in \mathbb{D}.$$

For a general $f = K\mu$, the boundary function $\zeta \mapsto f(\zeta)$, although belonging to L^p for $0 < p < 1$, need not be integrable and so the above Cauchy integral representation may not make sense. A result of Riesz says that the Cauchy integral formula holds if and only if f belongs to the Hardy space H^1, that is,

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta)| \, \mathrm{d}m(\zeta) < \infty.$$

Interestingly enough, there is a substitute Cauchy 'A-integral formula' due to Ul′yanov which says that if $\mu \ll m$ and $f = K\mu$, then

$$f(z) = \lim_{L \to \infty} \int_{|f| \leqslant L} \frac{f(\zeta)}{1 - \overline{\zeta} z} \, \mathrm{d}m(\zeta), \quad z \in \mathbb{D}.$$

This Cauchy A-integral formula has been recently used by Sarason and Garcia to further study the structure of certain H^p functions.

In Chapter 3 we treat the Cauchy transform not merely as an analytic function, but as a linear mapping $\mu \mapsto K\mu$ from the space of measures on the circle to the space of analytic functions on the disk. From Smirnov's theorem, we know that

$$K(M) \subsetneq \bigcap_{0<p<1} H^p.$$

In fact,

$$\|K\mu\|_{H^p} = O\left(\frac{1}{1-p}\right), \quad p \to 1^-.$$

We first cover the well-studied problem: if f belongs to a certain subclass of L^1, what type of analytic function is

$$f_+ := K(f\mathrm{d}m)?$$

Probably the earliest theorems here were those of Privalov (if f is a Lipschitz function on the circle, then f_+ is Lipschitz on $\mathbb{D}^-$), and of Riesz (if $1 < p < \infty$ and $f \in L^p$, then $f_+ \in H^p$). Then there are the more recent theorems of Spanne and Stein which say that if $f \in L^\infty$, then $f_+ \in BMOA$ (the analytic functions of bounded mean oscillation) while if f is continuous, then $f_+ \in VMOA$ (the analytic functions of vanishing mean oscillation). When $f \in L^2$ has Fourier series

$$f \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n)\zeta^n,$$

then

$$f_+(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n, \quad z \in \mathbb{D},$$

belongs to the Hardy space H^2 and the mapping $f \mapsto f_+$ is the orthogonal projection, the 'Riesz projection', of L^2 onto H^2.

Riesz's theorem says that the Riesz projection operator $f \mapsto f_+$ and the associated conjugation operator $f \mapsto \widetilde{f} := -2if_+ + i\widehat{f}(0) + if$ are continuous on L^p for $1 < p < \infty$, that is to say,

$$\|\widetilde{f}\|_{L^p} \leqslant A_p\|f\|_{L^p}, \quad \|f_+\|_{H^p} \leqslant B_p\|f\|_{L^p}, \quad f \in L^p,$$

for some constants A_p and B_p that are independent of f. An old theorem of Pichorides identifies the best constant A_p as $\tan(\pi/2p)$ if $1 < p \leqslant 2$ and $\cot(\pi/2p)$ if $p > 2$, while a relatively recent theorem of Hollenbeck and Verbitsky identifies the best constant B_p as $1/\sin(\pi p)$.

This chapter also covers the important weak-type theorem of Kolmogorov

$$m(|K\mu| > y) = O(1/y), \quad y \to \infty,$$

that gives an estimate of the distribution function for $K\mu$. It will turn out, quite amazingly, that one can recover information about the measure from this distribution function. For example, Tsereteli's theorem says

$$\mu \ll m \Leftrightarrow m(|K\mu| > y) = o(1/y), \quad y \to \infty.$$

Other work of Hruščev and Vinogradov, covered in Chapter 7, as well as some relatively recent work of A. Poltoratski, covered in Chapter 9, shows even more is true.

