A Brief History

Differential geometry began as the study of curves and surfaces in 3-space. The concept of a Riemannian manifold, an abstract manifold with a metric structure, was first formulated by Riemann in 1868 to generalize these classical objects. Naturally there arose the question of whether an abstract Riemannian manifold is simply a submanifold of some Euclidean space with its induced metric. This is the isometric embedding question. It has assumed a position of fundamental conceptual importance in differential geometry.

We briefly review four important aspects of the field of isometric embeddings of Riemannian manifolds in Euclidean space.

1. General Isometric Embedding of Riemannian Manifolds.

In 1873, Schlaefli made the following conjecture: *Every n-dimensional smooth Riemannian manifold admits a smooth local isometric embedding in $\mathbb{R}^{s_n}$, with $s_n = n(n+1)/2$.* It was more than 50 years later that an affirmative answer was given for the analytic case successively by Janet and Cartan; they proved in 1926-1927 that any analytic n-dimensional Riemannian manifold has a local analytic isometric embedding in $\mathbb{R}^{s_n}$. Schlaefli's question for the smooth case when $n = 2$ was given renewed attention by Yau in the 1980's and 1990's.

For the global isometric embedding, Nash in 1954 and Kuiper in 1955 proved the existence of a global C^1 isometric embedding of n-dimensional Riemannian manifolds in $\mathbb{R}^{2n+1}$. For smooth isometric embeddings, the difficulty arises from the loss of derivatives in the attempt to solve the nonlinear equations corresponding to the isometric embedding. In an outstanding paper published in 1956, Nash introduced an important technique of using smoothing operators to make up for the loss of derivatives. He proved that any smooth n-dimensional Riemannian manifold admits a (global) smooth isometric embedding in the Euclidean space $\mathbb{R}^N$, for $N = 3s_n + 4n$ in the compact case and $N = (n+1)(3s_n + 4n)$ in the noncompact case. The technique proves to be extremely useful in solving nonlinear differential equations. It has been modified by many people, including Moser and Hörmander, and is now known as the *hard* implicit function theorem, or Nash-Moser iteration.

Following Nash, one naturally looks for the smallest N. In his book *Partial Differential Relations*, published in 1986, Gromov studied various problems related to the isometric embedding of Riemannian manifolds. He proved that $N = s_n + 2n + 3$ is enough for the compact case. Then in 1989, Günther vastly simplified Nash's original proof. By rewriting the differential equations cleverly, he was able to employ the contraction mapping principle, instead of the Nash-Moser iteration, to construct solutions. Günther also improved the dimension of the target space to

$N = \max\{s_n + 2n, s_n + n + 5\}$. It is still not clear whether this is the best possible result on the dimension of the ambient space.

In 1970, Gromov and Rokhlin and Greene, independently, proved that any n-dimensional smooth Riemannian manifold admits a smooth isometric embedding in $\mathbb{R}^{s_n+n}$ *locally*. The proof is based on the iteration scheme introduced by Nash.

For 2-dimensional Riemannian manifolds, better results are available. First, according to Gromov or Günther, any compact 2-dimensional smooth Riemannian manifold can be isometrically embedded in $\mathbb{R}^{10}$ smoothly. When the manifolds have some special property, the dimension of the ambient space can be lowered. On the other hand, in 1973 Poznyak proved that any smooth 2-dimensional Riemannian manifold can be locally isometrically embedded in $\mathbb{R}^4$ smoothly. We are more interested in the question of whether we can isometrically embed a 2-dimensional Riemannian manifold in $\mathbb{R}^3$, locally or globally.

2. Local Isometric Embedding of Surfaces in $\mathbb{R}^3$.

It was known to Darboux in 1894 that isometrically embedding a surface locally in $\mathbb{R}^3$ is equivalent to finding a local solution of some nonlinear equation of the Monge-Ampère type. Such an equation is now called the Darboux equation; its type is determined by the sign of the Gauss curvature K. It is elliptic if K is positive and hyperbolic if K is negative. It is degenerate if K vanishes. Remarkably, even today, the local solvability of the Darboux equation in the general case is not covered by any known theory of partial differential equations.

The first attempt to establish the local isometric embedding of surfaces in $\mathbb{R}^3$ was not through the Darboux equation. In 1908, Levi proved the local isometric embedding in $\mathbb{R}^3$ of surfaces with negative curvature by using the equations of virtual asymptotes. It was several decades later that the Darboux equation attracted the attention of those interested in the isometric embedding. In the early 1950's, Hartman and Winter studied the Darboux equation in the case when the Gauss curvature K does not vanish and proved the existence of local solutions to the Darboux equation and hence the local isometric embedding in $\mathbb{R}^3$ in that case.

For a long time the case when K vanishes did not give way to the efforts of mathematicians. In 1985 and 1986, Lin made important breakthroughs. By a delicate analysis, he obtained the existence of sufficiently smooth local solutions of the Darboux equation and hence a sufficiently smooth isometric embedding in a neighborhood of p for the following two cases: $K(p) = 0$ and $dK(p) \neq 0$, or $K \geq 0$ in a neighborhood of the point p. Later, in 1987, Nakamura proved the existence of the smooth local isometric embedding if $K(p) = 0$, $dK(p) = 0$ and $\text{Hess}K(p) < 0$. Evidently, K is nonpositive near the point p and the leading part of K is an irreducible quadratic polynomial. For the case of nonpositive Gauss curvature, Hong in 1991 also proved the existence of a sufficiently smooth local isometric embedding in a neighborhood of p if $K = hg^{2m}$, where h is a negative function and g is a function with $g(p) = 0$ and $dg(p) \neq 0$. In 2005, Han gave a simple proof of Lin's result that g admits a sufficiently smooth local isometric embedding in a neighborhood of p if $K(p) = 0$ and $dK(p) \neq 0$. All these results are based on a careful study of the Darboux equation.

In 2003, Han, Hong and Lin studied the local isometric embedding of surfaces in $\mathbb{R}^3$ by a different method. Instead of the Darboux equation, they studied a quasilinear differential system equivalent to the Gauss-Codazzi system and proved the local isometric embedding for a large class of metrics with nonpositive Gauss curvature. They established the isometric embedding if some directional derivative of the Gauss curvature has a simple characterization for its zero set. This gives the results of Nakamura and Hong as special cases.

On the other hand, Pogorelov in 1972 constructed a $C^{2,1}$ metric g in $B_1 \subset \mathbb{R}^2$ with a sign-changing Gauss curvature such that (B_r, g) cannot be realized as a C^2 surface in $\mathbb{R}^3$ for any $r > 0$.

3. Global Isometric Embedding of Surfaces in $\mathbb{R}^3$.

In 1916, Weyl posed the following problem. Does every smooth metric on $\mathbb{S}^2$ with pointwise positive Gauss curvature admit a smooth isometric embedding in $\mathbb{R}^3$? The first attempt to solve the problem was made by Weyl himself. He used the continuity method and obtained a priori estimates up to the second derivatives. Twenty years later, Lewy solved the problem for an analytic metric g. In 1953, Nirenberg gave a complete solution under the very mild hypothesis that the metric g has continuous fourth derivatives. The result was extended to the case of continuous third derivatives of the metric by Heinz in 1962. In a completely different approach to the problem, Alexandroff in 1942 obtained a generalized solution of Weyl's problem as a limit of polyhedra. The regularity of this generalized solution was proved by Pogorelov in the late 1940's. In 1994 and 1995, Guan and Li, and Hong and Zuily independently generalized Nirenberg's result for metrics on $\mathbb{S}^2$ with nonnegative Gauss curvature.

The study of negatively curved surfaces in $\mathbb{R}^3$ is closely related to the interpretation of non-Euclidean geometry. The investigation of the isometric immersion of metrics with negative curvature goes back to Hilbert. He proved in 1901 that the full hyperbolic plane cannot be isometrically immersed in $\mathbb{R}^3$. A next natural step is to extend such a result to complete surfaces whose Gauss curvature is bounded above by a negative constant. The final solution of this problem was obtained by Efimov in 1963, more than sixty years later. Efimov proved that any complete negatively curved smooth surface does not admit a C^2 isometric immersion in $\mathbb{R}^3$ if its Gauss curvature is bounded away from zero. Efimov's proof is very delicate and complicated. In the years following, Efimov extended his result in several ways.

