CHAPTER 1

Introduction and overview of the results

The asymptotic (long-time) behavior of dynamical systems may be very arbitrary. In recent works [**64, 65**], the author has shown by use of the construction method that the attractors for the two typical models arising from applied sciences, the Navier-Stokes (NS) equations $u_t + (u \cdot \nabla)u = \nu\Delta u - \nabla p + f$, $\operatorname{div} u = 0$ and the Reaction-Diffusion (RD) systems $u_t = D\Delta u + F(u)$ [**118**], may be very complicated if the time-dependent external force f for the (NS) equation (resp. the vector supply term F for the (RD) system) has been applied from some infinite-dimensional and complicated source. In general, attractors are considered as the ensemble of infinitely many trajectories. However, the attractor we have constructed for the (NS) equation (resp. for the (RD) system) is generated by a single trajectory. Moreover, the very complicated attractor for the autonomous (RD) system is a part of the equilibria set of that system.

Although the afore mentioned (RD) system does not belong to the class of so-called *gradient-like* systems, it shares the typical property of a gradient-like system that every relatively compact trajectory asymptotically approaches a set of equilibria [**51**, Lemma 3.8.2, p. 50]. This raises an interesting question as to whether the situation would be better in *gradient systems.* Unfortunately, an example due to P. Poláčik and F. Simondon [**96**] (see also [**95**], [**29**]), by modifying a construction of P. Poláčik and K. P. Rybakowski [**95**] (the latter is based on an example given by J. Palis and W. de Melo [**93**]), gives a C^∞ function f such that the following nonlinear heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - f(x, u(t,x)), \quad x \in \Omega, \quad t > 0; \\ u(t,x) = 0 \quad x \in \partial\Omega, \quad t > 0; \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

in the bounded domain $\Omega \subset \mathbb{R}^N$ (with $N \geq 2$ and a C^2 boundary $\partial\Omega$), has a bounded solution that asymptotically approaches a continuum of stationary solutions of that equation. Eq. (1.1) can be rewritten in its gradient system form

$$\dot{u} = -\mathcal{E}'(u) \quad (\dot{u} \equiv \partial u/\partial t), \tag{1.2a}$$

where $\mathcal{E}'(u) := -\Delta u + f(x,u)$ is the *gradient* of the *energy functional*

$$\mathcal{E}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega F(x,u)\,dx \quad (F_u = f). \tag{1.2b}$$

There are two ingredients in the above example of Poláčik and Simondon [**96**]: one is the higher dimension of the domain Ω and the other one is the smoothness of the function f. In fact, if $\Omega = (a, b) \subset \mathbb{R}^1$ is an interval, then convergence to a single equilibrium holds under very general hypotheses on f and the boundary conditions; see T. J. Zelenyak [**124**] and H. Matano [**86**] (also [**53**]). On the other hand, analyticity of the function f helps to overcome the difficulties encountered in higher-dimensional cases. It was L. Simon [**107**] who first established the long-time convergence to one single equilibrium of solutions of Eq. (1.1), for any bounded domain $\Omega \subset \mathbb{R}^N$, under the assumption that the nonlinearity $f(x, u)$ is (real) analytic with respect to u.

Simon's proof of convergence to one single equilibrium is deduced from a **gradient inequality** which compares the *gradient* of $\mathcal{E}$ to an *increment* of $\mathcal{E}$, and it takes the form

$$\begin{gathered} |\mathcal{E}(u) - \mathcal{E}(\varphi)|^\theta \leq \|\Delta u - f(\cdot, u)\|_{L^2(\Omega)} \\ \forall u \in C^{2,\alpha}(\bar{\Omega}), \|u - \varphi\|_{C^{2,\alpha}} < \sigma, \end{gathered} \tag{1.3}$$

where $\theta \in [1/2, 1)$ and $\sigma > 0$ are constants, and $\varphi \in C^{2,\alpha}(\bar{\Omega})$ is a classical solution of the Dirichlet boundary value problem

$$\begin{cases} -\mathcal{E}'(\varphi) = \Delta\varphi - f(x, \varphi(x)) = 0, \quad x \in \Omega \\ \varphi(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{1.4}$$

In proving the gradient inequality (1.3), Simon used a deep result of S. Łojasiewicz [**81, 82, 83**] (Łojasiewicz's theorem), which asserts that if $\Gamma : \mathbb{R}^N \to \mathbb{R}$ is a function that is (real) analytic in a neighborhood of a given critical point $a \in \mathbb{R}^N$ (i.e., $\nabla\Gamma(a) = 0$), then there exist some constants $\theta \in [1/2, 1)$ and $\sigma > 0$ such that

$$|\Gamma(x) - \Gamma(a)|^\theta \leq |\nabla\Gamma(x)| \quad \forall x \in \mathbb{R}^N, |x - a| < \sigma. \tag{1.5}$$

The constants θ and σ depend on the function Γ and the point a.

The above cited work of L. Simon was published in 1983. It seems that it was forgotten for a long time. It is only very recently (1997) that F.-H. Lin and Q. Du [**78**] used the same idea as L. Simon to prove the convergence of solutions to the (autonomous) Ginzburg-Landau (GL) model equations for superconductivity. One of the reasons for this delay may be that the original work of Simon is very difficult to read. This situation changed with the works of A. Haraux and M. A. Jendoubi [**58, 71, 72, 55**]. Haraux and Jendoubi (especially Jendoubi [**72**]) have simplified the original proof procedures and further developed Simon's method. There are several subsequent works in extending the above Łojasiewicz-Simon-type (LS-type) gradient inequalities and deducing convergence results from these gradient inequalities in other systems, e.g., E. Feireisl and F. Simondon [**40**] in porous medium equations, E. Feireisl and P. Takáč [**41**] and S.-Z. Huang and P. Takáč [**70**] in time-dependent (GL) equations, P. Rybka and K.-H. Hoffmann [**101, 102**] in

Cahn-Hilliard equations and equations of viscoelasticity with capillarity but without initial effect, R. Chill, E. Fašangová and J. Prüss [**20**] in Cahn-Hilliard equations with dynamic boundary conditions, P. Takáč [**115**] in (RD) systems of the form (1.1) where the Laplacian has been replaced by a general strongly elliptic operator and the nonlinearity f is also assumed to be analytic in the second argument u but is allowed to have bad growth conditions for the first argument x near the boundary of the domain, A. Haraux and M. A. Jendoubi [**59, 60, 61**], especially on damped semilinear wave equations, S. Aizicovici, E. Feireisl, F. Issard-Roch and H. Petzeltová [**2, 3, 4, 37, 38**] on phase-field systems, the recent work of R. Chill and E. Fašangová [**21**] on semilinear wave equations with dissipation of memory type, and the works of R. Chill [**18**] and R. Chill and M. A. Jendoubi [**21**] on abstract evolutionary equations.

1. The methodology

Before proceeding, we sum up the methodology used by all of the above cited works.

Suppose we are given a C^1 functional

$$\mathcal{E} : U \subset X \to \mathbb{R} \tag{1.6a}$$

on an open subset U of a (real) Banach space X. Then its gradient

$$\mathcal{M} := \mathcal{E}' : U \to X' \tag{1.6b}$$

is a continuous map from U to X'. Suppose that a stronger continuity of the gradient map $\mathcal{M}$ can be gained in the sense that there exists a subspace Y of X', which becomes a Banach space under its own norm $\|\cdot\|_Y$, such that

$$\|y\|_Y \geq \|y\|_{X'} \quad \forall y \in Y, \tag{1.6c}$$

and such that $\mathcal{M}$ maps U into Y, and the map

$$\mathcal{M} = \mathcal{E}' : U \to Y \tag{1.6d}$$

is continuous, i.e., the map $U \ni u \mapsto \mathcal{M}(u) \in Y$ is continuous with respect to the stronger Y norm topology. Consider a given subset $V \subset U$. We say that $\mathcal{M} = \mathcal{E}'$ satisfies a **gradient inequality** in V, with respect to the Y norm, if there exists a function ϕ with $0 \leq \frac{1}{\phi} \in L^1_{loc}(\mathbb{R})$ (in notation, $\phi \in \mathcal{G}$) such that

$$\phi(\mathcal{E}(u)) \leq \|\mathcal{E}'(u)\|_Y \quad \forall u \in V. \tag{GI}$$

Given a point $\varphi \in U$, we say that $\mathcal{M} = \mathcal{E}'$ satisfies a gradient inequality near φ if it satisfies a gradient inequality in some neighborhood of φ.