In Chapter 4 we treat the Cauchy transforms $\mathcal{K} = \{K\mu : \mu \in M\}$ as a Banach space. Since

$$K\mu_1 = K\mu_2 \Leftrightarrow \mu_1 - \mu_2 \in \overline{H_0^1},$$

where $\overline{H_0^1}$ are the measures $\{\overline{f}\,\mathrm{d}m : f \in H^1, f(0) = 0\}$, $\mathcal{K}$ can be identified in a natural way with the quotient space $M/\overline{H_0^1}$, by means of the mapping $K\mu \mapsto [\mu]$. Here $[\mu]$ is the coset in $M/\overline{H_0^1}$ represented by μ. One defines the norm of $K\mu$ to be the norm of the coset $[\mu]$ in the quotient space topology of $M/\overline{H^1_0}$. Equivalently, the norm of an $f \in \mathcal{K}$ is

$$\|f\| = \inf\{\|\mu\| : f = K\mu\}.$$

Equipped with this norm, $\mathcal{K}$ becomes a Banach space and furthermore, the previous growth estimate can be improved to

$$|f(z)| \leqslant \frac{\|f\|}{1-|z|}, \quad z \in \mathbb{D}.$$

Thus $\mathcal{K}$ becomes a Banach space of analytic functions in that the natural injection $i : \mathcal{K} \to \mathrm{Hol}(\mathbb{D})$ (the analytic functions on $\mathbb{D}$ with the topology of uniform convergence on compact sets) is continuous. From here, one can ask some natural questions. Is $\mathcal{K}$ separable? Is it reflexive? What is its dual (predual)? How do the weak and weak-$*$ topologies act on $\mathcal{K}$? Is $\mathcal{K}$ weakly complete? Is $\mathcal{K}$ weakly sequentially complete? Does $\mathcal{K}$ have a basis? What type? These questions are thoroughly addressed in this chapter.

So far, we have discussed the basic properties of a Cauchy transform $f = K\mu$. An interesting and still open question is to determine whether or not a given analytic function f on the disk takes the form $f = K\mu$. From what was said above, certain necessary conditions hold. For example, a Cauchy transform f must have bounded Taylor coefficients; must satisfy the growth condition $|f(z)| = O((1-|z|)^{-1})$; the boundary values of the function f must satisfy the L^p condition $\|f\|_{L^p} = O((1-p)^{-1})$ for $0 < p < 1$; the boundary values for f must also satisfy the weak-type inequality $m(|f| > y) = O(1/y)$. Unfortunately, none of these conditions is sufficient.

A more tractable question is: if f is not merely analytic on $\mathbb{D}$ but instead is analytic on the larger set $\widehat{\mathbb{C}} \setminus \mathbb{T}$ with $f(\infty) = 0$, when is f equal to

$$\int \frac{1}{1-\overline{\zeta}z}\, \mathrm{d}\mu(\zeta), \quad z \in \widehat{\mathbb{C}} \setminus \mathbb{T},$$

for some measure μ on the circle? Tumarkin answered this question with the following theorem: if f is analytic on $\widehat{\mathbb{C}} \setminus \mathbb{T}$ with $f(\infty) = 0$, then f is the Cauchy integral of a measure on the circle if and only if

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(r\zeta) - f(\zeta/r)|\, \mathrm{d}m(\zeta) < \infty.$$

Aleksandrov refined this theorem and identified the type of measure (whether absolutely continuous or singular with respect to Lebesgue measure) needed to represent f. These representation theorems are covered in Chapter 5.

At the end of this chapter we examine the question: which Riemann maps $\psi : \mathbb{D} \to \Omega$ are Cauchy transforms? For example, it is relatively easy to see that if $\psi(\mathbb{D})$ is contained in a half-plane, then ψ is a Cauchy transform. What is more difficult to see is that ψ is a Cauchy transform whenever $\psi(\mathbb{D})$ omits two oppositely pointing rays. What happens when $\psi(\mathbb{D})$ is a domain that spirals out towards infinity?