Before the 1970's, the study of negatively curved surfaces was largely directed at nonexistence of isometric immersions in $\mathbb{R}^3$. As to existence, no result for complete negatively curved surfaces was known. In the 1980's, Yau proposed to find a sufficient condition for complete negatively curved surfaces to be isometrically immersed in $\mathbb{R}^3$. In 1993, Hong proved that complete negatively curved surfaces can be isometrically immersed in $\mathbb{R}^3$ if the Gauss curvature decays at a certain rate at infinity. His discussion was based on a differential system equivalent to the Gauss-Codazzi system.

Closely related to the global isometric embedding problem is the rigidity question. The first rigidity result was proved by Cohn-Vossen in 1927; this states that any two closed isometric analytic convex surfaces are congruent to each other. His proof was later considerably shortened by Zhitomirsky. In 1943, Herglotz gave a very short proof of the rigidity, assuming that the surfaces are three times continuously differentiable. Finally in 1962 it was extended by Sacksteder to surfaces with no more than two times continuously differentiable metrics. For compact surfaces with Gauss curvature of mixed sign, Alexandrov in 1938 introduced a class of surfaces satisfying some integral condition for its Gauss curvature and proved that any compact analytic surface with this condition is rigid. In 1963, Nirenberg generalized this result to smooth surfaces. To do this, he needed some extra conditions, one of which is not intrinsic.

4. Local Isometric Embedding of n-Dimensional Riemannian Manifolds in $\mathbb{R}^{s_n}$.

For the local isometric embedding of smooth n-dimensional Riemannian manifolds in $\mathbb{R}^{s_n}$, the case $n \geq 3$ is sharply different from the case $n = 2$. For $n = 2$, there is only one curvature function, and it determines the type of the Darboux equation, that is, the equation for the isometric embedding of 2-dimensional Riemannian manifolds in $\mathbb{R}^3$. For $n \geq 3$, the role of curvature functions is not clear.

In 1983, Bryant, Griffiths and Yang studied the characteristic varieties associated with the differential systems for the isometric embedding in $\mathbb{R}^{s_n}$ of smooth n-dimensional Riemannian manifolds. They proved that these characteristic varieties are never empty if $n \geq 3$. This implies in particular that the differential systems for the isometric embedding in $\mathbb{R}^{s_n}$ of n-dimensional Riemannian manifolds are never elliptic for $n \geq 3$, no matter what assumptions are put on curvatures. This is a sharp difference from the case $n = 2$. A related result is the local rigidity proved by Berger, Bryant and Griffiths in 1983.

For $n = 3$, Bryant, Griffiths and Yang in 1983 studied the characteristic varieties in detail. They were able to classify the type of differential system for the isometric embedding by its curvature functions. Here an important quantity is the signature of the curvature tensor viewed as a symmetric linear operator acting on the space of 2-forms. They proved that any smooth 3-dimensional Riemannian manifold admits a smooth local isometric embedding in $\mathbb{R}^6$ if the signature is different from (0,0) and (0,1). Then in 1989, Nakamura and Maeda proved the existence of the smooth local isometric embedding in $\mathbb{R}^6$ of smooth 3-dimensional Riemannian manifolds if the curvature tensors are not zero. The key step in the proof is the local existence of solutions to differential systems of principal type.

CHAPTER 5

Gauss Curvature Changing Sign Cleanly

In this chapter, we discuss the local isometric embedding in $\mathbb{R}^3$ of surfaces with Gauss curvature changing sign cleanly. The main result is the following theorem of Lin [**155**].

THEOREM 5.0.1. *Suppose g is a C^r metric, $r \geq 9$, in a neighborhood of $0 \in \mathbb{R}^2$ with $K(0) = 0$ and $\nabla K(0) \neq 0$. Then g admits a C^{r-6} isometric embedding of a neighborhood of 0 into $\mathbb{R}^3$.*

The proof we present is due to Han [**80**]. As mentioned in the introduction to Chapter 4, the proof of any local isometric embedding result consists of four steps. For the proof of Theorem 5.0.1, we will be considering the Darboux equation. In the second step, we introduce an initial approximate solution and by an appropriate scaling we transform the Darboux equation into a symmetric positive differential system. The introduction of the initial approximate solution involves some technical calculations and, at first, looks complicated. It yields the most efficient way to transform the Darboux equation into a symmetric positive differential system such that an admissible boundary condition can be easily prescribed. In the third step, we study the corresponding symmetric positive linear differential system and derive a priori estimates for its solutions. In the final step, we use the contraction mapping principle to solve the nonlinear symmetric differential system, derived from the Darboux equation, and hence prove the existence of the local isometric embedding.

We introduce the initial approximate solution in Section 5.1 and carry out the iteration process in Section 5.2. We discuss the existence and the regularity of solutions to general symmetric positive linear differential systems in Section 5.3.

In this chapter, we denote by $\|\cdot\|_s$ the H^s-norm. Thus $\|\cdot\|_0$ is the L^2-norm.

5.1. The Setting

Suppose $g = g_{ij}dx_i dx_j$ is a C^r metric in a neighborhood of $0 \in \mathbb{R}^2$. We consider the Darboux equation

$$\tilde{\mathcal{F}}(z) \equiv \det(\partial_{ij}z - \Gamma_{ij}^k \partial_k z) - K \det(g_{ij})(1 - g^{ij}\partial_i z \partial_j z) = 0, \tag{5.1.1}$$

with the condition $g^{ij}\partial_i z \partial_j z < 1$. The coefficients in (5.1.1) are C^{r-2}. By choosing a normal coordinate system (x_1, x_2), we have

$$g_{ij}(0) = \delta_{ij}, \quad \partial_k g_{ij}(0) = 0 \tag{5.1.2}$$

and

$$K(x_1, x_2) = ax_2 + O(x_1^2 + x_2^2),$$

for some constant $a > 0$. We look for a trial solution of the form

$$z(x) = \hat{w}(x) + \varepsilon^\beta w\left(\frac{x}{\varepsilon^\alpha}\right),$$

for some positive constants α and β.

We first choose $\hat{w}$. An obvious choice for $\hat{w}$ is

$$\hat{w}(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{a}{6}x_2^3. \tag{5.1.3}$$

This is based on the following simple observation

$$\det(\hat{w}_{ij}) = ax_2,$$

where the right side is the leading part of K. Such a choice of $\hat{w}$ has a geometric interpretation. The graph of the function

$$z = \frac{1}{2}x_1^2 + \frac{a}{6}x_2^3$$

is the surface obtained by a translation of the parabola $z = x_1^2/2$ along the cubic parabola $z = ax_2^3/6$. The parabola

$$z = \frac{1}{2}x_1^2,\ x_2 = 0$$

lying on this surface is a zero-curvature line that divides the surface into an elliptic ($x_2 > 0$) part and a hyperbolic ($x_2 < 0$) part.

For a technical reason, which will become apparent later, we need to modify the above expression. To be precise, we set

$$\hat{w}(x_1, x_2) = \frac{1}{2}x_1^2 + \gamma x_1 x_2 + \frac{\gamma^2}{2}x_2^2 + \frac{a}{6}x_2^3 + P_4,$$

where γ is a constant and P_4 is a homogeneous polynomial of degree 4, to be determined. Obviously, if $P_4 = 0$, we would have

$$\tilde{\mathcal{F}}(\hat{w}) = O(x_1^2 + x_2^2).$$

In the following, we choose P_4 such that

$$\tilde{\mathcal{F}}(\hat{w}) = O((x_1^2 + x_2^2)^{\frac{3}{2}}).$$

To see this, we expand the expression for $\tilde{\mathcal{F}}(\hat{w})$. By a straightforward calculation and (5.1.2), it is easy to see that the constant term and the homogeneous linear term in $\tilde{\mathcal{F}}(\hat{w})$ are zero and that the homogeneous quadratic term is given by

$$\begin{aligned}
&\partial_{22}P_4 - l(\Gamma_{22}^1)(x_1 + \gamma x_2) - l(\Gamma_{22}^2)(\gamma x_1 + \gamma^2 x_2)\\
&+ \gamma^2\left(\partial_{11}P_4 - l(\Gamma_{11}^1)(x_1 + \gamma x_2) - l(\Gamma_{11}^2)(\gamma x_1 + \gamma^2 x_2)\right)\\
&- 2\gamma\left(\partial_{12}P_4 - l(\Gamma_{12}^1)(x_1 + \gamma x_2) - l(\Gamma_{12}^2)(\gamma x_1 + \gamma^2 x_2)\right) - g_0 q(K)\\
=&\partial_{22}P_4 - 2\gamma\partial_{12}P_4 + \gamma^2\partial_{11}P_4 + \text{a known homogenous quadratic polynomial},
\end{aligned}$$

where $l(f)$ denotes the homogeneous linear part of f and $q(f)$ the homogeneous quadratic part of f. Now for any fixed γ, we can easily find P_4 to make such a quadratic part vanish. The choice of P_4 is not unique, there being three algebraic equations for the five coefficients of P_4. Note $\tilde{\mathcal{F}}(\hat{w})$ is C^{r-5} in a neighborhood of 0, since it involves the cubic part of K.