Clearly, $\mathcal{M} = \mathcal{E}'$ satisfies a gradient inequality near every regular point of $\mathcal{E}$. Hence, a gradient inequality is a matter concerning the behavior of $\mathcal{E}$ near its critical points.

Two more remarks should be taken into consideration.

First, we note that the norms of $\mathcal{E}'(u)$ are not evaluated in the X' norm but in the bigger Y norm. This follows from an easy observation: the larger the norms on the right-hand side of the inequality (GI) being evaluated, the better the chance of the existence of a function $\phi \in \mathcal{G}$ satisfying that inequality is. However, for the purpose of deducing convergence results from gradient inequalities, one has to try to estimate the norms in a relatively large subspace of X', because such a convergence occurs in general only with respect to a topology very similar to the topology induced by the norm in X'. This is a crucial point in establishing gradient inequalities.

Second, the condition that $\frac{1}{\phi}$ must be locally integrable is essentially important for applications in proving convergence to one single equilibrium of trajectories. To explain this point, we review the following results in Hilbert spaces, the proofs of which illustrate the methodology used originally by L. Simon and later by others and which will be developed in our present work.

Let H be a Hilbert space, and let $\mathcal{E} : X \to \mathbb{R}$ be a functional that is defined in a subspace X of H and has a gradient $\mathcal{E}'$ with $\mathcal{E}'(X) \subset H$ such that the map $X \ni u \mapsto \mathcal{E}'(u) \in H$ is continuous. We consider the following gradient system in H :

$$\dot{u}(t) = -\mathcal{E}'(u(t)), \quad t \geq 0. \tag{1.7}$$

The general working schema of L. Simon and others can be sketched as follows: Assuming that u is a global solution of (1.7), which is compact in the sense that its orbit $O(u) := \{u(t) : t \geq 0\}$ is contained in some compact subset of X, the goal is to establish the long-time convergence of u to some single point in X. To do so, it suffices to verify the weaker convergence that $\|u(t) - \varphi\|_H \to 0$ as $t \to \infty$ for some $\varphi \in H$. The reason is that the convergence $\|u(t) - \varphi\|_H \to 0$ implies that if $\tilde{\varphi} \in X$ is any cluster point of $O(u)$ with respect to the X norm, then we must also have $\|u(t) - \tilde{\varphi}\|_H \to 0$ as $t \to \infty$ and thus $\tilde{\varphi} = \varphi$. Therefore, $\varphi \in X$ and φ is the unique cluster point of the precompact trajectory u; thus we must have $u(t) \to \varphi$ in X as $t \to \infty$.

One condition that implies the convergence of u in H is the following:

$$I(u) := \int_0^\infty \|\dot{u}(t)\|_H \, dt < \infty.$$

This integral $I(u)$ can be thought of as the *length* of the curve $u : [0, \infty) \to X \subset H$ evaluated in the H norm. In all of the afore mentioned works, the convergence of u in the H norm topology is exactly this kind.

Our first result (Proposition 4.1 of Chapter 3) shows that the finiteness $I(u) < \infty$ is equivalent to the validity of a gradient inequality in the orbit $O(u)$, or equivalent to an appropriate convergence rate of the energies.

Theorem 1.1. *Assume that $u : \mathbb{R}_+ \to X$ is a classical solution of (1.7) such that*

$$\sup\{|\mathcal{E}(u(t))| : t \geq 0\} < \infty.$$

Then the following assertions **(i)-(iii)** *are equivalent:*

(i) *$\mathcal{E}'$ satisfies a gradient inequality in the orbit $O(u) := \{u(t) : t \geq 0\}$ with respect to the H norm; i.e., there holds the inequality*

$$\phi(\mathcal{E}(v)) \leq \|\mathcal{E}'(v)\|_H \quad \forall v \in O(u) \tag{1.8}$$

with some function ϕ such that $0 \leq \frac{1}{\phi} \in L^1_{loc}(\mathbb{R})$.

(ii) *There holds*

$$I(u) = \int_0^\infty \|\dot{u}(t)\|_H \, dt < \infty. \tag{1.9}$$

(iii) *There exists some function $\tilde{\phi}$ with $0 \leq \frac{1}{\tilde{\phi}} \in L^1_{loc}(\mathbb{R})$ such that*

$$\int_0^\infty \tilde{\phi}(\mathcal{E}(u(t))) \, dt < \infty. \tag{1.10}$$

We add some comments to these conditions. Note that there holds

$$\frac{d}{dt}\mathcal{E}(u(t)) = -\|\mathcal{E}'(u(t))\|_H^2 = -\|\dot{u}(t)\|_H^2 \leq 0. \tag{$*$}$$

Therefore, the energy function $\mathcal{E}(u(t))$ is nonincreasing, and thus the given boundedness assumption $\sup\{|\mathcal{E}(u(t))| : t \geq 0\} < \infty$ implies that the limit $a := \lim_{t\to\infty} \mathcal{E}(u(t))$ exists and is finite. From this we see that we can always find a nonnegative function $\tilde{\phi}(x)$ whose decay to zero as $x \to a$ is so rapid that the convergence in (1.10) occurs.

The crucial requirement for such a function $\tilde{\phi}$ is the extra local integrability $0 \leq 1/\tilde{\phi} \in L^1_{loc}(\mathbb{R})$ such that the decay $\tilde{\phi}(x) \to 0$ as $x \to a$ is not allowed to be too fast. Therefore, the condition in **(iii)** will be satisfied only if the energy function $\mathcal{E}(u(t))$ converges fast enough as $t \to \infty$.

On the other hand, by integrating $(*)$ we obtain that

$$\mathcal{E}(u(t)) - a = \int_t^\infty \|\mathcal{E}'(u(s))\|_H^2 \, ds = \int_t^\infty \|\dot{u}(s)\|_H^2 \, ds \tag{$**$}$$

for all $t \geq 0$. From $(**)$ we see that the requirement that the energy function $\mathcal{E}(u(t))$ converges fast enough as $t \to \infty$ is equivalent to the requirement that the norm function $\|\mathcal{E}'(u(t))\|_H = \|\dot{u}(t)\|_H$ $(t \geq 0)$ decays to zero fast enough as $t \to \infty$. The implication **(ii)**$\Longrightarrow$**(iii)** states that this is the case if $\int_0^\infty \|\dot{u}(t)\|_H \, dt < \infty$, while the other implication **(i)**$\Longrightarrow$**(iii)** says that this is also the case if the decay $\|\mathcal{E}'(u(t))\|_H \to 0$ as $t \to \infty$ dominates the decay $\mathcal{E}(u(t)) \to a$ as $t \to \infty$ in such a way that the gradient inequality (1.8) holds true for some $\phi \in \mathcal{G}$.

Now we turn to the convergence implications of gradient inequalities. As previously shown, the convergence $I(u) < \infty$ in Theorem 1.1-**(ii)** implies the usual convergence $\|u(t) - \varphi\|_H \to 0$ as $t \to \infty$ for some $\varphi \in H$. If in addition we know that the orbit $O(u)$ is contained in a compact subset of X, then we have the stronger convergence $\|u(t) - \varphi\|_X \to 0$ as $t \to \infty$ and $\mathcal{E}'(\varphi) = 0$. The following Theorem 1.2 (a special case of Theorem 3.3 of Chapter 3) has extended the above result in the sense that the validity of a gradient inequality in the orbit $O(u)$ can be replaced by the validity of a gradient inequality in the neighborhood of some cluster point of the orbit.

