Preface

A control system is a dynamical system on which one can act by using suitable *controls*. There are a lot of problems that appear when studying a control system. But the most common ones are the *controllability* problem and the *stabilization* problem.

The controllability problem is, roughly speaking, the following one. Let us give two states. Is it possible to move the control system from the first one to the second one? We study this problem in Part 1 and in Part 2. Part 1 studies the controllability of *linear* control systems, where the situation is rather well understood, even if there are still quite challenging open problems in the case of linear partial differential control systems. In Part 2 we are concerned with the controllability of *nonlinear* control systems. We start with the case of finite-dimensional control systems, a case where quite powerful geometrical tools are known. The case of nonlinear partial differential equations is much more complicated to handle. We present various methods to treat this case as well as many applications of these methods. We emphasize control systems for which the nonlinearity plays a crucial role, in particular, for which it is the nonlinearity that gives the controllability or prevents achieving some specific interesting motions.

The stabilization problem is the following one. We have an equilibrium which is unstable without the use of the control. Let us give a concrete example. One has a stick that is placed vertically on one of his fingers. In principle, if the stick is exactly vertical with a speed exactly equal to 0, it should remain vertical. But, due to various small errors (the stick is not exactly vertical, for example), in practice, the stick falls down. In order to avoid this, one moves the finger in a suitable way, depending on the position and speed of the stick; we use a "*feedback law*" (or "*closed-loop control*") which stabilizes the equilibrium. The problem of the stabilization is the existence and construction of such stabilizing feedback laws for a given control system. We study it in Part 3, both for finite-dimensional control systems and for systems modeled by partial differential equations. Again we emphasize the case where the nonlinear terms play a crucial role.

Let us now be more precise on the contents of the different parts of this book.

Part 1: Controllability of linear control systems

This first part is devoted to the controllability of *linear* control systems. It has two chapters: The first one deals with finite-dimensional control systems, the second one deals with infinite-dimensional control systems modeled by partial differential equations.

Let us detail the contents of these two chapters.

Chapter 1. This chapter focuses on the *controllability* of *linear finite-dimensional control systems*. We first give an integral necessary and sufficient condition for a linear time-varying finite-dimensional control system to be controllable. For a special quadratic cost, it leads to the optimal control. We give some examples of applications.

These examples show that the use of this necessary and sufficient condition can lead to computations which are somewhat complicated even for very simple control systems. In particular, it requires integrating linear differential equations. We present the famous Kalman rank condition for the controllability of linear *time-invariant* finite-dimensional control systems. This new condition, which is also necessary and sufficient for controllability, is purely algebraic: it does not require integrations of linear differential equations. We turn then to the case of linear *time-varying* finite-dimensional control systems. For these systems we give a sufficient condition for controllability, which turns out to be also necessary for analytic control systems. This condition only requires computing derivatives; again no integrations are needed.

We describe, in the framework of linear time-varying finite-dimensional control systems, the Hilbert Uniqueness Method (HUM), due to Jacques-Louis Lions. This method is quite useful in infinite dimension for finding numerically optimal controls for linear control systems.

Chapter 2. The subject of this chapter is the controllability of some classical linear partial differential equations. For the reader who is familiar with this subject, a large part of this chapter can be omitted; most of the methods detailed here are very well known. One can find much more advanced material in some references given throughout this chapter. We study a transport equation, a Korteweg-de Vries equation, a heat equation, a wave equation and a Schrödinger equation.

We prove the well-posedness of the Cauchy problem associated to these equations. The controllability of these equations is studied by means of various methods: explicit methods, extension method, moments theory, flatness, Hilbert Uniqueness Method, duality between controllability and observability. This duality shows that the controllability can be reduced to an observability inequality. We show how to prove this inequality by means of the multiplier method or Carleman inequalities. We also present a classical abstract setting which allows us to treat the well-posedness and the controllability of many partial differential equations in the same framework.

Part 2: Controllability of nonlinear control systems

This second part deals with the controllability of *nonlinear* control systems.

We start with the case of nonlinear *finite-dimensional* control systems. We recall the linear test and explain some geometrical methods relying on iterated Lie brackets when this test fails.