To proceed, we temporarily replace $x \in \mathbb{R}^2$ by $\tilde{x} \in \mathbb{R}^2$ and write $\widetilde{\partial}_i$ instead of $\partial_{\widetilde{x}_i}$. Then (5.1.1) has the form

$$\tilde{\mathcal{F}}(z) = \det(\widetilde{\partial}_{ij} z - \Gamma_{ij}^k \widetilde{\partial}_k z) - K \det(g_{ij}) \left(1 - g^{ij} \widetilde{\partial}_i z \widetilde{\partial}_j z\right). \tag{5.1.4}$$

All functions are evaluated at $\tilde{x}$.

For some positive constants α and β, we set

$$\tilde{x} = \varepsilon^\alpha x,$$

and

$$z(\tilde{x}) = \hat{w}(\tilde{x}) + \varepsilon^\beta w(x) = \hat{w}(\tilde{x}) + \varepsilon^\beta w\left(\frac{\tilde{x}}{\varepsilon^\alpha}\right),$$

where

$$\hat{w}(\tilde{x}_1, \tilde{x}_2) = \frac{1}{2}\tilde{x}_1^2 + \gamma \tilde{x}_1 \tilde{x}_2 + \frac{\gamma^2}{2}\tilde{x}_2^2 + \frac{a}{6}\tilde{x}_2^3 + P_4(\tilde{x}_1, \tilde{x}_2). \tag{5.1.5}$$

This satisfies

$$\tilde{\mathcal{F}}(\hat{w}) = O((\tilde{x}_1^2 + \tilde{x}_2^2)^{\frac{3}{2}}). \tag{5.1.6}$$

Now we evaluate $\tilde{\mathcal{F}}(z)$ in terms of w. By a straightforward calculation, we obtain

$$\begin{aligned} \tilde{\mathcal{F}}(z) =& \varepsilon^{\beta-2\alpha} a_{ij}(\partial_{ij} w - \varepsilon^\alpha \Gamma_{ij}^k \partial_k w) + \varepsilon^{2\beta-4\alpha} \det(\partial_{ij} w - \varepsilon^\alpha \Gamma_{ij}^k \partial_k w) \\ &+ \tilde{\mathcal{F}}(\hat{w}) + \varepsilon^{\beta-\alpha} K \det(g_{ij}) \left(2 g_{ij} \widetilde{\partial}_i \hat{w} \partial_j w + \varepsilon^{\beta-\alpha} g^{ij} \partial_i w \partial_j w\right), \end{aligned} \tag{5.1.7}$$

where (a_{ij}) is the cofactor matrix of $(\tilde{\partial}_{ij}\hat{w} - \Gamma_{ij}^k \tilde{\partial}_k \hat{w})$. For later purposes, we record the expressions for the a_{ij}:

$$\begin{aligned} a_{11} &= \gamma^2 + a\varepsilon^\alpha x_2 + \widetilde{\partial}_{22} P_4 - \Gamma_{22}^k \widetilde{\partial}_k \hat{w}, \\ a_{12} &= -(\gamma + \widetilde{\partial}_{12} P_4 - \Gamma_{12}^k \widetilde{\partial}_k \hat{w}), \\ a_{22} &= 1 + \widetilde{\partial}_{11} P_4 - \Gamma_{11}^k \widetilde{\partial}_k \hat{w}. \end{aligned} \tag{5.1.8}$$

To analyze (5.1.7), we first write

$$\varepsilon^{\beta-2\alpha} = \varepsilon^{\beta-\alpha} \cdot \frac{1}{\varepsilon^\alpha},$$

and then try to factor out the expression $\varepsilon^{\beta-\alpha}$ in $\tilde{\mathcal{F}}(z)$. For the ε factor in the second term in the right side of (5.1.7), we write

$$\varepsilon^{2\beta-4\alpha} = \varepsilon^{\beta-\alpha} \cdot \varepsilon^{\beta-3\alpha}.$$

If $\tilde{\mathcal{F}}(\hat{w})$ is quadratic, we would have

$$\tilde{\mathcal{F}}(\hat{w}) = O(\varepsilon^{2\alpha}) = \varepsilon^{\beta-\alpha} O(\varepsilon^{3\alpha-\beta}).$$

However, it is impossible to make both $\varepsilon^{\beta-3\alpha}$ and $\varepsilon^{3\alpha-\beta}$ small. We need a higher order of ε in $\tilde{\mathcal{F}}(\hat{w})$. It is for this reason that we introduced P_4 in $\hat{w}$ in (5.1.5)! With (5.1.6), we have

$$\tilde{\mathcal{F}}(\hat{w}) = O(\varepsilon^{3\alpha}) = \varepsilon^{\beta-\alpha} O(\varepsilon^{4\alpha-\beta}).$$

So we require

$$\beta - 3\alpha > 0 \text{ and } 4\alpha - \beta > 0,$$

or

$$3\alpha < \beta < 4\alpha.$$

Next set

$$\mathcal{F}(w) = \frac{1}{\varepsilon^{\beta-\alpha}}\tilde{\mathcal{F}}(z).$$

It has the following form

$$\begin{aligned}\mathcal{F}(w) =&\frac{1}{\varepsilon^{\alpha}}a_{ij}(\partial_{ij}w - \varepsilon^{\alpha}\Gamma_{ij}^{k}\partial_k w) + \varepsilon^{\beta-3\alpha}\det(\partial_{ij}w - \varepsilon^{\alpha}\Gamma_{ij}^{k}\partial_k w)\\ &+ \frac{1}{\varepsilon^{\beta-\alpha}}\tilde{\mathcal{F}}(\hat{w}) + K\det(g_{ij})\left(2g_{ij}\widetilde{\partial}_i\hat{w}\partial_j w + \varepsilon^{\beta-\alpha}g^{ij}\partial_i w\partial_j w\right).\end{aligned} \tag{5.1.9}$$

The last term, $K = K(\varepsilon^{\alpha}\tilde{x})$, has a factor of ε^{α}. For the first term, we write

$$\frac{1}{\varepsilon^{\alpha}}a_{ij}(\partial_{ij}w - \varepsilon^{\alpha}\Gamma_{ij}^{k}\partial_k w) = \frac{1}{\varepsilon^{\alpha}}a_{ij}\partial_{ij}w - a_{ij}\Gamma_{ij}^{k}\partial_k w,$$

and note that Γ_{ij}^{k} has a factor of ε^{α}, since $\Gamma_{ij}^{k}(0) = 0$. In (5.1.8), the expression $\widetilde{\partial}_{ij}P_4 - \Gamma_{ij}^{k}\widetilde{\partial}_k\hat{w}$ is evaluated at $\tilde{x}$ and hence has a factor of $\varepsilon^{2\alpha}$.

Henceforth, we always take

$$\alpha = 2,\ \beta = 7.$$

Then $\mathcal{F}(w)$ can be put into the form

$$\mathcal{F}(w) = \frac{1}{\varepsilon^2}\left\{(\gamma^2 + a\varepsilon^2 x_2)\partial_{11}w - 2\gamma\partial_{12}w + \partial_{22}w\right\} - \varepsilon\tilde{f}(\varepsilon, x, Dw, D^2w), \tag{5.1.10}$$

where $\tilde{f}$ is a quadratic polynomial of Dw and D^2w, with coefficients given by C^{r-5} functions of ε and x.

5.2. Iterations

In this section, we solve, in a neighborhood of the origin, the nonlinear differential equation

$$\frac{1}{\varepsilon^2}\left\{\partial_{tt}w - 2\gamma\partial_{xt}w + (\gamma^2 + a\varepsilon^2 t)\partial_{xx}w\right\} - \varepsilon\tilde{f}(\varepsilon, x, t, Dw, D^2w) = 0, \tag{5.2.1}$$

where $a > 0$ is a fixed constant, γ is a constant to be determined and $\tilde{f}$ is a function smooth in Dw, D^2w and C^{r-5} in ε, x and t. The equation (5.2.1) arises from (5.1.10) in the coordinate system (x,t) instead of (x_1, x_2).

We may solve $\partial_{tt}w$ from (5.2.1) in a neighborhood of the origin for small $\varepsilon > 0$ and then substitute it into $\tilde{f}$. Hence we assume there is no $\partial_{tt}w$ in the expression of $\tilde{f}$. This simple substitution is important for later discussions. In fact, it was already used in the discussion of the negative Gauss curvature in Section 4.2.