An important class of functions associated with a function space $\mathcal{X}$ are the 'multipliers'. Here we mean the set of functions ϕ for which $\phi\mathcal{X} \subset \mathcal{X}$. The multipliers constitute the complete set of multiplication operators $f \mapsto \phi f$ on $\mathcal{X}$ and there is quite a large literature on the subject. One can show that when $\mathcal{X}$ is a space of analytic functions, a multiplier of $\mathcal{X}$ must be a bounded analytic function. For the Hardy spaces H^p, the multipliers are precisely the bounded analytic functions. However, for other function spaces, such as the classical Dirichlet space or the analytic functions of bounded mean oscillation, not every bounded analytic function is a multiplier. Furthermore, even when a complete characterization of the multipliers is known, it is often difficult to apply to any particular circumstance.

Chapter 6 deals with the multipliers of $\mathcal{K}$. Despite some interesting results, these multipliers are still not thoroughly understood. For example, a multiplier of $\mathcal{K}$ must be bounded, must have radial limits everywhere (not just almost everywhere), and the partial sums of its Taylor series must be uniformly bounded. However, these conditions do not characterize the multipliers.

In this chapter we also cover the $\mathcal{F}$-property for $\mathcal{K}$. A space of functions $\mathcal{X}$ contained in the union of the H^p classes, as the Cauchy transforms are, satisfies the $\mathcal{F}$-property if whenever $f \in \mathcal{X}$ and ϑ is inner with $f/\vartheta \in H^p$ for some $p > 0$, then $f/\vartheta \in \mathcal{X}$. By the classical Nevanlinna factorization theorem, the Hardy spaces have the $\mathcal{F}$-property. It turns out that $\mathcal{K}$, as well as the multipliers of $\mathcal{K}$, enjoy the $\mathcal{F}$-property.

For the Hardy space, every inner function is a multiplier. On the other hand, there is the deep result of Hruščev and Vinogradov which says that an inner function is a multiplier of $\mathcal{K}$ if and only if it is a Blaschke product

$$z^m \prod_{n=1}^{\infty} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

whose zeros $(a_n)_{n \geqslant 1}$ satisfy the uniform Frostman condition

$$\sup_{\zeta \in \mathbb{T}} \sum_{n=1}^{\infty} \frac{1 - |a_n|}{|\zeta - a_n|} < \infty.$$

The proof of this is quite complicated but still worthwhile to present since it involves many earlier results about Cauchy transforms as well as the well-known Carleson interpolation theorem.

There is also an interesting connection between multipliers and co-analytic Toeplitz operators, namely, a bounded analytic function ϕ on $\mathbb{D}$ is a multiplier of $\mathcal{K}$ if and only if the co-analytic Toeplitz operator

$$(T_{\overline{\phi}} f)(z) := \int_{\mathbb{T}} \frac{\overline{\phi}(\zeta) f(\zeta)}{1 - \overline{\zeta} z} \, dm(\zeta) = (\overline{\phi} f)_+(z)$$

is a bounded operator from the space of bounded analytic functions to itself.

Kolmogorov's weak-type estimate $m(|K\mu| > y) = O(1/y)$ has been re-examined recently yielding some fascinating results on how this distribution function $y \mapsto m(|K\mu| > y)$ can be used to recover the singular part of the measure μ. Chapter 7 is devoted to these ideas. For example, it is relatively easy to show that when $\mu \ll m$, the Kolmogorov estimate can be improved from

$$m(|K\mu| > y) = O(1/y)$$

to

$$m(|K\mu| > y) = o(1/y).$$

Tsereteli proved the converse, namely,

$$m(|K\mu| > y) = o(1/y) \Leftrightarrow \mu \ll m.$$

The relationship between the distribution function and the singular part of the measure goes well beyond the improved Kolmogorov estimate. The first of two important theorems here is one of Hruščev and Vinogradov which says that

