Prologue

What is this book about and why is it so long? Parametrized homotopy theory concerns systems of spaces and spectra that are parametrized as fibers over points of a given base space B. Parametrized spaces, or "ex-spaces", are just spaces over and under B, with a projection, often a fibration, and a section. Parametrized spectra are analogous but considerably more sophisticated objects. They provide a world in which one can apply the methods of stable homotopy theory without losing track of fundamental groups and other unstable information. Stable homotopy theory has tended to ignore such intrinsically unstable data. This has the effect of losing contact with more geometric branches of mathematics in which the fundamental group cannot be ignored.

Parametrized homotopy theory is a natural and important part of homotopy theory that is implicitly central to all of bundle and fibration theory. Results that make essential use of it are widely scattered throughout the literature. For classical examples, the theory of transfer maps is intrinsically about parametrized homotopy theory, and Eilenberg-Moore type spectral sequences are parametrized Künneth theorems. Several new and current directions, such as "twisted" cohomology theories and parametrized fixed point theory cry out for the rigorous foundations that we shall develop.

On the foundational level, homotopy theory, and especially stable homotopy theory, has undergone a thorough reanalysis in recent years. Systematic use of Quillen's theory of model categories has illuminated the structure of the subject and has done so in a way that makes the general methodology widely applicable to other branches of mathematics. The discovery of categories of spectra with associative and commutative smash products has revolutionized stable homotopy theory. The systematic study and application of equivariant algebraic topology has greatly enriched the subject.

There has not been a study of parametrized homotopy theory that takes these developments into account, and we shall provide one. We shall also give some direct applications, especially to equivariant stable homotopy theory where the new theory is particularly essential. However, we shall leave many interesting loose ends, and we shall end the book with just glimpses of several new directions that are only beginning to be mapped out.

One reason this study is so lengthy is that, rather unexpectedly, many seemingly trivial nonparametrized results fail to generalize, and many of the conceptual and technical obstacles to a rigorous treatment have no nonparametrized counterparts. Another reason is that new general concepts are required to understand the full structure present in the parametrized setting and, in particular, to understand parametrized duality theory. For these reasons, the resulting theory is considerably more subtle than its nonparametrized precursors. We indicate some of these problems and phenomena here.

How to read this book. Before getting to this, we offer some words of advice on reading this book. There is a lot of technical material that most readers will want to skip at a first reading. The first three parts comprise the lengthy justification of results that can be summarized quite briefly. Part I introduces the basic categories of spaces, spaces over spaces, and ex-spaces in which we shall work, describing the closed symmetric monoidal category of ex-spaces (in §1.3), the basic triple of base change functors (in §§2.1 and 2.2), and many other interrelated functors. Part II explains in careful detail how all structure in sight passes to homotopy categories unstably. Part III explains how to do all of this stably. Along the way, we also explain how to do everything equivariantly, at least for actions by compact Lie groups. In the end, everything works out as well as can be expected, despite the unexpected technicalities that we encounter. Accepting this, much of Parts IV and V, which treat duality and homology and cohomology, should make sense without a careful reading of Parts I–III. We have tried to signpost where things are going with introductions to each of the five Parts and to each of the twenty-four Chapters. We urge the reader to peruse these introductions, as well as this Prologue, before plunging into the details.

Base change functors. A central conceptual subtlety in the theory enters when we try to prove that structure enjoyed by the point-set level categories of parametrized spaces descends to their homotopy categories. Many of our basic functors occur in Quillen adjoint pairs, and such structure descends directly to homotopy categories. Recall that an adjoint pair of functors (T, U) between model categories is a Quillen adjoint pair, or a Quillen adjunction, if the left adjoint T preserves cofibrations and acyclic cofibrations or, equivalently, the right adjoint U preserves fibrations and acyclic fibrations. It is a Quillen equivalence if, further, the induced adjunction on homotopy categories is an adjoint equivalence. We originally hoped to find a model structure on parametrized spaces in which all of the relevant adjunctions are Quillen adjunctions. It eventually became clear that there can be no such model structure, for altogether trivial reasons. Therefore, it is intrinsically impossible to lay down the basic foundations of parametrized homotopy theory using only the standard methodology of model category theory.