Set

$$u = \partial_t w, \quad v = \partial_x w.$$

Then the equation (5.2.1) is equivalent to the following system:

$$\begin{aligned}&\frac{1}{\varepsilon^2}\left\{\partial_t u - 2\gamma\partial_x u + (\gamma^2 + a\varepsilon^2 t)\partial_x v\right\} - \varepsilon\tilde{f}(\varepsilon, x, t, u, v, \partial_x u, \partial_x v) = 0,\\ &\quad\partial_t v - \partial_x u = 0.\end{aligned}$$

We put it in the matrix form

$$\frac{1}{\varepsilon^2}\partial_t\begin{pmatrix}u\\ v\end{pmatrix} + \frac{1}{\varepsilon^2}\begin{pmatrix}-2\gamma & \gamma^2 + a\varepsilon^2 t\\ -1 & 0\end{pmatrix}\partial_x\begin{pmatrix}u\\ v\end{pmatrix} - \varepsilon\begin{pmatrix}\tilde{f}\\ 0\end{pmatrix} = 0, \tag{5.2.2}$$

where $\tilde{f} = \tilde{f}(\varepsilon, x, t, u, v, \partial_x u, \partial_x v)$. We need to emphasize that no t-derivatives of u and v appear in $\tilde{f}$.

Set

$$\Omega = \{(x,t); |x| \le 1, \ |t| \le 1\}.$$

Then the system (5.2.2) is well defined in Ω if ε is small. Since we are concerned only with local solutions, we replace $\tilde{f}$ in (5.2.2) by $f = \chi \tilde{f}$ for some cutoff function $\chi = \chi(x) \in C_0^\infty(\mathbb{R})$ with $\chi(x) = 1$ for $|x| \le 1/2$ and $\chi(x) = 0$ for $|x| \ge 1$.

In the following, we solve the boundary value problem

$$\frac{1}{\varepsilon^2}\partial_t \begin{pmatrix} u \\ v \end{pmatrix} + \frac{1}{\varepsilon^2}\begin{pmatrix} -2\gamma & \gamma^2 + a\varepsilon^2 t \\ -1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix} - \varepsilon \begin{pmatrix} f \\ 0 \end{pmatrix} = 0 \ \text{ in } \Omega, \tag{5.2.3}$$

$$v = 0 \text{ for } t = 1 \text{ and } u = 0 \text{ for } t = -1, \tag{5.2.4}$$

$$u \text{ and } v \text{ are 2-periodic in } x.$$

The choice of the boundary condition will be apparent later.

Through the end of the chapter, all functions in Ω will be 2-periodic in x.

The system (5.2.3) is a fully nonlinear differential system, and we transform it into a quasilinear system. Such a trick was also employed in the previous chapter. To do this, we set

$$\tilde{u} = \partial_x u, \ \tilde{v} = \partial_x v.$$

Then f in (5.2.3) can be expressed as

$$f = f(\varepsilon, x, t, u, v, \tilde{u}, \tilde{v}).$$

Differentiating (5.2.3) with respect to x, we get

$$\frac{1}{\varepsilon^2}\partial_t \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \frac{1}{\varepsilon^2}\begin{pmatrix} -2\gamma - \varepsilon^3 \partial_{\tilde{u}} f & \gamma^2 + a\varepsilon^2 t - \varepsilon^3 \partial_{\tilde{v}} f \\ -1 & 0 \end{pmatrix} \partial_x \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} - \varepsilon \begin{pmatrix} f_1 \\ 0 \end{pmatrix} = 0, \tag{5.2.5}$$

where

$$f_1 = \partial_x f + \partial_u f \tilde{u} + \partial_v f \tilde{v}$$

is smooth in $u, v, \tilde{u}$ and $\tilde{v}$ and C^{r-6} in ε, x and t.

Now we couple (5.2.3) and (5.2.5) as follows. Set

$$U = (u, v, \tilde{u}, \tilde{v})^T. \tag{5.2.6}$$

Then U satisfies

$$\frac{1}{\varepsilon^2}\partial_t U + \frac{1}{\varepsilon^2}\tilde{B}_\varepsilon(U)\partial_x U - \varepsilon \tilde{F}_\varepsilon(U) = 0, \tag{5.2.7}$$

where

$$\tilde{B}_\varepsilon(U) = \begin{pmatrix} -2\gamma & \gamma^2 + a\varepsilon^2 t & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -2\gamma - \varepsilon^3 \partial_{\tilde{u}} f & \gamma^2 + a\varepsilon^2 t - \varepsilon^3 \partial_{\tilde{v}} f \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and

$$\tilde{F}_\varepsilon(U) = \begin{pmatrix} f & 0 & f_1 & 0 \end{pmatrix}^T.$$

This is a quasilinear differential system! All the known functions in (5.2.7) are 2-periodic in x and C^{r-6} in ε, x and t.

Next, we transform (5.2.7) to a symmetric differential system. To this end, we multiply (5.2.7) by the diagonal matrix

$$\mathrm{diag}\big(1, -(\gamma^2 + a\varepsilon^2 t), 1, -(\gamma^2 + a\varepsilon^2 t - \varepsilon^3 \partial_{\tilde v} f)\big).$$

Then we obtain

$$A_\varepsilon(U)\partial_t U + B_\varepsilon(U)\partial_x U = \varepsilon \tilde F_\varepsilon(U), \tag{5.2.8}$$

where

$$A_\varepsilon(U) = \frac{1}{\varepsilon^2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(\gamma^2 + a\varepsilon^2 t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(\gamma^2 + a\varepsilon^2 t - \varepsilon^3 \partial_{\tilde v} f) \end{pmatrix}, \tag{5.2.9}$$

and

$$\begin{aligned} B_\varepsilon(U) \\ = \frac{1}{\varepsilon^2}\begin{pmatrix} -2\gamma & \gamma^2 + a\varepsilon^2 t & 0 & 0 \\ \gamma^2 + a\varepsilon^2 t & 0 & 0 & 0 \\ 0 & 0 & -2\gamma - \varepsilon^3 \partial_{\tilde u} f & \gamma^2 + a\varepsilon^2 t - \varepsilon^3 \partial_{\tilde v} f \\ 0 & 0 & \gamma^2 + a\varepsilon^2 t - \varepsilon^3 \partial_{\tilde v} f & 0 \end{pmatrix}. \end{aligned} \tag{5.2.10}$$

Both matrices $A_\varepsilon(U)$ and $B_\varepsilon(U)$ are symmetric. To proceed, we consider the transform

$$U \mapsto e^{\varepsilon^2 t} U.$$

Then (5.2.8) is changed to

$$A_\varepsilon(U)\partial_t U + B_\varepsilon(U)\partial_x U + C_\varepsilon(U)U = \varepsilon F_\varepsilon(U),$$

where

$$C_\varepsilon(U) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(\gamma^2 + a\varepsilon^2 t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -(\gamma^2 + a\varepsilon^2 t - \varepsilon^3 \partial_{\tilde v} f) \end{pmatrix}. \tag{5.2.11}$$

Recalling (5.2.4), we see that the corresponding boundary value problem for U (as given in (5.2.6)) is

$$A_\varepsilon(U)\partial_t U + B_\varepsilon(U)\partial_x U + C_\varepsilon(U)U = \varepsilon F_\varepsilon(U), \tag{5.2.12}$$

$$v = \tilde v = 0 \text{ for } t = 1 \text{ and } u = \tilde u = 0 \text{ for } t = -1. \tag{5.2.13}$$

Our goal here is to solve (5.2.12)-(5.2.13).

THEOREM 5.2.1. *For some fixed constant $M > 0$ and any small $\varepsilon > 0$, there exists a unique solution $U = (u, v, \tilde u, \tilde v)^T \in H^{r-6}$ of the system (5.2.12)-(5.2.13) satisfying $\|U\|_{r-6} \le M$.*

This then allows us to prove Theorem 5.0.1.