THEOREM 1.2. *Assume that* $u : \mathbb{R}_+ \to X$ *is a classical solution of (1.7) such that its orbit* $O(u)$ *is contained in a compact subset of* X *and has a cluster point* φ *near which* $\mathcal{E}'$ *satisfies a gradient inequality (with respect to the* H *norm), i.e., there exist a positive constant* $\sigma > 0$ *and a function* ϕ *with* $0 \leq \frac{1}{\phi} \in L^1_{loc}(\mathbb{R})$ *such that*

$$\phi(\mathcal{E}(v)) \leq \|\mathcal{E}'(v)\|_H \quad \forall v \in X, \|v - \varphi\|_X < \sigma. \tag{1.11}$$

Then $u(t)$ *converges as* $t \to \infty$ *to* φ *with respect to the* X *norm, and* φ *is an equilibrium of (1.7). Moreover,* $I(u) = \int_0^\infty \|\dot{u}(t)\|_H \, dt < \infty$.

Sketch of the proof of Theorem 1.2. Let

$$H(t) := \mathcal{E}(u(t)), \quad t \geq 0.$$

We have

$$-\dot{H}(t) = \langle \mathcal{E}'(u(t)), -\dot{u}(t) \rangle = \|\mathcal{E}'(u(t))\|_H \cdot \|\dot{u}(t)\|_H \tag{1.12a}$$

for all $t \geq 0$. This implies that $H(t)$ is nonincreasing and is bounded by the precompactness of the orbit $O(u)$. Let $a := \lim_{t\to\infty} H(t)$ and consider the function

$$\Phi(x) := \int_a^x \frac{1}{\phi(s)} \, ds \quad (x \in \mathbb{R}).$$

Φ is absolutely continuous, since $\frac{1}{\phi}$ is locally integrable (this explains the main point of why we must assume the local integrability of $\frac{1}{\phi}$). Moreover, we have, for the composition $G(t) := \Phi(H(t))$ $(t \geq 0)$, that

$$-\dot{G}(t) = -\dot{H}(t)/\phi(H(t)) \quad \text{a.e.} \quad t \geq 0. \tag{1.12b}$$

We choose an increasing unbounded sequence $\{t_n\}_{n\geq 1} \subset \mathbb{R}_+$ such that $\|u(t_n) - \varphi\|_X \to 0$ as $n \to \infty$.

Let $\varepsilon \in (0, \sigma)$, where $\sigma > 0$ is the number involved in the gradient inequality (1.11). Choose $N(\varepsilon) \in \mathbb{N}$ to be so large that $\|u(t_n) - \varphi\|_X < \varepsilon$ for all $n \geq N(\varepsilon)$. For each $n \geq N(\varepsilon)$, we let

$$\tilde{t}_n := \sup\{t \geq t_n : \quad \|u(s) - \varphi\|_X < \varepsilon \quad \forall t_n \leq s < t\}.$$

(Intuitively, $\tilde{t}_n$ is the first execution time during which the section $\{u(t) : t \geq t_n\}$ escapes from the neighborhood $\{v \in X : \|v - \varphi\|_X < \varepsilon\}$.)

We claim:

(C) For each $\varepsilon \in (0, \sigma)$ there exists an $N \geq N(\varepsilon)$ such that $\tilde{t}_N = \infty$, i.e.,

$$\|u(t) - \varphi\|_X < \varepsilon \quad \forall t \geq t_N. \tag{1.13}$$

Note that **(C)** implies the convergence $\lim_{t\to\infty} \|u(t) - \varphi\|_X = 0$ as well as the validity of the gradient inequality (1.11) in the section $\{u(t) : t \geq t_N\}$.

To establish **(C)**, we suppose, in contrast, that for some $\varepsilon_0 \in (0, \sigma)$ we can find for each $n \geq N_0 := N(\varepsilon_0)$ an s_n, $s_n > t_n$, such that

$$\|u(t) - \varphi\|_X < \varepsilon_0 \quad \forall t \in [t_n, s_n) \tag{1.14a}$$

but

$$\|u(s_n) - \varphi\|_X \geq \varepsilon_0. \tag{1.14b}$$

Fix $n, n \geq N_0$. For $t \in [t_n, s_n)$ we have $\|u(t) - \varphi\|_X < \varepsilon_0 < \sigma$. Therefore, the gradient inequality (1.11) applies to each $v = u(t)$ with $t \in [t_n, s_n)$ and yields

$$\phi(\mathcal{E}(u(t))) \leq \|\mathcal{E}'(u(t))\|_H \quad \forall t \in [t_n, s_n). \tag{1.15}$$

Using (1.12b) and (1.12a), (1.15) implies that

$$-\dot{G}(t) \geq \|\dot{u}(t)\|_H \quad \text{a.e.} \quad t \in [t_n, s_n) \tag{1.16}$$

and thus, by integration,

$$\int_{t_n}^{s_n} \|\dot{u}(t)\|_H \, dt \leq G(t_n) - G(s_n).$$

Consequently,

$$\begin{aligned} \|u(s_n) - \varphi\|_H &\leq \|u(t_n) - \varphi\|_H + \int_{t_n}^{s_n} \|\dot{u}(t)\|_H \, dt \\ &\leq \|u(t_n) - \varphi\|_H + (G(t_n) - G(s_n)) \to 0 \end{aligned}$$

as $n \to \infty$. This implies that if $\tilde{\varphi}$ is any cluster point of the precompact sequence $\{u(s_n) : n \geq N_0\}$ in X, then we must have $\|\tilde{\varphi} - \varphi\|_H = 0$ and thus $\tilde{\varphi} = \varphi$. Hence, φ is the unique cluster point of the precompact sequence $\{u(s_n) : n \geq N_0\}$ in X and thus $\|u(s_n) - \varphi\|_X \to 0$ as $n \to \infty$, contradicting (1.14b).

In conclusion, we have established **(C)**. As result, it implies that (1.15) holds true for all sufficiently large t. Again, this yields by integration the finiteness $I(u) < \infty$. Clearly, $\mathcal{E}'(\varphi) = 0$. □

Remark 1.3. The above also shows the following attraction property of an open subset $U_1 \subset X$ in which the gradient map $\mathcal{E}' : X \to H$ satisfies a gradient inequality: If a trajectory $u(t)$ of the gradient system (1.7) visits the neighborhood U_1 infinitely many times as $t \to \infty$, then the trajectory $u(t)$ will eventually be attracted to that neighborhood U_1.

As seen in Theorem 1.2, the validity of a gradient inequality near some cluster point of a precompact trajectory suffices to ensure the convergence to one single equilibrium of that trajectory. Thus, the preparing of suitable gradient inequalities is in general the first task before proving the convergence.

The idea of establishing gradient inequalities, which will be presented in the next chapter, can be sketched as follows: First, the involved gradient map must be a Fredholm map of index zero. This Fredholm property will be used (by the Lyapunov-Schmidt reduction) to reduce the involved infinite-dimensional problem to a corresponding finite-dimensional problem. Second, an expected infinite-dimensional gradient inequality will be deduced, provided that there holds an appropriate gradient inequality for the corresponding finite-dimensional problem. The best known and universal finite-dimensional gradient inequality is the above cited Łojasiewicz gradient inequality (1.5) for *analytic* functions. Because of this, we are able to show (in Chapter 2) that many analytic gradient maps satisfy a gradient inequality of Łojasiewicz-Simon type.