The force of parametrized theory largely comes from base change functors associated to maps $f\colon A \longrightarrow B$. The existing literature on fiberwise homotopy theory says surprisingly little about such functors. This is especially strange since they are the most important feature that makes parametrized homotopy theory useful for the study of ordinary homotopy theory: such functors are used to transport information from the parametrized context to the nonparametrized context. One of the goals of our work is to fill this gap.

On the point-set level, there is a pullback functor f^* from ex-spaces (or spectra) over B to ex-spaces (or spectra) over A. That functor has a left adjoint $f_!$ and a right adjoint f_*. We would like both of these to be Quillen adjunctions, but that is not possible unless the model structures lead to trivial homotopy categories. We mean literally trivial: one object and one morphism. We explain why. It will be clear that the explanation is generic and applies equally well to other situations where one encounters analogous base change functors.

Counterexample 0.0.1. Consider the following diagram.

$$\begin{array}{ccc} \emptyset & \xrightarrow{\phi} & B \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle i_0} \\ B & \xrightarrow[i_1]{} & B \times I \end{array}$$

Here $\emptyset$ is the empty set and ϕ is the initial (empty) map into B. This diagram is a pullback since $B \times \{0\} \cap B \times \{1\} = \emptyset$. The category of ex-spaces over $\emptyset$ is the trivial category with one object, and it admits a unique model structure. Let $*_B$ denote the ex-space B over B, with section and projection the identity map. Both $(\phi_!, \phi^*)$ and (ϕ^*, ϕ_*) are Quillen adjoint pairs for any model structure on the category of ex-spaces over B. Indeed, $\phi_!$ and ϕ_* preserve weak equivalences, fibrations, and cofibrations since both take $*_\emptyset$ to $*_B$. We have $(i_0)^* \circ (i_1)_! \cong \phi_! \circ \phi^*$ since both composites take any ex-space over B to $*_B$. If $(i_1)_!$ and $(i_0)^*$ were both Quillen left adjoints, it would follow that this isomorphism descends to homotopy categories. If, further, the functors $(i_1)_!$ and $(i_0)^*$ on homotopy categories were equivalences of categories, this would imply that the homotopy category of ex-spaces over B with respect to the given model structure is equivalent to the trivial category.

Information in ordinary homotopy theory is derived from results in parametrized homotopy theory by use of the base change functors $r_!$ and r_* associated to the trivial map $r\colon B \longrightarrow *$, as we shall illustrate shortly. For this and other reasons, we choose our basic model structure to be one such that $(f_!, f^*)$ is a Quillen adjoint pair for every map $f\colon A \longrightarrow B$ and is a Quillen equivalence when f is a homotopy equivalence. Then (f^*, f_*) cannot be a Quillen adjoint pair in general. However, it is essential that we still have the adjunction (f^*, f_*) after passage to homotopy categories. For example, taking f to be the diagonal map on B, this adjunction is used to obtain the adjunction on homotopy categories that relates the fiberwise smash product functor $\wedge_B$ on ex-spaces over B to the function ex-space functor F_B. To construct the homotopy category level right adjoints f_*, we shall have to revert to more classical methods, using Brown's representability theorem. However, it is not clear how to verify the hypotheses of Brown's theorem in the model theoretic framework.

Counterexample 0.0.1 also illustrates the familiar fact that a commutative diagram of functors on the point-set level need not induce a commutative diagram of functors on homotopy categories. When commuting left and right adjoints, this is a problem even when all functors in sight are parts of Quillen adjunctions. Therefore, proving that compatibility relations that hold on the point-set level descend to the homotopy category level is far from automatic. In fact, proving such "compatibility relations" is often a highly non-trivial problem, but one which is essential to the applications. We do not know how to prove the most interesting compatibility relations working only model theoretically.