PROOF OF THEOREM 5.0.1. If $U = (u, v, \tilde u, \tilde v)$ is a solution in H^{r-6}, then $(u, v, \partial_x u, \partial_x v)$ is also a solution. This is obvious from the derivation of the last two equations in (5.2.12) from the first two equations. Hence by the uniqueness, we have $\tilde u = \partial_x u$ and $\tilde v = \partial_x v$. This implies that (u, v) is an H^{r-5} solution of (5.2.3)-(5.2.4). We have found a solution $w \in H^{r-4} \subset C^{r-6}$ of (5.2.1) in a neighborhood of the origin, and then get a C^{r-6} solution z to (5.1.1). Obviously, z satisfies $g^{ij}\partial_i z \partial_j z < 1$ in a neighborhood of the origin for small ε. Therefore, by Lemma 3.1.1, there exists a C^{r-6} isometric embedding in $\mathbb{R}^3$ of g restricted to a neighborhood of the origin. □

In the rest of this section, we solve (5.2.12)-(5.2.13) and prove Theorem 5.2.1. We refer the reader to Section 5.3 for the existence and regularity of solutions to symmetric positive linear differential systems.

First we discuss the linear differential system corresponding to (5.2.12)-(5.2.13). For any fixed U, we consider

$$A_\varepsilon(U)\partial_t W + B_\varepsilon(U)\partial_x W + C_\varepsilon(U)W = \varepsilon F_\varepsilon(U). \tag{5.2.14}$$

Here, we write

$$W = (w_1, w_2, w_3, w_4)^T.$$

For simplicity, we write $A_\varepsilon, B_\varepsilon, C_\varepsilon$ instead of $A_\varepsilon(U), B_\varepsilon(U), C_\varepsilon(U)$, and when there is no ambiguity, we simply write A, B, C.

We verify that A, B and C satisfy the assumptions (H1)-(H3) of Theorem 5.3.7 (of Section 5.3) for any ε small. We set

$$\Theta_\varepsilon = C_\varepsilon + C_\varepsilon^T - \partial_t A_\varepsilon - \partial_x B_\varepsilon.$$

First we check (H1).

Lemma 5.2.2. *For any fixed γ with $\gamma^2 < a/2$, the matrix Θ_ε is positive definite in Ω for any small $\varepsilon > 0$ and any $U \in C^1(\Omega)$ with $|U|_{C^1} \le 1$.*

Proof. A straightforward calculation, with (5.2.9)-(5.2.11), shows

$$\Theta_\varepsilon = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & a - 2\gamma^2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & a - 2\gamma^2 \end{pmatrix} + O(\varepsilon).$$

This is positive definite if $a > 2\gamma^2$ and ε is small. □

Next, we check (H2).

Lemma 5.2.3. *For any fixed $\gamma > 0$, the matrix A_ε is invertible in Ω for any small $\varepsilon > 0$ and any $U \in L^\infty(\Omega)$ with $|U|_{L^\infty} \le 1$. Moreover,*

$$H^1_{\mathcal{L}} = \{W \in H^1; w_2 = w_4 = 0 \text{ on } t = 1, \quad w_1 = w_3 = 0 \text{ on } t = -1\}.$$

Proof. This is obvious from the definition of A_ε in (5.2.9). In fact, A_ε is diagonal and satisfies

$$\varepsilon^2 A_\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\gamma^2 \end{pmatrix} + O(\varepsilon^2),$$

if $|U|_{L^\infty} \le 1$ and ε is small. □

From now on, we fix γ such that

$$0 < \gamma < \sqrt{\frac{a}{2}}.$$

We study the following boundary value problem

$$A_\varepsilon(U)\partial_t W + B_\varepsilon(U)\partial_x W + C_\varepsilon(U)W = \varepsilon F_\varepsilon(U) \quad \text{in } \Omega, \tag{5.2.15}$$

$$w_2 = w_4 = 0 \text{ for } t = 1 \text{ and } w_1 = w_3 = 0 \text{ for } t = -1. \tag{5.2.16}$$

Setting

$$\tilde{s} = r - 6,$$

we note that $\tilde{s} \geq 3$. By the Sobolev embedding theorem, we may always take some constant $M > 0$ such that

$$\|U\|_{\tilde{s}} \leq M \quad \text{implies} \quad |U|_{C^1} \leq 1. \tag{5.2.17}$$

LEMMA 5.2.4. *There is a constant $\varepsilon_{\tilde{s}} > 0$ such that, for any $\varepsilon \in (0, \varepsilon_{\tilde{s}})$ and any $U \in H^{\tilde{s}}$ with $\|U\|_{\tilde{s}} \leq M$, there exists a unique solution $W \in H^{\tilde{s}}$ of (5.2.15)-(5.2.16). Moreover,*

$$\|W\|_{\tilde{s}} \leq M. \tag{5.2.18}$$

PROOF. By Lemma 5.2.2 and Lemma 5.2.3, the assumptions (H1) and (H2) are satisfied. We note that A and B are not uniformly bounded. However, their derivatives are. This is obvious from the definition of A and B in (5.2.9) and (5.2.10). Specifically, we have

$$\|\partial_x A\|_2 + \|\partial_x B\|_2 \leq c\varepsilon.$$

Moreover, we have

$$\|A^{-1}B\|_{\tilde{s}} + \|A^{-1}\|_{\tilde{s}} + \|C\|_{\tilde{s}} + \|\partial_x A\|_{\tilde{s}-1} + \|\partial_x B\|_{\tilde{s}-1} \leq c.$$

Therefore, by Theorem 5.3.7, for small ε there exists a unique solution $W \in H^{\tilde{s}}$ of (5.2.15)-(5.2.16) satisfying

$$\|W\|_{\tilde{s}} \leq c\varepsilon \|F_\varepsilon(U)\|_{\tilde{s}} \leq c\varepsilon(1 + \|U\|_{\tilde{s}}) \leq c\varepsilon(1 + M). \tag{5.2.19}$$

This implies (5.2.18) if ε is small. □

We now are ready to prove Theorem 5.2.1.

PROOF OF THEOREM 5.2.1. For any $U \in H^{\tilde{s}}$ with $\|U\|_{\tilde{s}} \leq M$, we define a map $\mathcal{T} : H^{\tilde{s}} \to H^{\tilde{s}}$ by $W = \mathcal{T}U$, where W is the unique solution of (5.2.15)-(5.2.16) as in Lemma 5.2.4. We have

$$\|\mathcal{T}U\|_{\tilde{s}} \leq M, \quad \text{if } \|U\|_{\tilde{s}} \leq M.$$

Now we claim that $\mathcal{T}$ is a contraction in the $H^{\tilde{s}-1}$-norm. Specifically, there holds for any $U_1, U_2 \in H^{\tilde{s}}$ with $\|U_i\|_{\tilde{s}} \leq M$

$$\|\mathcal{T}U_1 - \mathcal{T}U_2\|_{\tilde{s}-1} \leq \frac{1}{2}\|U_1 - U_2\|_{\tilde{s}-1}, \tag{5.2.20}$$

if ε is small.

To prove this, we consider the corresponding solution $W_i = \mathcal{T}U_i$, $i = 1, 2$, i.e.,

$$A(U_i)\partial_t W_i + B(U_i)\partial_x W_i + C(U_i)W_i = \varepsilon F(U_i) \quad \text{in } \Omega,$$
$$W_i \in H^{\tilde{s}}_{\mathcal{L}}.$$

By taking differences, we have

$$\begin{aligned} &A(U_1)\partial_t(W_1 - W_2) + B(U_1)\partial_x(W_1 - W_2) + C(U_1)(W_1 - W_2) \\ &= -\left(A(U_1) - A(U_2)\right)\partial_t W_2 - \left(B(U_1) - B(U_2)\right)\partial_x W_2 \\ &\quad - \left(C(U_1) - C(U_2)\right)W_2 + \varepsilon\left(F(U_1) - F(U_2)\right) \quad \text{in } \Omega, \end{aligned}$$
$$W_1 - W_2 \in H^{\tilde{s}}_{\mathcal{L}}.$$

We may apply $(5.3.7)_{\tilde{s}-1}$ to get

$$\begin{aligned} \|W_1 - W_2\|_{\tilde{s}-1} \le c\big\{&\varepsilon\|F(U_1) - F(U_2)\|_{\tilde{s}-1} + \|\left(C(U_1) - C(U_2)\right)W_2\|_{\tilde{s}-1} \\ &+ \|\left(A(U_1) - A(U_2)\right)\partial_t W_2\|_{\tilde{s}-1} + \|\left(B(U_1) - B(U_2)\right)\partial_x W_2\|_{\tilde{s}-1}\big\}. \end{aligned}$$

By a discussion similar to that in the proof of Theorem 4.2.1, we obtain

$$\|W_1 - W_2\|_{\tilde{s}-1} \le c\varepsilon\|U_1 - U_2\|_{\tilde{s}-1}.$$

We may choose ε small to get (5.2.20).