Let us review the gradient inequality (1.3) of L. Simon. It is not surprising, as compared to Łojasiewicz's inequality (1.5), that the function ϕ has the form $\phi_\theta(s) := |s - \mathcal{E}(\varphi)|^\theta$ for all $s \in \mathbb{R}$. The requirement $\theta < 1$ for the exponent corresponds exactly to the local integrability of $1/\phi_\theta$. It is also of little surprised why Simon chose $X = C^{2,\alpha}(\bar{\Omega})$. This choice makes the functional $\mathcal{E}$ an analytic functional from X to $\mathbb{R}$ and such that the gradient $\mathcal{E}'$ has a range contained in $C^\alpha(\bar{\Omega})$. Thus, any Banach space Y between $C^\alpha(\bar{\Omega})$ and the dual of $C^{2,\alpha}(\bar{\Omega})$ can be considered as a possible candidate for establishing the gradient inequality (GI) for this case. In the original proof of Simon, the gradient inequality with the choice $Y = C^\alpha(\bar{\Omega})$ is fairly easy to make. However, jumping from the space $C^\alpha(\bar{\Omega})$ to the Hilbert space $Y = L^2(\Omega)$ is certainly a subtle skill. What Simon needs for this jump is the classical regularity results shared by the Laplacians; this regularity property corresponds exactly to the Fredholm property of the semilinear elliptic operators. As seen in Theorem 1.2, the validity of a gradient inequality with respect to the $H = L^2(\Omega)$ norm suffices to guarantee the convergence to one single equilibrium for compact solutions of the nonlinear heat equation (1.1).

The remainder context of this chapter is concerned with the sketch of the convergence results deduced from gradient inequalities and applications of these convergence results, which we will elaborate on in the latter chapters. Each of these results is presented in both its abstract form and in its more concrete form for systems settled in Hilbert spaces. Moreover, we will give a brief synopsis for further studies in these topics.

2. Convergence results for gradient-like trajectories

Our setting is as follows. Let U be an open subset of a Banach space X and $\mathcal{E}' : U \to \mathbb{R}$ a C^1 functional. We assume further that there is a subspace Y of the dual X' such that $\mathcal{E}'(U) \subset Y$ and that the map $U \ni u \mapsto \mathcal{E}'(u) \in Y$ is continuous.

By a *trajectory* in U, we mean a continuous map $u : [0,T) \to U$ that has a weak derivative $\dot{u}$ taking values in Y'. Our study is concentrated in these *gradient-like trajectories* $u : [0,\infty) \to U$ that satisfy a growth condition of the form:

$$\begin{cases} \langle -\mathcal{E}'(u(t)), \dot{u}(t)\rangle_{Y\times Y'} \geq \|\mathcal{E}'(u(t))\|_Y \cdot \|\gamma\dot{u}(t)\|_{Y'} + F'(t), \\ \|\mathcal{E}'(u(t))\|_Y \geq \|\gamma\dot{u}(t)\|_{Y'} + G'(t) \end{cases} \tag{1.17a}$$

for all $t \geq 0$, where $\gamma : Y' \to Y'$ is an **injective** and continuous map (which could be nonlinear), and $F, G : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing C^1 functions satisfying

$$\lim_{t\to+\infty}(F(t)+G(t)) = 0 \quad \text{and} \quad \int_0^\infty \sqrt{F(t)}\,dt < \infty. \tag{1.17b}$$

The map γ as well as the functions F, G may depend on the given trajectory. A simplified version of (1.17a,b) is obtained by switching F, G into zero and weakening the inequality in the second line of (1.17a):

$$\begin{cases} \langle -\mathcal{E}'(u(t)), \dot{u}(t)\rangle_{Y\times Y'} \geq \|\mathcal{E}'(u(t))\|_Y \cdot \|\gamma\dot{u}(t)\|_{Y'}, \\ \mathcal{E}'(u(t)) = 0 \Longrightarrow \dot{u}(t) = 0 \end{cases} \qquad \forall t \geq 0. \tag{1.17$'$}$$

Observe that in (1.17a) and (1.17′), the norms for the gradient $\mathcal{E}'$ are not evaluated in the smaller X' norm but in the larger Y norm. The use of the larger Y norm is due to the fact that, in general, one can only have a gradient inequality in the larger Y norm rather than in the smaller X' norm.

Also note that the solutions considered in Theorem 1.2 satisfy (1.17′) under the choice of the space $Y = H \cong H' = Y'$ and $\gamma =$ identity operator on H. Therefore, Theorem 2.1 (corresponding to Theorem 3.3 of Chapter 3) extends Theorem 1.2.

THEOREM 2.1. *Let* $u : [0,\infty) \to U$ *be a trajectory satisfying* (1.17′). *Assume that the orbit* $O(u) = \{u(t) : t \geq 0\}$ *is precompact in* X *and has a cluster point* $\varphi \in X$ *near which the gradient map* $\mathcal{E}' : U \to Y$ *satisfies a gradient inequality. Then the trajectory* u *converges as* $t \to \infty$ *to* φ *in the* X *norm topology.*

For Theorem 2.2 (corresponding to Theorem 3.6 of Chapter 3), we need a new notion. For a constant $k \geq 1$, we denote by $\mathcal{G}_k$ the set of all functions $\phi \in \mathcal{G}$ (i.e., $0 \leq \frac{1}{\phi} \in L^1_{loc}(\mathbb{R})$) satisfying the following extra growth condition

$$\phi(x+y) \leq k\phi(x) + |y|^{1/2} \quad \forall x, y \in \mathbb{R}. \tag{1.18}$$

We say that a gradient inequality (GI) is of type $\mathcal{G}_k$ if the corresponding function ϕ involved in (GI) belongs to the class $\mathcal{G}_k$.

THEOREM 2.2. *Let* $u : [0,\infty) \to U$ *be a trajectory satisfying (1.17a) and (1.17b). Assume that the orbit* $O(u) = \{u(t) : t \geq 0\}$ *is precompact in* X *and has a cluster point* $\varphi \in X$ *near which the gradient map* $\mathcal{E}' : U \to Y$ *satisfies a gradient inequality of type* $\mathcal{G}_k$ *with some* $k \geq 1$. *Then the trajectory* u *converges as* $t \to \infty$ *to* φ *in the* X *norm topology.*

Comments. (a) We note that the functions

$$\tilde{\phi}_\theta(s) := \frac{1}{2}\min\{|s-a|^\theta, |s-a|^{1/2}\} \quad (s \in \mathbb{R})$$

with constants $a \in \mathbb{R}$ and $\theta \in [1/2, 1)$ belong to the class $\mathcal{G}_2$. Therefore, if we replace the function $\phi_\theta(s) = |s - \mathcal{E}(\varphi)|^\theta$ used in the gradient inequality (1.3) of L. Simon by the smaller $\tilde{\phi}_\theta$ with $a = \mathcal{E}(\varphi)$, then we have a gradient inequality of type $\mathcal{G}_2$. As it will be shown later, a gradient inequality of the type (1.3) will be shared by many other analytic gradient maps and thus Theorem 2.2 can be applied in a wide variety of applications.

(b) Observe that in (1.17a) as well as in (1.17′) we have included a continuous map $\gamma : Y' \to Y'$ that is only required to be injective. Hence the function $\|\gamma \dot{u}(t)\|_{Y'}$, compared to the usual norm function $\|\dot{u}(t)\|_{Y'}$, may have a very quick decay. It is exactly this point that opens up enormous possibilities for applying Theorem 2.2.

3. Applications to gradient-like systems in Hilbert spaces

Results presented below will be proved in the latter chapters, especially in §4 of Chapter 3 for the general case and in Chapter 4 for the special semilinear case.

3.1. General systems.

Our setting is as follows.

- H is a Hilbert space and $A : D(A) \subset H \to H$ is a linear self-adjoint and positive definite operator.
- Both X and $\tilde{X}$ are Banach spaces under their respective norms and such that the embeddings

$$X \hookrightarrow H_A \equiv (D(A), \langle\cdot,\cdot\rangle_A), \quad \tilde{X} \hookrightarrow H$$

are continuous. Moreover, $X \subset \tilde{X}$.