Poincaré duality. Before continuing our discussion of the foundations, we pause to whet the reader's appetite by pointing out how the parametrized theory sheds new light on even the most classical parts of algebraic topology. Nonequivariantly, we shall construct a good homotopy category of spectra over B for any space B. A spectrum k_B over B represents (reduced) homology and cohomology theories, denoted k_*^B and k_B^*, on ex-spaces over B and, more generally, on spectra over B. We shall have a smash product that assigns a spectrum $k \wedge X$ over B to an ordinary spectrum k and an ex-space or spectrum X over B. Taking k_* to mean reduced homology and implicitly stabilizing by applying suspension spectrum functors, we shall have the following conceptual variant of the usual homotopical proof of the Poincaré duality theorem. See §20.5 for details. It illustrates the use of the functor $r_!$ from spectra over B to spectra, which collapses sections to a point, and its proof features a comparison between $r_!$ and r_*; the latter is the "global sections" functor from spectra over B to spectra.

Example 0.0.2 (Poincaré duality). Let M be a smooth closed n-manifold and let S^τ denote the spherical fibration obtained from the tangent bundle τ of M by fiberwise one-point compactification; it is an ex-space over M. Let S_M^n denote the ex-space $S^n \times M$; it is a trivial spherical fibration, and $S_M^0 = M \amalg M$. For any spectrum k, the "parametrized Atiyah duality theorem" implies an isomorphism

$$k_q(M_+) \cong (k \wedge S^\tau)_M^{-q}(S_M^0).$$

Thus the parametrized theory implicitly gives a direct global homotopical construction of a version of "generalized cohomology with local coefficients" that gives Poincaré duality for any representing spectrum k, without orientation hypotheses. Now let k be a commutative ring spectrum. The Thom space of τ is $T\tau = r_! S^\tau$, $r\colon M \longrightarrow *$, while $r_! S_M^n = \Sigma^n(M_+)$. By definition, a k-orientation of M is a cohomology class $\mu \in k^n(T\tau)$ that restricts to a unit of $k^n(S^n) \cong k^0(S^0)$ on each fiber. We may view μ as a map

$$\mu\colon r_! S^\tau \longrightarrow \Sigma^n k$$

with adjoint

$$\tilde{\mu}\colon S^\tau \longrightarrow r^*(\Sigma^n k) \cong k \wedge S_M^n.$$

Smashing with k and using the product $k \wedge k \longrightarrow k$, we find that $\tilde{\mu}$ induces a map

$$\bar{\mu}\colon k \wedge S^\tau \longrightarrow k \wedge S_M^n$$

of k-module spectra over M. The unit property of μ is exactly the statement that $\bar{\mu}$ restricts to an equivalence on each fiber, and this implies that $\bar{\mu}$ is an equivalence of spectra over M. This is a precise mathematical formulation of the intuition that the tangent bundle of a k-orientable manifold is stably trivial when viewed through the eyes of k-theory. The equivalence $\bar{\mu}$ induces an "untwisting" isomorphism

$$(k \wedge S^\tau)_M^{-q}(S_M^0) \cong k^{n-q}(M_+).$$

This completes a proof of the Poincaré duality theorem as a formal implication of parametrized Atiyah duality and the definition of an orientation. The proof does not use the Thom isomorphism directly, but the equivalence of spectra $r_!\bar{\mu}$ implies the isomorphism $k_q(T\tau) \cong k_{q-n}(M_+)$ on passage to homotopy groups, and similarly for cohomology.

Here is another example of something that should be an old result and is intuitively very plausible, but seems to be new. Details are given in §18.6.

Example 0.0.3. As an application of a relative form of parametrized Atiyah duality, we prove that if M is a smooth closed manifold embedded in $\mathbb{R}^q$ and L is a smooth closed submanifold, then M/L is $(q-1)$-dual to the cofiber of the Pontryagin-Thom map $T\nu_M \longrightarrow T\nu_L$ of Thom spaces, where ν_M and ν_L are the normal bundles of M and L.