Next we consider nonlinear partial differential equations. For these *infinite-dimensional* control systems, we begin with the case where the linearized control system is controllable. Then we get local controllability results and even global controllability results if the nonlinearity is not too big. The case where the linearized control system is not controllable is more difficult to handle. In particular, the tool of iterated Lie brackets, which is quite useful for treating this case in finite dimension, turns out to be useless for many interesting infinite-dimensional control systems. We present three methods to treat some of these systems, namely the return method, quasi-static deformations and power series expansions. On various examples, we show how these three methods can be used.

Let us give more details on the contents of the seven chapters of Part 2.

Chapter 3. In this chapter we study the local controllability of *finite-dimensional nonlinear* control systems around a given equilibrium. One does not know any interesting necessary and sufficient condition for small-time local controllability, even for analytic control systems. However, one knows powerful necessary conditions and powerful sufficient conditions.

We recall the classical “linear test”: If the linearized control system at the equilibrium is controllable, then the nonlinear control system is locally controllable at this equilibrium.

When the linearized control system is not controllable, the situation is much more complicated. We recall the Lie algebra condition, a necessary condition for local controllability of (analytic) control systems. It relies on iterated Lie brackets. We explain why iterated Lie brackets are natural for the problem of controllability.

We study in detail the case of the driftless control systems. For these systems, the above Lie algebra rank condition turns out to be sufficient, even for global controllability.

Among the iterated Lie brackets, we describe some of them which are “good” and give the small-time local controllability, and some of them which are “bad” and lead to obstructions to small-time local controllability.

Chapter 4. In this chapter, we first consider the problem of the controllability around an equilibrium of a *nonlinear partial differential equation* such that the linearized control system around the equilibrium is controllable. In finite dimension, we have already seen that, in such a situation, the nonlinear control system is locally controllable around the equilibrium. Of course in infinite dimension one expects that a similar result holds. We prove that this is indeed the case for various equations: A nonlinear Korteweg-de Vries equation, a nonlinear hyperbolic equation and a nonlinear Schrödinger equation. For the first equation, one uses a natural fixed-point method. For the two other equations, the situation is more involved due to a problem of loss of derivatives. For the hyperbolic equation, one uses, to take care of this problem, an ad-hoc fixed-point method, which is specific to hyperbolic systems. For the case of the Schrödinger equation, this problem is overcome by the use of a Nash-Moser method.

Sometimes these methods, which lead to *local* controllability results, can be adapted to give a *global* controllability result if the nonlinearity is not too big at infinity. We present an example for a nonlinear one-dimensional wave equation.

Chapter 5. We present an application of the use of iterated Lie brackets for a nonlinear partial differential equation (a nonlinear Schrödinger equation). We also explain why iterated Lie brackets are less powerful in infinite dimension than in finite dimension.

Chapter 6. This chapter deals with the *return method*. The idea of the return method goes as follows: If one can find a trajectory of the nonlinear control system such that:

- it starts and ends at the equilibrium,
- the linearized control system around this trajectory is controllable,

then, in general, the implicit function theorem implies that one can go from every state close to the equilibrium to every other state close to the equilibrium. In Chapter 6, we sketch some results in flow control which have been obtained by this method, namely:

- global controllability results for the Euler equations of incompressible fluids,
- global controllability results for the Navier-Stokes equations of incompressible fluids,
- local controllability of a 1-D tank containing a fluid modeled by the shallow water equations.

Chapter 7. This chapter develops the *quasi-static deformation method*, which allows one to prove in some cases that one can move from a given equilibrium to another given equilibrium if these two equilibria are connected in the set of equilibria. The idea is just to move very slowly the control (quasi-static deformation) so that at each time the state is close to the curve of equilibria connecting the two given equilibria. If some of these equilibria are unstable, one also uses suitable feedback laws in order to stabilize them; without these feedback laws the quasi-static deformation method would not work. We present an application to a semilinear heat equation.

Chapter 8. This chapter is devoted to the *power series expansion method*: One makes some power series expansion in order to decide whether the nonlinearity allows us to move in every (oriented) direction which is not controllable for the linearized control system around the equilibrium. We present an application to a nonlinear Korteweg-de Vries equation.

Chapter 9. The previous three methods (return, quasi-static deformations, power series expansion) can be used together. We present in this chapter an example for a nonlinear Schrödinger control equation.

