Preface

The present monograph is an attempt to a better understanding of an interdisciplinary question, namely the impact of foliation theory on the geometry and analysis on CR manifolds. To start with, any Levi-flat CR manifold M carries a complex foliation $\mathcal{F}$ (the *Levi foliation*) tangent to the null space of the Levi form of the manifold. At least in the real analytic case, if M is embedded then $\mathcal{F}$ extends to a holomorphic foliation of an open neighborhood of M (*Rea's theorem*, [**203**]). Complex foliations occur in a natural way on certain nondegenerate CR manifolds, as well. To give a simple example, R. Penrose's manifold $\mathbb{P}(\mathbb{T}_0)$ (the boundary of the manifold $\mathbb{P}(\mathbb{T}_+)$ of all right-handed spinning photons, cf. [**199**]) is a nondegenerate CR manifold of hypersurface type foliated by $\mathbb{C}P^1$'s and this situation generalizes to C. Le Brun's twistor CR manifolds (cf. [**165**]-[**166**]). As shown in [**84**], there are also natural CR analogues of complex Monge-Ampére foliations (in the sense of [**30**], for instance) occurring on strictly pseudoconvex CR manifolds. Each leaf of such a tangential Monge-Ampére foliation is a CR manifold and the inclusion in the ambient space is a pseudohermitian immersion. Finally, let us mention that each nondegenerate CR manifold carries a flow defined by its contact vector field. This is evidence enough to the interrelation between foliation theory and CR geometry, and that an overall use of the former is liable to clear up certain questions in complex analysis. For instance, let $\Omega \subset \mathbb{C}^{n+1}$ be a strictly pseudoconvex domain with real analytic boundary $\partial\Omega$. Let $\mathcal{O}(\overline{\Omega})$ be the algebra of functions on $\overline{\Omega}$ which admit a holomorphic extension to some neighborhood of $\overline{\Omega}$. Let $M \subset \partial\Omega$ be a real analytic submanifold which is not $\mathbb{C}$-tangent at any of its points. By a result of L. Boutet de Monvel and A. Iordan (cf. [**56**]) M is locally a maximum modulus set for $\mathcal{O}(\overline{\Omega})$ (in the sense of T. Duchamp and E.L. Stout, [**93**]) if and only if $\mathcal{L}(X, Y)$ is real valued for any sections X, Y in $L = T(M) \cap H(\partial\Omega)$. Here $\mathcal{L}$ is the Levi form of $\partial\Omega$ and $H(\partial\Omega)$ is its maximal complex distribution. If this is the case then L is completely integrable and gives rise to a $\mathbb{C}$-tangent foliation $\mathcal{F}$ of M of codimension one and the paper [**22**] studies the interplay between the properties of $\mathcal{F}$ and the geometry of the second fundamental form of M in $\partial\Omega$. M turns out to be a Levi flat contact CR submanifold of $\partial\Omega$ and $\mathcal{F}$ is its Levi foliation. When M is minimal $\mathcal{F}$ is harmonic.

Let us add that, besides from the very interest in interdisciplinary problems, in a series of papers (cf. [**16**], [**19**] and [**21**]) the first two authors developed an idea of E.M. Chirka, [**67**], regarding foliations with transverse CR structure (which contain the class of CR manifolds as the special case of transversally CR foliations by points) which led to Chapter 6 of this monograph.

Sections 1.1 and 1.2 review the notions of foliation theory needed through the text. We only sketch the essentials, as many monographs on the subject have been available for quite a few years (such as [**179**], or [**243**], which are the most frequently referred to).

The next seven chapters form the main core of this book. The case of foliated CR manifolds is considered in Chapter 2. Sections 2.3 to 2.5 are imitative of P. Tondeur's exposition of the geometry of foliations on Riemannian manifolds, cf. [**243**], p. 47-73, and the similarity comes from the fact that in the nondegenerate case CR manifolds possess a canonical metric (the *Webster metric*) and connection (the *Tanaka-Webster connection*) of which the latter resembles the Chern connection in Hermitian geometry and the Levi-Civita connection in Riemannian geometry.

Chapter 3 is dedicated to Levi foliations and their holomorphic extendibility. We give a proof of a beautiful result referred to as *Rea's theorem*. It is based on a theorem by F. Severi and G. Tomassini (cf. [**219**] and [**242**]) about holomorphic extension of CR functions in the real analytic case. There are many other CR extension theorems available in today's mathematical literature (cf. [**50**] and references therein) yet it seems to the authors that Rea's is the only attempt (cf. [**203**]) to apply a CR extension result in order to get a holomorphic extension of a Levi (or semi-holomorphic) foliation.

Related to Rea's theorem we present the solution (due to D.E. Barrett, [**28**]) to the problem of the existence of a pluriharmonic defining function for a Levi-flat real analytic hypersurface in a complex manifold. Next, we exhibit a characterization of Levi flatness of real analytic hypersurfaces in $\mathbb{C}^n$ in terms of *holomorphic degeneracy*, cf. Theorem 3.22 in Section 3.4 (due to N.K. Stanton, [**227**]).