Model structures. Returning to our discussion of the foundations, we shall of course use model structures wherever we can. However, even in the part of the theory in which model theory works, it does not work as expected. There is an obvious naive model structure on ex-spaces over B in which the weak equivalences, fibrations, and cofibrations are the ex-maps whose maps of total spaces are weak equivalences, fibrations, and cofibrations of spaces in the usual Quillen model structure. This "q-model structure" is the natural starting point for the theory, but it turns out to have severe drawbacks that limit its space level utility and bar it from serving as the starting point for the development of a useful spectrum level stable model structure. In fact, it has two opposite drawbacks. First, it has too many cofibrations. In particular, the model theoretic cofibrations need not be cofibrations in the intrinsic homotopical sense. That is, they fail to satisfy the fiberwise homotopy extension property (HEP) defined in terms of parametrized mapping cylinders. This already fails for the sections of cofibrant objects and for the inclusions of cofibrant objects in their cones. Therefore the classical theory of cofiber sequences fails to mesh with the model category structure.

Second, it also has too many fibrations. The fibrant ex-spaces are Serre fibrations, and Serre fibrations are not preserved by fiberwise colimits. Such colimits are preserved by a more restrictive class of fibrations, namely the well-sectioned Hurewicz fibrations, which we call ex-fibrations. Such preservation properties are crucial to resolving the problems with base change functors that we have indicated.

In model category theory, decreasing the number of cofibrations increases the number of fibrations, so that these two problems cannot admit a solution in common. Rather, we require two different equivalent descriptions of our homotopy categories of ex-spaces. First, we have another model structure, the "qf-model structure", which has the same weak equivalences as the q-model structure but has fewer cofibrations, all of which satisfy the fiberwise HEP. Second, we have a description in terms of the classical theory of ex-fibrations, which does not fit naturally into a model theoretic framework. The former is vital to the development of the stable model structure on parametrized spectra. The latter is vital to the solution of the intrinsic problems with base change functors.

Other foundational issues. Before getting to the issues just discussed, we shall have to resolve various others that also have no nonparametrized analogues. Even the point set topology requires care since function ex-spaces take us out of the category of compactly generated spaces. Equivariance raises further problems, although most of our new foundational work is already necessary nonequivariantly. Passage to the spectrum level raises more serious problems. One main source of difficulty is that the underlying total space functor is too poorly behaved, especially with respect to smash products and fibrations, to give good control of homotopy groups as one passes from parametrized spaces to parametrized spectra. Moreover, since the underlying total space functor does not commute with suspension, it does not give a forgetful functor from parametrized spectra to nonparametrized spectra.

The resolution of base change problems requires a different set of details on the spectrum level than on the space level.

This theory gives perhaps the first worked example in which a model theoretic approach to derived homotopy categories is intrinsically insufficient and must be blended with a quite different approach even to establish the essential structural features of the derived category. Such a blending of techniques seems essential in analogous sheaf theoretic contexts that have not yet received a modern model theoretic treatment. Even nonequivariantly, the basic results on base change, smash products, and function ex-spaces that we obtain do not appear in the literature. Such results are essential to serious work in parametrized homotopy theory.

Much of our work should have applications beyond the new parametrized theory. The model theory of topological enriched categories has received much less attention in the literature than the model theory of simplicially enriched categories. Despite the seemingly equivalent nature of these variants, the topological situation is actually quite different from the simplicial one, as our applications make clear. In particular, the interweaving of h-type and q-type model structures that pervades our work seems to have no simplicial counterpart. Such interweaving does also appear in algebraic contexts of model categories enriched over chain complexes, where foundations analogous to ours can be developed. One of our goals is to give a thorough analysis and axiomatization of how this interweaving works in general in topologically enriched model categories.