$$\lim_{y\to\infty} \pi y m(|K\mu| > y) = \|\mu_s\|,$$

where μ_s is the singular part of μ with respect to Lebesgue measure. Notice that when $\mu \ll m$, or equivalently $\mu_s = 0$, we obtain Tsereteli's theorem. The other more striking, and more recent, theorem of Poltoratski says that

$$\lim_{y\to\infty} \pi y m(|K\mu| > y) \cdot m = \mu_s, \quad \text{weak-}*,$$

thus recovering the actual singular part of the measure and not merely its total variation norm.

These distributional results are closely related to the distribution functions

$$y \mapsto m(|Q\mu| > y) \quad \text{and} \quad y \mapsto m_1(|\mathcal{H}\mu| > y)$$

of the conjugate function

$$(Q\mu)(e^{i\theta}) = P.V. \int \cot\left(\frac{\theta - t}{2}\right) \mathrm{d}\mu(e^{it})$$

and the Hilbert transform

$$(\mathcal{H}\mu)(x) = P.V. \int_{\mathbb{R}} \frac{1}{x - t}\, \mathrm{d}\mu(t),$$

where μ is a finite measure on $\mathbb{R}$. Some of these distribution theorems are quite classical. For instance, an 1857 theorem of Boole says that if

$$g(x) := \sum_{j=1}^{n} \frac{c_j}{x - a_j}, \quad a_j \in \mathbb{R}, \quad c_j > 0,$$

which is just the Hilbert transform of the positive discrete measure

$$\mu := \sum_{j=1}^{n} c_j\, \delta_{a_j},$$

then

$$m_1(\{x \in \mathbb{R} : g(x) > y\}) = \frac{1}{y} \sum_{j=1}^{n} c_j,$$

where m_1 is Lebesgue measure on $\mathbb{R}$.

Though the material in the first several chapters is certainly both elegant and important, our real inspiration for writing this monograph is the relatively recent work beginning with a seminal paper of Clark which relates the Cauchy transform to perturbation theory. Due to recent advances of Aleksandrov and Poltoratski, this remains an active area of research rife with many interesting problems. Chapters 8, 9, and 10 cover this connection between Cauchy transforms and perturbation theory.

Let us take a few moments to describe the basics of Clark's results. According to Beurling's theorem, the subspaces ϑH^2, where ϑ is an inner function, are all of the (non-trivial) invariant subspaces of the shift operator $Sf = zf$ on H^2. Consequently, the invariant subspaces of the backward shift operator

$$S^* f = \frac{f - f(0)}{z}$$

are of the form $(\vartheta H^2)^\perp$.

The functions

$$k_\lambda(z) := \frac{1 - \overline{\vartheta(\lambda)}\vartheta(z)}{1 - \overline{\lambda} z}, \quad \lambda, z \in \mathbb{D},$$

are the reproducing kernels for $(\vartheta H^2)^\perp$ in the sense that $k_\lambda \in (\vartheta H^2)^\perp$ and

$$f(\lambda) = \langle f, k_\lambda \rangle \quad \forall\, f \in (\vartheta H^2)^\perp.$$

Here we are using the usual 'Cauchy' inner product

$$\langle f, g \rangle := \int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)} \mathrm{d}m(\zeta)$$

on H^2. Clark's work was inspired by the question as to whether or not a given sequence of kernel functions $(k_{\lambda_n})_{n \geqslant 1}$ has dense linear span in $(\vartheta H^2)^\perp$. Clark showed that for certain $\zeta \in \mathbb{T}$, the kernels k_ζ belong to $(\vartheta H^2)^\perp$ and are eigenvectors for an associated unitary operator U_α on $(\vartheta H^2)^\perp$. Using the spectral properties of U_α, Clark determined when these eigenvectors k_ζ form a spanning set for $(\vartheta H^2)^\perp$ and then used a Paley-Wiener type theorem to say when the k_{λ_n}'s were 'close enough' to the k_ζ's to form a spanning set.