- $\mathcal{M} = \mathcal{E}' : X \to \tilde{X}$ is an **analytic** gradient map with the following properties: $\mathcal{M}$ is a Fredholm map of index zero; i.e., for each $u \in X$ the linear bounded operator $\mathcal{M}'(u) \in L(X, \tilde{X})$ is a Fredholm operator of index zero. Moreover, for each fixed $u \in X$ the bounded linear symmetric operator $\mathcal{M}'(u) : X \to \tilde{X}$ has an extension $\mathcal{M}_1(u) : H_A \to H$, which is also a symmetric Fredholm operator of index zero, and such that the map

$$X \ni u \mapsto \mathcal{M}_1(u)A^{-1} \in L(H)$$

is continuous.

Theorem 3.1. **(Gradient inequalities; Theorem 4.2 of Chapter 2)** *Under the above circumstances, there hold the following assertions:*

(i) The analytic gradient map $\mathcal{E}' : X \to \tilde{X}$ satisfies a gradient inequality of LS-type near every critical point; i.e., if $\varphi \in X$ is a critical point of $\mathcal{E}$, $\mathcal{E}'(\varphi) = 0$, then there exist positive constants c, σ and $\theta \in [1/2, 1)$ such that

$$c \cdot |\mathcal{E}(u) - \mathcal{E}(\varphi)|^\theta \le \|\mathcal{E}'(u)\|_H \quad \forall u \in X, \|u - \varphi\|_X < \sigma. \tag{1.19a}$$

(ii) The analytic gradient map $\mathcal{E}' : X \to \tilde{X}$ *satisfies a gradient inequality of type* $\mathcal{G}_2$ *in every compact subset of* X; *i.e., for every compact subset* $K \subset X$ *there exists some* $\phi \in \mathcal{G}_2$ *such that*

$$\phi(\mathcal{E}(u)) \leq \|\mathcal{E}'(u)\|_H \quad \forall u \in K. \tag{1.19b}$$

Our first convergence result is concerned with the following first order gradient-like system in the Hilbert space H:

$$\begin{cases} \dot{u} = -\mathcal{N}(u) + g(t,u), \quad t \geq 0, \\ u \equiv u(t) \in H, \dot{u} \equiv \dot{u}(t) \in H, \end{cases} \tag{1.20a}$$

where $\mathcal{N} : X \to H$ is a continuous map for which there exist a **strictly positive** continuous function $\eta : X \to (0,1)$ and an **injective** linear and bounded operator $\tilde{J} : H \to H$ such that

$$\begin{cases} \langle \mathcal{E}'(u), \mathcal{N}(u) \rangle_{H\times H} \geq \eta(u) \cdot \|\mathcal{E}'(u)\|_H^2 \\ \|\tilde{J}\mathcal{N}(u)\|_H \leq (1/\eta(u)) \cdot \|\mathcal{E}'(u)\|_H \end{cases} \qquad \forall u \in X. \tag{1.20b}$$

Moreover, the time-dependent perturbation $g : \mathbb{R}_+ \times X \to H$ is a continuous map satisfying

$$\begin{cases} \|g(t,u)\|_H \leq g(t) \quad \forall t \geq 0, u \in X; \\ \displaystyle\int_0^\infty \rho(t) g(t)^2 \, dt + \int_0^\infty \frac{1}{\rho(t)} < \infty \\ \text{with some increasing positive function} \quad \rho. \end{cases} \tag{1.20c}$$

Roughly speaking, condition (1.20b) states that $\mathcal{N}$ is a gradient-like map associated to the gradient map $\mathcal{E}'$, while (1.20c) is a condition implying that system (1.20a) will become asymptotically autonomous in a collective way. It is worth noting that condition (1.20c) does not imply the decay $g(t) \to 0$ as $t \to \infty$. In fact, there are functions that satisfy (1.20c), but they have arbitrarily large bounds in a suitable sequence of small intervals.

THEOREM 3.2. **(Theorem 4.2 of Chapter 3)**
Under the above situations we let $u : [0,\infty) \to H$ *be a solution of (1.20a) such that the orbit* $O(u)$ *is contained in a compact subset of the Banach space* X. *Then* $u(t)$ *converges as* $t \to \infty$ *to some* $\varphi \in X$ *in the* X *norm topology and* $\mathcal{E}'(\varphi) = 0$.

The tool for proving Theorem 3.2 is the gradient inequality (1.19a) or (1.19b).

There are also analogous convergence results (Theorem 4.4 and Theorem 4.5 of Chapter 3) for the following second-order gradient-like systems in the Hilbert space H :

$$\begin{cases} \ddot{u} + \kappa B(t,u,\dot{u}) = \pm\mathcal{E}'(u) + g(t,u,\dot{u}), \quad t \geq 0, \\ u \equiv u(t) \in H, \dot{u} \equiv \dot{u}(t) \in H, \ddot{u} \equiv \ddot{u}(t) \in H, \end{cases} \tag{1.21}$$

where the constant κ is either equal to -1 or 1, and $B : \mathbb{R}_+ \times X \times H \to H$ and $g : \mathbb{R}_+ \times X \times H \to H$ are continuous maps satisfying appropriate coerciveness and boundedness conditions. The choice $\kappa = 1$ (resp. $\kappa = -1$) corresponds to the case where the term $\kappa B(t, u, \dot{u})$ works as a structural damping (resp. a structural excitation) of the system (1.21).

If $\mathcal{E}'(u) = -\Delta u + f(x, u)$ is a semilinear elliptic operator, then (1.21) becomes $\ddot{u}(t) + \kappa B(t, u, \dot{u}) = \pm(-\Delta u + f(x, u)) + g(t, u, \dot{u})$. Under the choice of the negative sign, it is simply the *semilinear wave equation*; while under the choice of the positive sign, it is an equation of *elliptic type*.

We make several remarks.

(a) Theorem 3.2 and Theorems 4.4 and 4.5 of Chapter 3 apply particularly to the semilinear case where $\mathcal{E}'(u) = Au + f(x, u)$ and to the variational case where the functional $\mathcal{E}$ has the form $\mathcal{E}(u) = \int_\Omega F(x, u, Du)\, dx$ with a Carathéodory function $F(x, u, \xi)$ which is uniformly analytic in the u variable and strictly convex in the ξ variable.

(b) Evolutionary equations corresponding to variational functionals are, in general, fully nonlinear. Because of the lack of a powerful theory about the existence of global solutions to such fully nonlinear equations, our results presented in §5 of Chapter 3 for the variational case are somewhat restrictive.

(c) In contrast to the variational case, our results presented in Chapter 4 for the semilinear case are very fruitful. The reason is that there is a complete and powerful theory [**63, 94**] for the existence of global solutions to the corresponding semilinear Cauchy problems. We will sketch some of them below.

(d) Since we have also proved in Theorem 4.3 of Chapter 2 the validity of gradient inequalities of LS-type for some special class of **non-analytic** gradient maps, there are also convergence results analogous to those mentioned above for some special non-analytic cases.

(e) Theorem 3.2 applies to the following special choice of the map $\mathcal{N} = J\mathcal{E}'$; i.e., it applies to the following systems

$$\dot{u} = -J\mathcal{E}'(u) + g(t, u), \tag{1.22}$$

where the operator

$$J : D(J) \subset H \to H$$

is linear, closed, surjective and coercive in the sense that

$$\langle Jy, y\rangle_{H\times H} \geq c \cdot \|y\|_H^2 \quad (y \in D(J)).$$

Typical examples of systems (1.22) are the classical Cahn-Hilliard equations $\dot{u} = \Delta(-\Delta u + f(x, u))$ corresponding to the choices $J = -\Delta$ and $\mathcal{E}'(u) = -\Delta u + f(x, u)$.

(f) Note that the operator J involved in (1.22) is allowed to be **nonlinear** and in some sense to be very arbitrary.

(g) Theorem 3.2 applies particularly to perturbations that satisfy the condition

$$\int_0^\infty t|\log t|^\tau g(t)^2\,dt < \infty \quad \text{with some} \quad \tau > 1.$$

It seems that convergence results for perturbations satisfying the above simple growth condition are sufficient for many application situations.