The foundational issues that we have been discussing occupy the first three parts of this book. Part I gives basic preliminaries, Part II develops unstable parametrized homotopy theory, and Part III develops stable parametrized homotopy theory. The end result of this foundational work may seem intricate, but it gives a very powerful framework in which to study homotopy theory, as we illustrate in the last two parts.

Parametrized duality theory. In Part IV, we develop parametrized duality theory. This has three aspects. First, there is a fiberwise duality theory that leads to a smooth general treatment of transfer maps. There are two ways of thinking about transfer maps. For fibrations in general, they are best thought of as instances of generalized trace maps present in any closed symmetric monoidal category. For bundles, they are best thought of fiberwise, with transfer maps on fibers inserted fiberwise into bundles of spectra. It is not obvious that these give equivalent constructions when both apply, and our fiberwise duality theory makes that comparison transparent.

Second, there is a new kind of parametrized duality theory that was first discovered by Costenoble and Waner. It, rather than fiberwise duality, is the appropriate parametrized analogue of Spanier-Whitehead duality, and it is the kind of duality that is used in the proof of Poincaré duality described in Example 0.0.2. These two notions of duality are quite different. Parametrized sphere spectra are invertible and therefore fiberwise dualizable, but they are not Costenoble-Waner dualizable in general. Parametrized finite cell spectra are Costenoble-Waner dualizable, but they are not fiberwise dualizable in general. The previous sentence hides another subtlety. Finite cell objects in topological model categories such as ours are elusive structures because their fibrant approximations are no longer finite. In our triangulated stable categories, the parametrized finite cell spectra and their retracts do not seem to give the objects of a thick subcategory, which is contrary to all previously encountered situations.

More centrally, conceptual understanding of the new duality theory requires the new categorical notion of a closed symmmetric bicategory and a formal duality theory for 1-cells in such a bicategory. Bimodules over varying rings, their derived and brave new counterparts, and parametrized spectra over varying base spaces all give examples. Dual pairs of "base change bimodules" and "base change spectra" encode base change functors in terms of the bicategory operations. These categorical foundations promise to have significant applications in other fields and will be more fully developed elsewhere. The basic treatment here, in Chapter 16, can be read independently of everything else in the book.

Third, there is a way to insert parametrized Atiyah duality fiberwise into bundles of spectra to develop a fiberwise Costenoble-Waner duality theory. The basic change of groups isomorphisms of equivariant stable homotopy theory, namely the generalized Wirthmüller and Adams isomorphisms, are very special cases of our duality theorems, which are already of considerable interest nonequivariantly. These applications depend on a clear and precise definition of a bundle of spectra and an analysis of how such bundles behave homotopically. This notion has appeared sporadically in the literature, although without rigorous foundations. It seems certain to become important.

Parametrized homology and cohomology. The first three parts, and most of the fourth, give reasonably complete treatments of the topics they cover, but Part V has a different character. Its main focus is the definition of parametrized homology and cohomology theories and the beginning of their study. It seems to us that another book this length could well be written on this topic, which we believe will come to play an increasingly important role in algebraic topology and its applications. We just scratch the surface. In Chapters 20 and 21, we show how to axiomatize and represent parametrized homology and cohomology theories, and we say a little about duality, base change, coefficient systems, products, and the Serre and parametrized Atiyah-Hirzebruch spectral sequences. We give separate treatments of the nonequivariant and equivariant theory for the reader's convenience. While we describe various calculational tools, we do not turn to explicit computations here.

We observe in Chapter 22 that twisted K-theory is an example of a particular kind of parametrized cohomology theory, thereby making its associated homology theory precise and making all of the standard tools of algebraic topology readily available. We also explain a Čech local to global (or descent) spectral sequence, the twisted Rothenberg-Steenrod spectral sequence, and a construction of the Eilenberg–Moore spectral sequence, viewed as a parametrized Künneth spectral sequence in the stable homotopy category of spectra over B.