Now we are in a position to apply the contraction mapping principle to $\mathcal{T}$ to conclude the existence of a solution $U \in H^{\tilde{s}}$ to (5.2.12)-(5.2.13) with $\|U\|_{\tilde{s}} \le M$. □

5.3. Symmetric Positive Linear Differential Systems

In this section, we prove the existence of solutions of the symmetric positive linear differential systems. We assume $N \ge 1$ is an integer and

$$\Omega = \{(x, t); |x| \le 1, \ |t| \le 1\} \subset \mathbb{R}^2.$$

All functions on Ω *considered here are assumed 2-periodic in* x. We set

$$(U, V) = \int_\Omega U \cdot V \quad \text{for any functions } U, V : \Omega \to \mathbb{R}^N.$$

We suppose that A, B and C are $N \times N$ matrices whose entries are functions on Ω, at least C^1, and in addition that A and B are symmetric. We consider the following symmetric linear differential system for the N-vector W:

$$\mathcal{L}W \equiv A\partial_t W + B\partial_x W + CW = F. \tag{5.3.1}$$

Our basic assumption is the following. By introducing

$$\Theta \equiv \Theta_{\mathcal{L}} = C + C^T - \partial_t A - \partial_x B,$$

we assume Θ is positive definite in Ω; i.e., for some positive constant θ,

$$\Theta \ge \theta I. \tag{$H1$}$$

Then (5.3.1) is called a *symmetric positive linear differential system.* Next, we assume

$$A \text{ is not singular in } \Omega. \tag{$H2$}$$

Suppose the eigenvalues of A are given by $\lambda_1, \cdots, \lambda_N$. Then λ_i is never zero and has a fixed sign in Ω, for each $i = 1, \cdots, N$. Set

$$\Lambda_+ = \{i; \lambda_i > 0 \text{ in } \Omega\}, \quad \Lambda_- = \{i; \lambda_i < 0 \text{ in } \Omega\}.$$

For any N-vector W, there holds

$$W^T AW = \sum_{i=1}^{N} \lambda_i \left(l_i(W)\right)^2,$$

where $l_i(W)$ is a linear combination of components of W, with coefficients given by functions in Ω. Now we consider

$$H^1_{\mathcal{L}} = \{W \in H^1; l_i(W)(\cdot,1) = 0 \text{ if } i \in \Lambda_-,\ l_i(W)(\cdot,-1) = 0 \text{ if } i \in \Lambda_+\},$$

and for any integer $s \geq 1$,

$$H^s_{\mathcal{L}} = H^1_{\mathcal{L}} \cap H^s, \quad C^s_{\mathcal{L}} = H^1_{\mathcal{L}} \cap C^s.$$

Functions in $H^1_{\mathcal{L}}$ are said to satisfy an *admissible boundary condition* for $\mathcal{L}$. The reason for such a term arises from the following inequality:

$$W^T AW(\cdot,1) - W^T AW(\cdot,-1) \geq 0, \quad \text{for any } W \in H^1_{\mathcal{L}}. \tag{5.3.2}$$

This expression originates from integrating by parts $(W, \mathcal{L}W)$. In the following, we consider the boundary value problem

$$\mathcal{L}W \equiv A\partial_t W + B\partial_x W + CW = F \quad \text{in } \Omega, \tag{5.3.3}$$

$$W \in H^1_{\mathcal{L}}. \tag{5.3.4}$$

We also assume

$$\|\partial_x A\|_2 + \|\partial_x B\|_2 \leq \eta, \tag{$H3$}$$

where η is a small positive constant, yet to be determined.

We do not assume a uniform bound on A and B. For applications in Section 5.2, the L^∞-norms of A and B could be large.

In the following, we also need the adjoint operator $\mathcal{L}^*$ of $\mathcal{L}$. By a simple integration by parts, we note that $\mathcal{L}^*$ is of the form

$$\begin{aligned} \mathcal{L}^* V &\equiv -\partial_t(AV) - \partial_x(BV) + C^T V \\ &= -A\partial_t V - B\partial_x V + (C^T - \partial_t A - \partial_x B)V. \end{aligned} \tag{5.3.5}$$

Then we have

$$\begin{aligned} \Theta^* &= (C^T - \partial_t A - \partial_x B) + (C^T - \partial_t A - \partial_x B)^T + \partial_t A + \partial_x B \\ &= C^T + C - \partial_t A - \partial_x B = \Theta. \end{aligned}$$

Hence $\mathcal{L}$ and $\mathcal{L}^*$ are symmetric positive simultaneously. Corresponding to $\mathcal{L}^*$, we have

$$H^1_{\mathcal{L}^*} = \{W \in H^1; l_i(W)(\cdot,1) = 0 \text{ if } i \in \Lambda_+,\ l_i(W)(\cdot,-1) = 0 \text{ if } i \in \Lambda_-\},$$

and for $s \geq 1$

$$H^s_{\mathcal{L}^*} = H^1_{\mathcal{L}^*} \cap H^s, \quad C^s_{\mathcal{L}^*} = H^1_{\mathcal{L}^*} \cap C^s.$$

It is easy to see

$$(\mathcal{L}W, V) = (W, \mathcal{L}^* V) \quad \text{for any } W \in H^1_{\mathcal{L}}, V \in H^1_{\mathcal{L}^*}.$$

In fact, a simple integration by parts shows that for any $W, V \in H^1$

$$
\begin{aligned}
&(\mathcal{L}W, V) - (W, \mathcal{L}^* V) \\
&= \int_{|x|\le 1} W^T A V(x,1)dx - \int_{|x|\le 1} W^T A V(x,-1)dx \\
&= \sum_{i=1}^N \int_{\{|x|\le 1\}} \big(\lambda_i l_i(W) l_i(V)\big)(x,1)dx \\
&\quad - \sum_{i=1}^N \int_{\{|x|\le 1\}} \big(\lambda_i l_i(W) l_i(V)\big)(x,-1)dx.
\end{aligned} \tag{5.3.6}
$$

Now we derive a priori estimates for solutions to (5.3.3)-(5.3.4).

Lemma 5.3.1. *Suppose $m \ge 3$ is an integer and A, B, C are H^m matrices satisfying (H1)-(H3). Then there exists an $\eta_m > 0$ depending only on θ and m such that for any $\eta \in (0, \eta_m)$ and any $W \in H^{m+1}_{\mathcal{L}}$, $s = 0, 1, \cdots, m$,*

$$\|W\|_s \le c_s \|\mathcal{L}W\|_s, \tag{5.3.7}$$

where c_s is a positive constant which, for $s \le 3$, depends only on θ and the H^3-norms of $A^{-1}B$, A^{-1}, C, and, for $3 < s \le m$, depends only on θ, the H^s-norms of $A^{-1}B, A^{-1}, C$ and the H^{s-1}-norms of $\partial_x A$ and $\partial_x B$.

Proof. Set $F = \mathcal{L}W$. By taking the inner product of W and $\mathcal{L}W$ and integrating by parts, we get

$$(W, \Theta W) \le 2(W, F),$$

where there is no boundary integral on $|x| = 1$, since all functions are periodic in x, and there is also no boundary integral on $|t| = 1$ by (5.3.2) since $W \in H^1_{\mathcal{L}}$. Therefore we obtain

$$\frac{\theta}{2}\|W\|_0^2 \le (W, F). \tag{5.3.8}$$

Note that (5.3.8) implies $(5.3.7)_0$ for c_0 depending only on θ.