3.2. Semilinear systems. Below, $(\Omega, \mathcal{B}, \mu)$ denotes a measure space of finite total measure $|\Omega| \equiv \mu(\Omega) < \infty$ and d is a positive integer. We will use $L^p(\Omega; \mathbb{R}^d)$ to denote the L^p spaces of vector-valued functions of d components over the given measure space Ω.

Our study focuses on the following first-order semilinear systems

$$\begin{cases} \dot{u} = -\mathcal{E}_F'(u) + g(t,u), \\ \mathcal{E}_F'(u) \equiv Au + f(x,u), \quad f \equiv \nabla_u F \end{cases} \tag{1.23a}$$

over the Hilbert space

$$\mathcal{H} := L^2(\Omega; \mathbb{R}^d), \tag{1.23b}$$

where $\mathcal{E}_F$ is a functional of the form

$$\mathcal{E}_F(u) := \frac{1}{2}\|Bu\|_{L^2}^2 + \int_\Omega F(x, u(x))\,d\mu(x) \quad (B := \sqrt{A}) \tag{1.23c}$$

for which the linear part A and the nonlinearity F satisfy the following conditions (i)-(ii):

(i) **(Compactness and submarkovian property).** The linear operator

$$A = \operatorname{diag}(A_1, ..., A_d) : D(A) \subset L^2(\Omega; \mathbb{R}^d) \to L^2(\Omega; \mathbb{R}^d) \tag{1.24a}$$

is a linear, diagonal, positive semidefinite and symmetric submarkovian generator such that the inverse

$$(1 + A)^{-1} : L^2(\Omega; \mathbb{R}^d) \to L^2(\Omega; \mathbb{R}^d) \tag{1.24b}$$

is compact. Moreover, there exist $\beta \in (0, 1)$ and $q \in [2, \infty)$ such that the embedding

$$D(A^\beta \restriction L^q) \cap L^\infty(\Omega; \mathbb{R}^d) \hookrightarrow L^\infty(\Omega; \mathbb{R}^d) \tag{1.24c}$$

is compact.

(ii) **(Analyticity and growth condition).** $F : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is a Carathéodory function that is uniformly *analytic* with respect to the second argument and its gradient $(f_1, ..., f_d) \equiv f := \nabla_u F$ satisfies the following growth condition: There exist three constant vectors $M = (M_1, ..., M_d) \geq 0$, $\tau_+ = (\tau_+^1, ..., \tau_+^d) \geq 0$ and $\tau_- = (\tau_-^1, ..., \tau_-^d) \leq 0$ such that, for all boundary points $u \equiv (u_1, ..., u_d) \in \partial Q$ of the "rectangle" $Q := \{u \in \mathbb{R}^d : \tau_- \leq u \leq \tau_+\}$, there holds

$$\begin{cases} f_j(\cdot, u) \geq M_j \quad \text{if} \quad u_j = \tau_+^j; \\ f_j(\cdot, u) \leq -M_j \quad \text{if} \quad u_j = \tau_-^j; \end{cases} \qquad j = 1, 2, ..., d. \tag{1.25}$$

Particularly, for the scalar case $d = 1$, condition (1.25) is equivalent to the following very transparent one:

$$f(\cdot, \tau_+) \geq M, \quad f(\cdot, \tau_-) \leq -M. \tag{1.25$'$}$$

It is very important to note that the growth condition (1.25) (cf. (1.25′) for the scalar case) does not imply, in general, the coercivity of the functional $\mathcal{E}_F$. In fact, it is not difficult to construct functions F that satisfy the above growth condition (1.25′), but the corresponding functionals $\mathcal{E}_F$ are not bounded from below.

Our final condition is on the perturbation

$$g : \mathbb{R}_+ \times L^\infty(\Omega; \mathbb{R}^d) \to L^\infty(\Omega; \mathbb{R}^d).$$

We assume that g is a locally Lipschitz map satisfying

$$\begin{cases} \|g(t,u)\|_{L^\infty} \leq M, \ \|g(t,u)\|_{L^2} \leq g(t) \quad \forall (t,u) \in \mathbb{R}_+ \times L^\infty; \\ \displaystyle\int_0^\infty \rho(t) g(t)^2 \, dt + \int_0^\infty \frac{1}{\rho(t)} < \infty \\ \text{with some increasing positive function} \quad \rho. \end{cases} \tag{1.26}$$

We have the following convergence and stability results.

Theorem 3.3. **(Convergence; Theorem 1.8 of Chapter 4)** *Under the above circumstances we set* $X := D(\sqrt{A}) \cap L^\infty(\Omega; \mathbb{R}^d)$ *and*

$$X_\tau := \{\varphi \in X : \tau_- \leq \varphi \leq \tau_+\}.$$

Then for each initial value $u_0 \in L^\infty$ *with*

$$\tau_- \leq u_0 \leq \tau_+,$$

the Cauchy problem corresponding to (1.23a) has a unique classical solution

$$u \in C([0,\infty); L^2) \cap C((0,\infty); D(A)) \cap C^1((0,\infty); L^2)$$

for which there exists an equilibrium $\varphi \in X_\tau$ *such that*

$$\lim_{t \to \infty} \|u(t) - \varphi\|_X = 0.$$

Moreover, the part of equilibria

$$\mathcal{S}_\tau := \{\varphi \in X_\tau : \mathcal{E}_F'(\varphi) = 0\}$$

is a compact subset of X.

Theorem 3.4. **(Stability of ground state; Theorem 1.9 of Chapter 4)** *Under the above circumstances there exists some* $\psi \in \mathcal{S}_\tau$ *such that*

$$\mathcal{E}_F(\psi) \leq \mathcal{E}_F(u) \quad \forall u \in X_\tau.$$

Moreover, this local ground state ψ*, as a stationary solution of the gradient system* $\dot{u} = -\mathcal{E}_F'(u)$*, is Lyapunov stable, and it is uniformly asymptotically stable if it is an isolated stationary solution.*

Both Theorems 3.3 and 3.4 apply to the case where A is a *Dirichlet form operator* [**84**], and especially to the case where $-A$ are Laplacian operators. Related results about semilinear evolutionary problems around the Laplacian operators will be presented in §3, §4 and §5 of Chapter 4. There we will show the convergence and stability of the following model equations arising from applied sciences: the Cahn-Hilliard equations, the Swift-Hohenberg equations, the Reaction-Diffusion equations in gradient forms, the Ginzburg-Landau equations for superconductivity and the porous medium equations.

4. Application to the stability problem

Let $\mathcal{E} : U \subset X \to \mathbb{R}$ be a C^1 functional. Here U is an open subset of the real Banach space X. We consider a critical point $\varphi \in U$ which is a *ground state* in the sense that $\mathcal{E}$ attains its minimum at this point:

$$\mathcal{E}(\varphi) = \inf_{u \in U} \mathcal{E}(u).$$

For $r > 0$ we use U_r to denote the r-neighborhood of φ :

$$U_r := \{u \in X : \|u - \varphi\|_X < r\}.$$

Our first result reads as follows.

THEOREM 4.1. **(Stability Theorem; Theorem 1.2 of Chapter 5)** *Let φ be a ground state of $\mathcal{E}$ with $\mathcal{E}(\varphi) = 0$ and assume the following gradient inequality:*

$$\phi(\mathcal{E}(u)) \le \|\mathcal{E}'(u)\|_{X'} \quad \forall u \in U_\sigma \cap U, \tag{1.27}$$

with some positive constant σ and some ϕ such that $0 \le \frac{1}{\phi} \in L^1_{loc}(\mathbb{R})$.