The unitary operator U_α mentioned above is the following: let S_ϑ be the compression of the shift S to $(\vartheta H^2)^\perp$; that is,

$$S_\vartheta := P_\vartheta S | (\vartheta H^2)^\perp$$

where P_ϑ is the orthogonal projection of H^2 onto $(\vartheta H^2)^\perp$. All possible rank-one unitary perturbations of S_ϑ, under the simplifying assumption that $\vartheta(0) = 0$, are given by

$$U_\alpha f := S_\vartheta f + \left\langle f, \frac{\vartheta}{z} \right\rangle \alpha, \quad \alpha \in \mathbb{T}.$$

It turns out that U_α is also cyclic and hence the spectral theorem for unitary operators says that U_α is unitarily equivalent to the operator 'multiplication by z', $(Zg)(\zeta) \mapsto \zeta g(\zeta)$, on the space $L^2(\sigma_\alpha)$, where σ_α is a certain positive singular measure on $\mathbb{T}$. It is quite remarkable, as we shall discuss in a moment, that σ_α can be computed from the inner function ϑ.

The unitary equivalence of Z on $L^2(\sigma_\alpha)$ and U_α on $(\vartheta H^2)^\perp$ is realized by the unitary operator

$$\mathcal{F}_\alpha : (\vartheta H^2)^\perp \to L^2(\sigma_\alpha),$$

which maps the reproducing kernel

$$k_\lambda(z) := \frac{1 - \overline{\vartheta(\lambda)}\vartheta(z)}{1 - \overline{\lambda} z}$$

for $(\vartheta H^2)^\perp$ to the function

$$\zeta \mapsto \frac{1 - \overline{\vartheta(\lambda)}\alpha}{1 - \overline{\lambda}\zeta}$$

in $L^2(\sigma_\alpha)$ and extends by linearity and continuity. Clark uses this unitary equivalence, as well as the structure of the associated space $L^2(\sigma_\alpha)$, to further examine whether or not the kernels $(k_{\lambda_n})_{n\geqslant 1}$ form a spanning set for $(\vartheta H^2)^\perp$.

This spectral measure σ_α for U_α arises as follows: for each fixed $\alpha \in \mathbb{T}$ the function

$$z \mapsto \Re\left(\frac{\alpha + \vartheta(z)}{\alpha - \vartheta(z)}\right)$$

is a positive harmonic function on $\mathbb{D}$, which, by Herglotz's theorem, takes the form

$$\Re\left(\frac{\alpha + \vartheta(z)}{\alpha - \vartheta(z)}\right) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2}\, \mathrm{d}\sigma_\alpha(\zeta),$$

where the right-hand side of the above equation is the Poisson integral $(P\sigma_\alpha)(z)$ of a positive measure σ_α on $\mathbb{T}$. Without too much difficulty, one can show that the measure σ_α is carried by the set $\{\zeta \in \mathbb{T} : \vartheta(\zeta) = \alpha\}$ and hence $\sigma_\alpha \perp m$. Furthermore, $\sigma_\alpha \perp \sigma_\beta$ for $\alpha \neq \beta$. Though many mathematicians, and some physicists, have used the measures described above, we think it is appropriate to call such measures 'Clark measures' since they are frequently referred to as such in the literature.