Next, we prove $(5.3.7)_s$ for $1 \le s \le m$. Note that $\partial_x^s W$ satisfies (4.4.2). We first estimate

$$\Theta_s = C_s + C_s^T - \partial_t A - \partial_x B.$$

By (4.4.3), we have

$$\Theta_s = \Theta + s(\partial_x B - \partial_x A A^{-1} B) + s(\partial_x B - \partial_x A A^{-1} B)^T.$$

By (H3) we may choose η small, depending only on θ and m, such that Θ_s is positive definite for any $s \le m$. Therefore (4.4.2) is a symmetric positive linear differential system for each such η. By applying $(5.3.7)_0$ to the solution $\partial_x^s W$ of (4.4.2), we obtain

$$\|\partial_x^s W\|_0 \le c_0 \|F_s\|_0,$$

where F_s is as in (4.4.4). By Lemma 4.4.4, we have

$$\|\partial_x^s W\|_0 \le c_s(\eta\|W\|_s + \|W\|_{s-1} + \|F\|_s), \tag{5.3.9}$$

where c_s is a positive constant which, for $s \le 3$, depends only on θ and the H^3-norms of $A^{-1}B, A^{-1}, C$, and, for $3 < s \le m$, depends only on θ, the H^s-norms of

$A^{-1}B, A^{-1}, C$ and the H^{s-1}-norms of $\partial_x A$ and $\partial_x B$. For other derivatives of order s, we simply apply Lemma 4.4.7 to get for each $s = 1, \cdots, m$

$$\sum_{j=1}^{s} \|\partial_x^{s-j}\partial_t^j W\|_0 \le c_s(\|\partial_x^s W\|_0 + \|W\|_{s-1} + \|F\|_{s-1}), \tag{5.3.10}$$

where c_s depends on H^3-norms of $A^{-1}B, A^{-1}C, A^{-1}$ for $s \le 3$ and on H^s-norms of $A^{-1}B, A^{-1}C, A^{-1}$ for $s > 3$. With (5.3.9) and (5.3.10), we get for $1 \le s \le m$

$$\|W\|_s \le c_s(\eta\|W\|_s + \|W\|_{s-1} + \|F\|_s)$$

or

$$\|W\|_s \le c_s(\|W\|_{s-1} + \|F\|_s),$$

if η is sufficiently small. Now a simple induction argument yields $(5.3.7)_s$ for $1 \le s \le m$. $\square$

REMARK 5.3.2. We need to emphasize that c_s in Lemma 5.3.1 does not depend on either $|A|_{L^\infty}$ or $|B|_{L^\infty}$. See Remark 4.4.2.

REMARK 5.3.3. In Lemma 5.3.1, the assumption that $A, B, C \in H^m$ is essential. This is the minimal regularity assumption on A, B, C for the desired result. However, the assumption on W can be improved. In fact, Lemma 5.3.1 still holds if we merely assume $W \in H^m_{\mathcal{L}}$ and $\mathcal{L}W \in H^m$. This is easily proved by the difference quotient or using mollifiers. Later on, we get the a priori estimate (5.3.7) under a weaker assumption as a corollary of the existence result. See Theorem 5.3.7.

Now we consider transformations for the system (5.3.1). For a nonsingular matrix M in Ω, we set

$$W = MU. \tag{5.3.11}$$

Then we have

$$\mathcal{L}(MU) = AM\partial_t U + BM\partial_x U + (A\partial_t M + B\partial_x M + CM)U.$$

To make this system symmetric, we multiply by M^T to get

$$\mathcal{L}_M U \equiv M^T\mathcal{L}(MU) = M^T AM\partial_t U + M^T BM\partial_x U + C_M U, \tag{5.3.12}$$

where

$$C_M = M^T A\partial_t M + M^T B\partial_x M + M^T CM.$$

Then we have

$$\begin{aligned}\Theta_M =& C_M + C_M^T - \partial_t(M^T AM) - \partial_x(M^T BM)\\ =& M^T(C + C^T - \partial_t A - \partial_x B)M.\end{aligned}$$

This implies

$$\Theta_M = M^T\Theta M.$$

Therefore symmetric positive linear differential systems are preserved by a change of type (5.3.11).

We may check that the admissible boundary condition is also preserved. Specifically, we have

$$W \in H^1_{\mathcal{L}} \quad \text{if and only if} \quad U \in H^1_{\mathcal{L}_M}.$$

REMARK 5.3.4. When the smoothness of the coefficient matrices A, B, C is not a concern, we may choose M such that $M^T AM$ has a simple form. For example, we may assume $M^T AM$ is diagonal. Further, we might even assume $M^T AM$ is diagonal and constant. This is always possible since $\det A \neq 0$. However, this is not adequate when the minimal regularity of coefficient matrices is assumed, since the new coefficients in $\mathcal{L}_M$ involve derivatives of old coefficients in $\mathcal{L}$.

Next, we derive an estimate on Sobolev norms of the negative index. We define for any integer $s \geq 1$

$$H^{*s} = \{u \in L^2; \partial_x^i u \in L^2, \text{ for any } i \leq s\},$$

and for any function $w \in H^{*s}$

$$\|w\|_s^* = \|\partial_x^s w\|_0 + \|w\|_0.$$

Note derivatives are taken only in the x-direction. Obviously, $H^s \subset H^{*s}$.

LEMMA 5.3.5. *Suppose $m \geq 1$ is an integer and A, B, C are C^∞ satisfying (H1)-(H3). Then there exists an $\eta_m > 0$ depending only on m and θ such that for any $\eta \in (0, \eta_m)$ and any $W \in C_{\mathcal{L}}^\infty$,*

$$\|W\|_{-m}^* \leq c_m \|\mathcal{L}W\|_{-m}^*, \tag{5.3.13}$$

where c_m is a positive constant depending only on m and A, B, C.

PROOF. We first assume A is a constant matrix. For the given W, we solve the equation

$$(-1)^m \partial_x^{2m} U + \lambda U = W \quad \text{in } \Omega,$$

where $\lambda \geq 1$ is a positive constant to be determined. It has a unique solution $U \in C_{\mathcal{L}}^\infty$ for $W \in C_{\mathcal{L}}^\infty$. In order to verify U satisfies the same boundary condition at $t = 1$ and $t = -1$ as W does, we use here the fact that the eigenvectors of A are constant vectors. Note

$$\|W\|_{-m}^* = \sup_{\varphi \in H^{*m}} \frac{|(\varphi, W)|}{\|\varphi\|_m^*},$$

where we calculate the numerator in the following way:

$$\begin{aligned}(\varphi, W) &= (\varphi, (-1)^m \partial_x^{2m} U + \lambda U) \\ &= (\partial_x^m \varphi, \partial_x^m U) + \lambda(\varphi, U) \leq \lambda \|\varphi\|_m^* \|U\|_m^*.\end{aligned}$$

Hence we get

$$\|W\|_{-m}^* \leq \lambda \|U\|_m^*. \tag{5.3.14}$$

On the other hand, we have

$$\|\mathcal{L}W\|_{-m}^* = \sup_{\varphi \in H^{*m}} \frac{|(\varphi, \mathcal{L}W)|}{\|\varphi\|_m^*} \geq \frac{(U, \mathcal{L}W)}{\|U\|_m^*},$$

where we write

$$(U, \mathcal{L}W) = (-1)^m (U, \mathcal{L}\partial_x^{2m} U) + \lambda(U, \mathcal{L}U).$$

By (5.3.8), we have for $U \in C_{\mathcal{L}}^\infty$

$$(U, \mathcal{L}U) \geq \frac{1}{2}\theta \|U\|_0^2. \tag{5.3.15}$$

To consider the first term, we apply (4.4.2) and (4.4.6) to $\partial_x^{2m}U = \partial_x^m(\partial_x^m U)$ to get

$$\begin{aligned}\mathcal{L}\partial_x^{2m}U &= \partial_x^m(\mathcal{L}\partial_x^m U) - m\partial_x B\partial_x^{2m}U \\ &+ \sum_{i=0}^{m-2} b_{m,i}\partial_x^{m-i}B\partial_x^{m+i+1}U + \sum_{i=0}^{m-1} c_{m,i}\partial_x^{m-i}C\partial_x^{m+i}U,\end{aligned} \tag{5.3.16}$$

where $b_{m,i}$ and $c_{m,i}$ are constants. Now, we write (5.3.16) as

$$\mathcal{L}\partial_x^{2m}U = \partial_x^m(\mathcal{L}\partial_x^m U) - m\partial_x^m(\partial_x B\partial_x^m U) + I, \tag{5.3.17}$$

where in each term of I the total number of derivatives on U has an order $< 2m$. First, we obtain by (5.3.8) for $\partial_x^m U \in C_L^\infty$

$$(-1)^m(U, \partial_x^m\mathcal{L}\partial_x^m U) = (\partial_x^m U, \mathcal{L}\partial_x^m U) \geq \frac{\theta}{2}\|\partial_x^m U\|_0^2. \tag{5.3.18}$$

Next, we have by $(H3)$

$$|(U, \partial_x^m(\partial_x B\partial_x^m U)| = |(\partial_x^m U, \partial_x B\partial_x^m U)| \leq \eta\|\partial_x^m U\|_0^2. \tag{5.3.19}$$

Finally, in each term in the inner product (U, I) the total number of derivatives on U has an order $< 2m$. Hence by integrating by parts finitely many times and applying (4.3.6), we obtain for small $\delta > 0$

$$|(U, I)| \leq c\|\partial_x^m U\|_0 \left(\sum_{i=0}^{m-1}\|\partial_x^i U\|_0\right) \leq \delta\|\partial_x^m U\|_0^2 + c(\delta)\|U\|_0^2. \tag{5.3.20}$$

Now we sum (5.3.18)-(5.3.20) to get

$$(-1)^m(U, \mathcal{L}\partial_x^{2m}U) \geq (\frac{1}{2}\theta - c\eta - \delta)\|\partial_x^m U\|_0^2 - c(\delta)\|U\|_0^2. \tag{5.3.21}$$

From (5.3.15) and (5.3.21), we obtain

$$\begin{aligned}&(-1)^m(U, \mathcal{L}\partial_x^{2m}U) + \lambda(U, \mathcal{L}U) \\ \geq&(\frac{1}{2}\theta - c\eta - \delta)\|\partial_x^m U\|_0^2 + (\frac{1}{2}\theta\lambda - c(\delta))\|U\|_0^2.\end{aligned}$$

Hence, if η and δ are chosen small and λ large, we have

$$\|\mathcal{L}W\|_{-m}^* \geq c\|U\|_m^*.$$

This, together with (5.3.14), finishes the proof if A is a constant matrix.