Part 3: Stabilization

The two previous parts were devoted to the controllability problem, which asks if one can move from a first given state to a second given state. The control that one gets is an open-loop control: it depends on time and on the two given states, but it *does not* depend on the state during the evolution of the control system. In many practical situations one prefers closed-loop controls, i.e., controls which do not depend on the initial state but depend, at time t, on the state x at this time. One requires that these closed-loop controls (asymptotically) stabilize the point one wants to reach. Usually such closed-loop controls (or feedback laws) have the advantage of being be more robust to disturbances (recall the experiment of the stick on the finger). The main issue discussed in this part is the problem of deciding whether a controllable system can be (asymptotically) stabilized.

This part is divided into four chapters: Chapter 10, Chapter 11, Chapter 12 and Chapter 13, which we now briefly describe.

Chapter 10. This chapter is mainly concerned with the stabilization of *finite-dimensional linear* control systems. We first start by recalling the classical pole-shifting theorem. A consequence of this theorem is that every controllable linear system can be stabilized by means of linear feedback laws. This implies that, if the linearized control system at an equilibrium of a nonlinear control system is controllable, then this equilibrium can be stabilized by smooth feedback laws.

Chapter 11. This chapter discusses the stabilization of *finite-dimensional nonlinear* control systems, mainly in the case where the nonlinearity plays a key role. In particular, it deals with the case where the linearized control system around the equilibrium that one wants to stabilize is no longer controllable. Then there are obstructions to stabilizability by smooth feedback laws even for controllable systems. We recall some of these obstructions. There are two ways to enlarge the class of feedback laws in order to recover stabilizability properties. The first one is the use of discontinuous feedback laws. The second one is the use of time-varying feedback laws. We give only comments and references on the first method, but we give details on the second one. We also show the interest of time-varying feedback laws for output stabilization: In this case the feedback laws depend only on the output, which is only part of the state.

Chapter 12. In this chapter, we present important tools for constructing explicit stabilizing feedback laws, namely:

1. control Lyapunov function,
2. damping,
3. homogeneity,
4. averaging,
5. backstepping,
6. forwarding,
7. transverse functions.

These methods are illustrated on various control systems, in particular, the stabilization of the attitude of a rigid spacecraft.

Chapter 13. In this chapter, we give examples of how some tools introduced for the stabilization of *finite-dimensional* control systems can be used to stabilize some partial differential equations. We treat the following four examples:

1. rapid exponential stabilization by means of Gramians for linear time-reversible partial differential equations,
2. stabilization of a rotating body-beam without damping,
3. stabilization of the Euler equations of incompressible fluids,
4. stabilization of hyperbolic systems.

Appendices

This book has two appendices. In the first one (Appendix A), we recall some classical results on semigroups generated by linear operators and classical applications to evolution equations. We omit the proofs but we give precise references where they can be found. In the second appendix (Appendix B), we construct the degree of a map and prove the properties of the degree we use in this book. As an application of the degree, we also prove the Brouwer and Schauder fixed-point theorems which are also used in this book.

Acknowledments. I thank the Rutgers University Mathematics Department, especially Eduardo Sontag and Héctor Sussmann, for inviting me to give the 2003 Dean Jacqueline B. Lewis Memorial Lectures. This book arose from these lectures. I am grateful to the Institute Universitaire de France for providing ideal working conditions for writing this book.

It is a pleasure to thank Azgal Abichou, Claude Bardos, Henry Hermes, Vilmos Komornik, Marius Tucsnak, and José Urquiza, for useful discussions. I am also especially indebted to Karine Beauchard, Eduardo Cerpa, Yacine Chitour, Emmanuelle Crépeau, Olivier Glass, Sergio Guerrero, Thierry Horsin, Rhouma Mlayeh, Christophe Prieur, Lionel Rosier, Emmanuel Trélat, and Claire Voisin who recommended many modifications and corrections. I also thank Claire Voisin for important suggestions and for her constant encouragement.

It is also a pleasure to thank my former colleagues at the Centre Automatique et Systèmes, Brigitte d'Andréa-Novel, François Chaplais, Michel Fliess, Yves Lenoir, Jean Lévine, Philippe Martin, Nicolas Petit, Laurent Praly and Pierre Rouchon, for convincing me to work in control theory.

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October, 2006