This idea extends beyond inner functions ϑ to any $\phi \in \mathrm{ball}(H^\infty)$ to create a family of positive measures $\{\mu_\alpha : \alpha \in \mathbb{T}\}$ associated with ϕ. It is becoming a tradition to call this family of measures the 'Aleksandrov measures' associated with ϕ. A beautiful theorem of Aleksandrov shows how this family of measures provides a disintegration of normalized Lebesgue measure m on the circle. Indeed,

$$\int_{\mathbb{T}} \mu_\alpha \, \mathrm{d}m(\alpha) = m,$$

where the integral is interpreted in the weak-$*$ sense; that is,

$$\int_{\mathbb{T}} \left(\int_{\mathbb{T}} f(\zeta)\, \mathrm{d}\mu_\alpha(\zeta) \right) \mathrm{d}m(\alpha) = \int_{\mathbb{T}} f(\zeta)\, \mathrm{d}m(\zeta)$$

for all continuous functions f on $\mathbb{T}$.

The identity

$$(K\sigma_\alpha)(z) = \frac{1}{1 - \overline{\alpha}\vartheta(z)}$$

produces the following formula for

$$\mathcal{F}_\alpha^* : L^2(\sigma_\alpha) \to (\vartheta H^2)^\perp$$

in terms of the 'normalized' Cauchy transform

$$\mathcal{F}_\alpha^* f = \frac{K(f\, \mathrm{d}\sigma_\alpha)}{K\sigma_\alpha}.$$

Poltoratski showed that several interesting things happen here. The first is that for σ_α-almost every $\zeta \in \mathbb{T}$, the non-tangential limit of the above normalized Cauchy transform exists and is equal to $f(\zeta)$. On the other hand, for $g \in (\vartheta H^2)^\perp$, the non-tangential limits certainly exist almost everywhere with respect to Lebesgue measure on the circle (since $(\vartheta H^2)^\perp \subset H^2$). But in fact, for σ_α-almost every ζ, the non-tangential limit of g exists and is equal to $(\mathcal{F}_\alpha g)(\zeta)$.

The compression S_ϑ and its rank-one unitary perturbation U_α are covered in Chapter 8. Clark measures, as well as Clark's theorem and Poltoratski's weak-type theorem

$$\lim_{y\to\infty} \pi y m(|K\mu| > y)\cdot m = \mu_s, \quad \text{weak-}*,$$

are covered in Chapter 9. Poltoratski's theorems on the normalized Cauchy transform

$$\mathcal{V}_\mu f = \frac{K(f\,\mathrm{d}\mu)}{K\mu}$$

are covered in Chapter 10.

At the end of Chapter 10, we briefly mention an independent and parallel 'Clark-type' theory, starting with some early papers of Aronszajn and Donoghue and continued in more recent papers of Simon and Wolff, involving the spectral measures for the rank-one perturbations

$$A_\lambda := A + \lambda\, v \otimes v, \quad \lambda \in \mathbb{R},$$

of a self-adjoint operator A with cyclic vector v. Here, the Borel transform

$$\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{t-z},$$

a close cousin to the Cauchy transform, comes into play.

In Chapter 11 we survey some results about the classical operators on $\mathcal{K}$. These operators, which have been studied quite extensively on the Hardy spaces H^p, include the shift, backward shift, composition, Toeplitz, and Cesàro operators. We also discuss versions of the Hardy space theorems, Beurling's theorem for example, in the setting of Cauchy transforms.

Conspicuously missing from this book is a discussion of the Cauchy transform

$$\int \frac{1}{w-z}\,\mathrm{d}\mu(w)$$

of a measure μ compactly supported in the plane. Certainly these Cauchy transforms are important. However, broadening this book to include these opens up a vast array of topics from so many other fields of analysis such as potential theory, partial differential equations, polynomial and rational approximation [**212, 213, 214**], the Painlevé problem, Tolsa's solution to the semi-additivity of analytic capacity [**216, 217**], as well as many others, that our original motivation for writing this monograph would be lost. Focusing on Cauchy transforms of measures on the circle links the classical function theory with the more modern applications to perturbation theory. If one is interested in exploring Cauchy transforms of measures on the plane, the books [**27, 50, 73, 78, 146, 154, 169**] as well as the survey papers [**32, 33**] are a good place to start. There is also a notion of fractional Cauchy transforms [**131**].