In general, we consider a nonsingular matrix M such that M^TAM is a constant matrix. In fact, we may choose M with bounded H^m-norms of M and M^{-1} such that M^TAM is a constant diagonal matrix. Then consider the operator $\mathcal{L}_M$ in (5.3.12). It is easy to see that $\mathcal{L}_M$ satisfies (H1)-(H3). By what has just been proved, we have for any $V \in C_{\mathcal{L}_M}^\infty$

$$\|V\|_{-m}^* \leq c\|\mathcal{L}_M V\|_{-m}^*.$$

This implies (5.3.13) easily. □

Remark 5.3.6. We did not keep track of the dependence of the constant c_m on the smoothness of A, B, C. The current form of Lemma 5.3.5 is sufficient for later applications.

Now we prove the main result in this section.

THEOREM 5.3.7. *Suppose $\tilde{s} \geq 3$ is an integer and A, B, C are $H^{\tilde{s}}$ satisfying (H1)-(H3). Then there exists an $\eta_{\tilde{s}} > 0$ depending only on $\tilde{s}$ and θ such that for any $\eta \in (0, \eta_{\tilde{s}})$ and any $F \in H^{\tilde{s}}$, there exists a unique solution $W \in H^{\tilde{s}}$ of (5.3.3)-(5.3.4). Moreover, for $s = 0, 1, \cdots, \tilde{s}$*

$$\|W\|_s \leq c_s \|F\|_s, \tag{5.3.22}$$

where c_s is a positive constant which, for $s \leq 3$, depends only on θ and the H^3-norms of $A^{-1}B$, A^{-1}, C, and, for $3 < s \leq \tilde{s}$, depends only on θ, the H^s-norms of $A^{-1}B, A^{-1}, C$ and the H^{s-1}-norms of $\partial_x A$ and $\partial_x B$.

PROOF. We prove Theorem 5.3.7 in two steps.

Step 1. First we assume A, B, C and F are smooth. By applying Lemma 5.3.5 to $\mathcal{L}^*$ with an integer $m \geq \tilde{s}$, we obtain

$$\|V\|^*_{-m} \leq c\|\mathcal{L}^*V\|^*_{-m} \text{ for any } V \in C^\infty_{\mathcal{L}^*}.$$

For the given F and any $V \in C^\infty_{\mathcal{L}^*}$

$$|(F, V)| \leq \|F\|^*_m \|V\|^*_{-m} \leq c\|F\|^*_m \|\mathcal{L}^*V\|^*_{-m}. \tag{5.3.23}$$

Define a linear functional l on $\mathcal{L}^*(C^\infty_{\mathcal{L}^*})$ by

$$l(\mathcal{L}^*V) = (F, V).$$

It is well-defined by (5.3.23) and is a bounded linear functional if $\mathcal{L}^*(C^\infty_{\mathcal{L}^*})$ is equipped with the norm $\|\cdot\|^*_{-m}$. By the Hahn-Banach Theorem, l can be extended to a bounded linear functional on $(H^{*m})'$, the dual space of H^{*m}. By the Riesz representation theorem, there exists a unique $W \in H^{*m}$ such that

$$l(\varphi) = (W, \varphi) \text{ for any } \varphi \in (H^{*m})',$$

and in particular

$$(F, V) = (W, \mathcal{L}^*V) \text{ for any } V \in C^\infty_{\mathcal{L}^*}.$$

This implies $W \in H^m_{\mathcal{L}}$ and $\mathcal{L}W = F$. In fact, we first have for any $V \in C^\infty_{\mathcal{L}^*}$

$$\begin{aligned}(F, V) &= (W, -\partial_t(AV) - \partial_x(BV) + C^TV) \\ &= \big(W, -\partial_t(AV)\big) + \big(B\partial_x W + CW, V\big).\end{aligned}$$

By taking $V \in C_0^\infty$, we get $W \in H^1$ and

$$A\partial_t W + B\partial_x W + CW = F \quad \text{in } \Omega.$$

This implies easily $W \in H^m$, since $W \in H^{*m}$ and $\det A \neq 0$. Next, by taking $V \in C^\infty_{\mathcal{L}^*}$ in (5.3.6), we obtain

$$V^TAW(\cdot, 1) - V^TAW(\cdot, -1) = 0 \text{ for any } V \in C^\infty_{\mathcal{L}^*}.$$

This implies $W \in H^1_{\mathcal{L}}$. If we take $m = \tilde{s} + 1$ and apply Lemma 5.3.1, we obtain (5.3.22) for any $s = 0, 1, \cdots, \tilde{s}$.

Step 2. Now suppose A, B, C and F are $H^{\tilde{s}}$ functions. Consider sequences of smooth $\{A_k\}$, $\{B_k\}$, $\{C_k\}$ and $\{F_k\}$ such that $A_k \to A$ in $H^{\tilde{s}}$ and likewise for the sequences $\{B_k\}$, $\{C_k\}$ and $\{F_k\}$. Furthermore, we assume that $\{A_k\}$, $\{B_k\}$, $\{C_k\}$ satisfy (H1)-(H3) and that the eigenvalues of A_k converge to the corresponding ones of A uniformly in Ω. Now we define

$$\mathcal{L}_k W = A_k \partial_t W + B_k \partial_x W + C_k W,$$

and with $H^1_{\mathcal{L}_k}$ defined accordingly. By Step 1, there exists a unique $H^{\tilde{s}}$ solution W_k of the problem

$$\mathcal{L}_k W_k = F_k,\ W_k \in H^1_{\mathcal{L}_k}.$$

Moreover, for $s = 0, 1, \cdots, \tilde{s}$

$$\|W_k\|_s \le c_s \|F_k\|_s,$$

where c_s is independent of k. By taking a subsequence if necessary, we may assume W_k converges to W weakly in $H^{\tilde{s}}$ and strongly in $H^{\tilde{s}-1}$. Since $H^{\tilde{s}-1} \subset L^\infty$, we also conclude W_k converges to W uniformly. Now it is easy to conclude that $\mathcal{L}W = F$, $W \in H^{\tilde{s}}_{\mathcal{L}}$, and (5.3.22) holds for $s = 0, 1, \cdots, \tilde{s}$. This finishes the proof. □

REMARK 5.3.8. With Theorem 5.3.7, the condition on W in Lemma 5.3.1 can be relaxed to $W \in H^m_{\mathcal{L}}$ and $\mathcal{L}W \in H^m$.

Notes

Theorem 5.0.1 was first proved by Lin in [**155**]. He reduced the Darboux equation to a Tricomi-type equation and then further, using a tricky transform, to a symmetric positive differential system. The presentation here is taken from Han [**80**], and it follows a similar idea. First, we choose a different initial approximate solution and a different scaling. The Darboux equation can then be changed directly into a symmetric positive differential system in a standard form. This makes our presentation easy to follow. However, the regularity is a little worse than that in [**155**]. We establish the C^{r-6} isometric embedding, instead of C^{r-3} by Lin originally. Günther [**78**] also solved the Tricomi-type equation by a modification of Galerkin's method.

In this chapter, we proved only the existence of a sufficiently smooth isometric embedding. Nakamura and Maeda [**164**] proved that the metric g admits a smooth local isometric embedding in $\mathbb{R}^3$ if g is smooth and its Gauss curvature changes its sign cleanly. Their proof is based on a discussion of equations of principal type.

Symmetric positive linear differential systems were introduced and studied by Friedrichs in [**61**]. Later on, they were studied by many people. In our presentation, we solve the symmetric positive linear differential system by establishing a priori estimates of solutions in Sobolev norms of negative index. This avoids the discussion of the equivalence of strong solutions and weak solutions.