Let $\mathcal{N} : U \to X$ be a locally Lipschitz map for which there exists a constant $\alpha > 0$ such that

$$\begin{cases} \langle \mathcal{E}'(u), \mathcal{N}(u)\rangle_{X' \times X} \ge \alpha \cdot \|\mathcal{E}'(u)\|_{X'} \cdot \|\mathcal{N}(u)\|_X, \\ \|\mathcal{E}'(u)\|_{X'} \ge \alpha \cdot \|\mathcal{N}(u)\|_X \end{cases} \tag{1.28}$$

for all $u \in U$. Choose $\varepsilon \in (0, \sigma/2)$ to be so small that

$$\int_0^{\mathcal{E}(u)} \frac{ds}{\phi(s)} < (\alpha\sigma)/2 \quad \forall u \in U_\varepsilon. \tag{1.29}$$

Then for each $u_0 \in U_\varepsilon$, the following autonomous Cauchy problem

$$\begin{cases} \dot{u}(t) = -\mathcal{N}(u(t)), t \ge 0; \\ u(0) = u_0 \end{cases} \tag{1.30}$$

admits a global solution $u : [0, \infty) \to U_{\sigma/2}$ that converges as $t \to \infty$ to some stationary point $u_\infty \in \mathcal{N}^{-1}(0) \cap U_\sigma$ with respect to the X norm. As a consequence, φ as an equilibrium of (1.30) is Lyapunov stable.

Moreover, φ is uniformly asymptotically stable if it is an isolated equilibrium and the inequality in the second line of (1.28) is strengthened by

$$(1/\alpha) \cdot \|\mathcal{N}(u)\|_X \ge \|\mathcal{E}'(u)\|_{X'} \ge \alpha \cdot \|\mathcal{N}(u)\|_X \quad (\forall u \in U). \tag{1.31}$$

Our second result is concerned with the convergence in systems generated by pseudo-gradient vector fields associated with the given functional $\mathcal{E}$. Such a system is used to construct solutions to variational problems [**112**] and is a variant of the usual steepest descent method for Hilbert spaces.

We recall [**112**, p. 78] that a C^1 functional $\mathcal{E} : X \to \mathbb{R}$ is said to satisfy the Palais-Smale (in short, (P.-S.)) condition if every sequence $\{u_n\}_{n\geq 1} \subset X$ such that $\sup_{n\geq 1} |\mathcal{E}(u_n)| < \infty$ and $\|\mathcal{E}'(u_n)\|_{X'} \to 0$ as $n \to \infty$ contains a convergent subsequence.

The following Theorem 4.2 (corresponding to Theorem 2.1 of Chapter 5) applies to the following Cauchy problem:

$$\dot{u}(t) = -\mathcal{N}(u(t)) + g(t, u(t)), t \geq 0; \quad u(0) = u_0, \tag{1.32a}$$

where $\mathcal{N} : X \to X$ is a locally Lipschitz map for which there exists a positive constant α such that

$$\begin{cases} \langle \mathcal{E}'(u), \mathcal{N}(u)\rangle_{X'\times X} \geq \alpha \cdot \|\mathcal{E}'(u)\|_{X'}^2, \\ \|\mathcal{E}'(u)\|_{X'} \geq \alpha \cdot \|\mathcal{N}(u)\|_X \end{cases} \qquad \forall u \in X. \tag{1.32b}$$

Moreover, $g : \mathbb{R}_+ \times X \to X$ is a locally Lipschitz map that satisfies the following smallness condition:

$$\begin{cases} \|g(s,u)\|_X \leq g(s) \quad \forall s \geq 0, u \in X; \\ \displaystyle\int_0^\infty \left(\int_t^\infty g(s)^2\, ds\right)^{1/2} dt < \infty. \end{cases} \tag{1.32c}$$

THEOREM 4.2. **(Theorem 2.1 of Chapter 5)**
Assume that the C^1 functional $\mathcal{E} : X \to \mathbb{R}$ is bounded below, satisfies the (P.-S.) condition, and satisfies a gradient inequality of type $\mathcal{G}_k$ near every point of X.

Then for any $u_0 \in X$, the Cauchy problem (1.32a) admits a global solution $u : [0,\infty) \to X$ that converges as $t \to \infty$ to some critical point of $\mathcal{E}$.

Theorem 4.2 implies particularly the **convergence of the steepest descent method** concerning a functional $\mathcal{E}$ that is bounded below, satisfies the (P.-S.) condition and is such that its gradient satisfies a gradient inequality of type $\mathcal{G}_k$ near any given point of X. More importantly, from the point of view of practical computations, Theorem 4.2 ensures that the afore mentioned convergence of the steepest descent method is stable under perturbations satisfying the smallness condition (1.32c). For the case where the gradient inequalities are of the special LS-type (this is the case if $\mathcal{E}'$ is analytic and Fredholm), then the above smallness condition (1.32c) can be greatly relaxed into the type of (1.20c); i.e., the second line in (1.32c) can be relaxed as

$$\int_0^\infty \rho(t) g(t)^2\, dt + \int_0^\infty \frac{1}{\rho(t)} < \infty, \tag{1.33}$$

where ρ is some increasing positive function. A smallness condition of the form (1.33) will be satisfied in many practical situations.

5. Additional remarks

(a) In Chapter 2 we will highlight many gradient inequalities for both finite- and infinite-dimensional cases. These gradient inequalities serve to ensure the applicability of the convergence results obtained in the present work.

We will close this chapter by reviewing some other convergence results obtained without using gradient inequalities. Moreover, we will give a short description of a program for further study.

(b) **(Convergence in dissipative systems).** The following convergence result should be known, but we had difficulty in finding suitable sources. A related result for Hilbert spaces without perturbation can be found in the monograph of J. W. Neuberger [**91**, Theorem 4.12, p. 26].

THEOREM 5.1. *Let E be a Banach space such that its dual E' is strictly convex. Let $\mathcal{M} : U \subset E \to E$ be an accretive map, i.e.,*

$$\langle \mathcal{M}(u) - \mathcal{M}(v), \mathcal{F}(u-v)\rangle \geq 0 \quad \forall u, v \in U, \tag{1.34a}$$

where $\mathcal{F} : E \to E'$ is the duality map given by

$$\mathcal{F}(w) := \{f \in E' : \langle w, f\rangle = \|w\|_E^2 = \|f\|_{E'}^2\} \quad (w \in E).$$

Let $u : [0,\infty) \to U$ be a global solution of the following perturbed dissipative system

$$\dot{u}(t) = -\mathcal{M}(u(t)) + g(t, u(t)),\ t \geq 0, \tag{1.34b}$$

where the perturbation $g : \mathbb{R}_+ \times E \to E$ is locally Lipschitz and satisfies the following smallness condition:

$$\sup_{u\in E} \|g(t,u)\|_E \leq g(t) \quad \forall t \geq 0; \qquad \int_0^\infty g(t)\,dt < \infty. \tag{1.34c}$$

If a subsequence of $\{u(t)\}_{t\geq 0}$ converges to some equilibrium of (1.34b), i.e., if there exists an unbounded increasing sequence $\{t_n\}_{n\geq 1}$ and some $\varphi \in E$ such that

$$\mathcal{M}(\varphi) = 0 \quad \text{and} \tag{1.35a}$$

$$\|u(t_n) - \varphi\|_E \to 0 \quad \text{as} \quad n \to \infty, \tag{1.35b}$$

then $\|u(t) - \varphi\|_E \to 0$ as $t \to \infty$.