Generalizations of Thom spectra. Another topic in Part V is the construction of generalized analogues of Thom spectra. We explain how the parametrized way of thinking leads directly to the construction of new nonparametrized orthogonal ring spectra and thus E_∞ ring spectra. Our iterated Thom spectra are examples. The construction is entirely elementary and only uses ex-spaces, not parametrized spectra. We urge the interested reader to turn directly to Chapter 23, since the construction is almost completely independent of everything else in the book. A more sophisticated application of parametrized methods constructs the Thom spectrum associated to a map $f: X \longrightarrow B$, where B is the classifying space for some class of bundles or fibrations, by pulling back the universal sphere bundle or fibration spectrum E over B along f and then pushing down along $r: X \longrightarrow *$ to obtain the ordinary spectrum $r_! f^* E$. We explain the idea briefly, but we do not pursue it here. It is the starting point of work in progress with Andrew Blumberg.

In fact, we have hardly begun the serious study of any of the topics in Part V. There are other areas, such as fixed point theory and Goodwillie calculus, where parametrized theory is expected to play an important role but has not yet been seriously applied due to the lack of firm foundations. It is time for this book to appear, but it is only a beginning. There is a great deal more work to be done in this emerging area of algebraic topology.

History. This project began with unpublished notes of the first author, dating from the summer of 2000 [111]. He put the project aside and returned to it in the fall of 2002, when he was joined by the second author. Some of Parts I and II was originally in a draft of the first author that was submitted and accepted for publication, but was later withdrawn. That draft was correct, but it did not include the "qf-model structure", which comes from the second author's 2004 PhD thesis [152]. The first author's notes [111] claimed to construct the stable model structure on parametrized spectra starting from the q-model structure on ex-spaces. Following [111], the monograph [78] of Po Hu also takes that starting point and makes that claim. The second author realized that, with the obvious definitions, the axioms for the stable model structure cannot be proven from that starting point and that any naive variant would be disconnected with cofiber sequences and other essential needs of a fully worked out theory. His qf-model structure is the crucial new ingredient that is used to solve this problem.

The new duality theory of Chapters 16, 17, and 18 was inspired by work of Costenoble and Waner [41]. The applications of Chapter 19 were inspired by Hu's work [78]. The implementation of her results as manifestations of fiberwise Costenoble-Waner duality came as a pleasant surprise.

Thanks. We thank the referee of the partial first version for several helpful suggestions. We thank Gaunce Lewis and Peter Booth for help with the point set topology, Mike Cole for sharing his remarkable insights about model categories, Mike Mandell for much technical help, and Brooke Shipley for an illuminating discussion of different types of generators. We thank Kathleen Lewis for working out the counterexample in Theorem 1.7.1, Victor Ginzburg for giving us the striking Counterexample 11.6.2, Amnon Neeman for giving us the useful Lemma 20.8.4, and Shmuel Weinberger for pointing out Proposition 21.7.4.

We are especially grateful to Kate Ponto for a meticulously careful reading that uncovered many obscurities and infelicities. Needless to say, she is not to blame for those that remain.

Some of the work on this book was done during the second author's visits to the University of Chicago and the Institut Mittag-Leffler and he gratefully acknowledges their hospitality and support. He would also like to thank the members of the homotopy group at the University of Sheffield for useful conversations and a very stimulating environment.

Finally, we would like to thank the Editorial Committee of the Mathematical Surveys and Monographs series of the AMS for accepting this book for publication unconditionally, while gently suggesting that we add more motivation and "an attempt at more examples" and also suggesting a tentative deadline for us not to meet. The freedom and pressure their decision gave us led directly to our working out and writing up the new material that begins in Chapter 16, despite the preliminary nature of much of it. Nearly all of this work postdates the acceptance of the book on April 13, 2005.

Dedication. On May 17, 2006, Gaunce Lewis died after a lengthy bout with brain cancer. This book is in large part the culmination of a long development of the foundations of equivariant stable homotopy theory that began with Lewis's 1978 Chicago PhD thesis and his joint work with the first author that appeared in 1986 [98]. His influence on our work will be evident. With sadness, we dedicate this book to his memory.

July 11, 2006