PROOF. The strict convexity of E' implies [**32**, Prop. 13.1, p. 124] that the function $h(t) := \frac{1}{2}\|u(t) - \varphi\|_E^2$ $(t \geq 0)$ is differentiable and

$$h'(t) = \langle \dot{u}(t), \mathcal{F}(u(t) - \varphi)\rangle.$$

Substituting $\dot{u} = -\mathcal{M}(u) + g(t,u)$ and using $\mathcal{M}(\varphi) = 0$, we find that

$$h'(t) = -\langle \mathcal{M}(u(t)) - \mathcal{M}(\varphi), \mathcal{F}(u(t) - \varphi)\rangle + \langle g(t,u), \mathcal{F}(u(t) - \varphi)\rangle.$$

The first sum term is non positive by the accretivity of $\mathcal{M}$, while the second sum term is bounded by $g(t)(2h(t))^{1/2}$. Therefore,

$$h'(t) \leq 2g(t)h(t)^{1/2} \quad \forall t \geq 0. \tag{1.36a}$$

Integrating (1.36a), we find that

$$h(t)^{1/2} \leq h(t_n)^{1/2} + \int_{t_n}^{t} g(s)\, ds \tag{1.36b}$$

for all $t \geq t_n$. Taking the limit $t \to \infty$ in (1.36b), we obtain that

$$\limsup_{t\to\infty} h(t)^{1/2} \leq h(t_n)^{1/2} + \int_{t_n}^{\infty} g(s)\, ds \tag{1.36c}$$

for all $n \in \mathbb{N}$. By (1.34c) and (1.35b) we have $h(t_n) \to 0$ and $\int_{t_n}^{\infty} g(s)\, ds \to 0$ as $n \to \infty$. Therefore, $h(t) \to 0$ as $t \to \infty$ by (1.36c). This is the desired convergence. □

(c) (**Other known convergence results**) For $\Omega \subset \mathbb{R}^N$ with $N \geq 2$, a number of sufficient conditions are known that guarantee the long-time convergence of solutions to the semilinear heat equation (1.1).

Let L_φ be the self-adjoint operator

$$L_\varphi := \Delta - f_u(x, \varphi(x)) : D(\Delta) \subset L^2(\Omega) \to L^2(\Omega),$$

which is obtained by linearizing the right-hand side of Eq. (1.1) about an equilibrium φ of (1.1). Let $\omega(u_0)$ be the ω-limit set of a given solution u of Eq. (1.1). If zero happens to be an eigenvalue of L_φ with multiplicity at most one for each $\varphi \in \omega(u_0)$, then P.-L. Lions [**79**] proves the long-time convergence. Lions' method takes advantage of comparison results using the maximum principle, and his idea is closely related to an abstract result due to J. K. Hale and P. Massatt [**52**], which has been proved by J. K. Hale and G. Raugel [**53**, Theorem 2.4] in a more rigorous manner. Theorem 2.4 in [**53**] applies to the so-called *thin domains* $\Omega \subset \mathbb{R}^2$ for which the property that the multiplicity is at most one holds true and thus the convergence follows; see also [**100**]. A thin domain in $\mathbb{R}^2$ is a sufficiently thin neighborhood of an arc.

When Ω is a ball and f is independent of $x \in \Omega$, A. Haraux and P. Poláčik [**62**] prove the long-time convergence of nonnegative solutions.

E. Feireisl [**36**] also obtains some results concerning the long-time behavior of solutions to Eq. (1.1) in the total space $\Omega = \mathbb{R}^N$.

The semilinear heat equation (1.1) belongs to the so-called class of *monotone dynamical systems* [**109, 125**]. M. Hirsch shows that the generic convergence holds for such systems. The interested reader should consult the monographs of Hal L. Smith [**109**] and X. Q. Zhao [**125**] and the references therein.

(d) (**A program for further studies**) Below we sketch a program for further studies concerning the applications in the following topics: geometric evolution problems, image processing, phase-field models, and optimization problems.

• Applications to the unique tangent cone problem for minimal surfaces and to the unique tangent problem for harmonic maps due to L. Simon [**107, 108**]. What Simon needs for these applications is the convergence result for the following second-order gradient-like and structurally excited systems

$$\ddot{u} - \dot{u} = -\mathcal{E}'(u) + g(t, u, \dot{u}), \quad t \geq 0, \tag{1.37a}$$

where the energy functional $\mathcal{E}$ is given by an analytic function $E : \bar{\Omega} \times \mathbb{R}^{1+n} \to \mathbb{R}$ as follows:

$$\mathcal{E}(u) = \int_{\Omega} E(x, u, \nabla u)\, dx. \tag{1.37b}$$

Moreover, $E(x, u, \xi)$ is assumed to be uniformly convex in the ξ variable so that the linearization of the gradient map $\mathcal{M}(u) := \mathcal{E}'(u)$ about any equilibrium of the system (1.37a) is a second-order linear elliptic operator. This ellipticity is very essential in establishing gradient inequalities for $\mathcal{M}$. Note that systems of the form (1.37a) fit to our systems (1.21) with more general functionals $\mathcal{E}$. Therefore, applications of Theorems 4.4 and 4.5 of Chapter 3 to other geometric problems should be expected.

We point out that M. F. Bidaut-Veron and L. Veron [**120, 121**] have used the ideas of L. Simon to study the conformal asymptotics of the isothermal gas spheres equations, and P. F. Guang and G. F. Wang [**50**] are able to prove their convergence result for a fully nonlinear conformal flow using a variant of Simon's convergence result given by B. Andrews [**10**].

• Application to the "geometric studies of trajectories of gradient fields". The geometric studies for trajectories of *analytic* gradient vector fields have been done by several authors; see e.g., K. Kurdyka, T. Mostowski and A. Parusiński [**76**], A. Nowel and Z. Szafraniec [**92**], F. Santz [**103**], and the references therein. One main tool used (and needed) by these authors is the afore mentioned gradient inequality of Łojasiewicz for analytic gradient maps. It seems that the ideas used by these authors can be extended to *non-analytic* cases where some suitable kinds of gradient inequalities (e.g., those used in our present work) are available.

• Applications to the "calculus of variations" [**112, 122, 43, 27**], in particular, applications to those problems arising from geometric interests [**9, 112, 107, 108**].

• Application to image processing. As proposed by L. Alvarez, F. Guichard, P.-L. Lions and J.-M. Morel [**5**] (the AGLM-theory), the mathematical equations involved in image processing are the following (non-linear) evolution equations:

$$\dot{u} = F(D^2 u, Du, u, t). \tag{1.38}$$

In [**5**], the forms for F are not specified and might be very arbitrary. In many practical uses [**12**], a gradient system (1.38) with an energy of the form

$$\mathcal{E}(u) = \int_\Omega |u_0 - Ru|^2\,dx + \lambda \int_\Omega \rho(|\nabla u|)\,dx \tag{1.39}$$

serves to ensure good digital quality. If the function ρ is convex, then our convergence results apply and ensure the convergence of compact solutions. However, as reported in [**12**], a surprising numerical fact is that one can observe very good numerical results using nonconvex functions ρ (e.g., $\rho(s) = s^2/(1+s^2)$) or functions ρ (e.g., $\rho(s) = s^p$ with $p \neq 2$) such that the corresponding gradient systems are degenerate. The failure of convexity of the functions ρ (and thus the failure of ellipticity of linearization of the gradient map about an equilibrium) makes the mathematical treatment of the convergence problem very difficult. Results in this direction are very rare. It seems that the result of E. Feireisl and F. Simondon [**40**] for the porous medium equations is the only such known results. Thus, a further deep study is needed.

- Applications to phase-field models [**15**]. In the very recent works [**2, 3, 4, 37, 38**] of S. Aizicovici, E. Feireisl, H. Petzeltová and F. Issard-Roch, the convergence to equilibria of solutions to many phase-field models with analytic nonlinearities are established, by virtue of certain suitable gradient inequalities. These results provide the prototype for further studies.
- Application to optimization problems. In solving optimization problems, there are two kinds of *steepest descent methods*: the continuous one and the discrete one. As seen in Theorem 4.2, the validity of gradient inequalities ensures the convergence to one single critical point of the continuous steepest descent method. However, from the point of view of practical computation, the problem concerning the convergence of the discrete steepest descent method is more interesting. In the recent work [**68**] the author himself has been able, for the finite-dimensional case, to establish the result that the validity of gradient inequalities implies the convergence to one single limit for minimizing sequences generated by discrete gradient methods. We expect that the ideas used in [**68**] can be extended to the infinite-dimensional case. It might be thought that the convergence result for the continuous gradient methods implies, to some extent, the convergence of discrete gradient methods. However, as shown in [**68**], there are very crucial differences between the continuous and discrete gradient methods.