An active research field in complex analysis (in several complex variables) is that related to the problem of global regularity of the Neumann operators N_q, $1 \leq q \leq n$, and of the Bergman projections P_q, $0 \leq q \leq n$, for a smoothly bounded pseudoconvex domain $\Omega \subset \mathbb{C}^n$. Precisely, the question is whether N_q and P_q are continuous on the space $W^s_{(0,q)}(\Omega)$, $s \geq 0$, of all $(0,q)$-forms with coefficients in the Sobolev space $W^s(\Omega)$, $s \geq 0$ (cf. e.g. [**48**]). The state of the art is represented by Theorem 3.32 (due to H.P. Boas and E.J. Straube, [**49**]) in Section 3.5 of this monograph. The estimates leading to the result in Theorem 3.32 were known (by a result of D. Catlin, [**61**]) at the points of finite type, yet required a new technique, based on the existence of complex vector fields commuting approximately with $\overline{\partial}$ (cf. Definition 3.33) on the set $K \subset \partial\Omega$ of all boundary points of infinite type. Such vector fields were shown to exist when Ω admits a plurisubharmonic defining function (cf. Definition 3.30), a fact which led to Theorem 3.31 (due again to H.P. Boas et al., [**48**]). When the set K of all infinite type points is contained in a real submanifold $M \subset \partial\Omega$ of the boundary which is tangent (i.e. $T(M) \subset \mathrm{Null}(G_\theta)$) to the Levi null distribution (e.g. when $K = \overline{\overset{\circ}{K}}$ and the Levi form of $\partial\Omega$ vanishes at each point of $\overset{\circ}{K}$) the beautiful (from a differential geometric viewpoint) finding by H.P. Boas et al., [**49**] (and further examined by E.J. Straube and M.K. Sucheston, [**233**]) is the existence of a de Rham cohomology class $a(M) \in H^1(M, \mathbb{R})$ (the *D'Angelo class*, under the terminology adopted in this monograph) which is an obstruction to the existence of the special vector fields mentioned above (cf. Theorem 3.36). Section 3.5 concludes with a discussion of the D'Angelo class within foliation theory (i.e. the relationship among $a(M)$ and the infinitesimal holonomy of the leaf M of the Levi foliation on $\overset{\circ}{K}$) and a few open problems.

Chapter 4 reports on the known results about the nonexistence of Levi flat CR submanifolds in a complex projective space, such as Y-T. Siu's result (cf. [**221**]-[**222**]) with the lower differentiability requirements due to J. Cao and M-C. Shaw and L. Wang, [**60**] (cf. Theorem 4.1 in Section 4.1 of this book), the result of L. Ni and J. Wolfson, [**188**] (based on a Lefschetz type result for CR submanifolds of a Kählerian manifold of positive holomorphic bisectional curvature, established by themselves, and the classical theorem of A. Haefliger, [**131**], on the inexistence of real analytic codimension one foliations on compact simply connected manifolds), and the purely differential geometric approach of M. Djorić and M. Okumura, [**81**].

Chapter 5 is about foliations with tangential CR structure i.e. each of whose leaves is a CR manifold. We look at foliations by level hypersurfaces of the defining function of a strictly pseudoconvex domain in $\mathbb{C}^n$ such as occurring in C.R. Graham and J.M. Lee's paper [**124**] (and studied by them in connection with the Dirichlet problem for certain degenerate Laplacians of which the prototype is the Bergman Laplacian on the unit ball in $\mathbb{C}^n$). We give a new axiomatic description of the canonical connection there (the *Graham-Lee connection*) and use it to look at the boundary values of a Yang-Mills field in a Hermitian holomorphic vector bundle $\pi : F \to \Omega$ over a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ (cf. [**26**]). Precisely we endow Ω with the Bergman metric and consider the Dirichlet problem for the Yang-Mills equations

$$\delta^D R^D = 0 \quad \text{in } \Omega, \quad D = D^0 \quad \text{on } \partial\Omega, \tag{0.1}$$

where the boundary data D^0 is a C^∞ Hermitian connection in the Hermitian CR-holomorphic vector bundle $E = \pi^{-1}(\partial\Omega) \to \partial\Omega$. It is then shown that the boundary values D^0 of a solution D to (0.1) must be a pseudo Yang-Mills field on $\partial\Omega$ (cf. our Theorem 5.22). Section 5.6 is based on our own work on tangential Monge-Ampére foliations (cf. [**84**] and fitting into the theory of pseudohermitian immersions, cf. also [**89**]).

Chapters 6 to 8 are based on work on transversally CR foliations by the first two authors (cf. *op. cit.*). Chapter 6 is devoted to the basics while Chapters 7 and 8 present two main applications. The first regards the interrelation between G. Gigante and G. Tomassini's theory of CR Lie algebras (cf. [**116**]) and F. Fedida's $\mathcal{G}$-Lie foliations (cf. [**103**]) and includes a homotopy classification of transverse f-structures. The second is devoted to solving a transverse Beltrami equation, which is a foliated analogue of the Beltrami equation in the work of A. Korányi and H.M. Reimann, [**159**]. The effect is producing foliated quasiconformal mappings (cf. E. Barletta, [**16**]). These results extend A. Korányi and H.M. Reimann's considerations - originally holding only on strictly pseudoconvex CR manifolds - to certain degenerate CR manifolds where the degeneracy may be 'factored out' by an algebraic process leading to a strictly pseudoconvex transversally CR foliation. The authors hope that Section 6.4 may contribute to a better understanding of the features of degenerate CR manifolds.

At least for compact Hausdorff foliations i.e. foliations with all leaves compact and the leaf space Hausdorff, the leaf space has (by a result of J. Girbau and M. Nicolau, [**120**], relying itself on a result by D.B.A. Epstein, [**100**]) a natural structure of an *orbifold* (or *V-manifold* in the terminology of I. Satake, [**213**], to whom the notion is due). If this is the case a given transverse CR structure induces a CR structure (in the sense of Chapter 11) on the leaf space, the latter becoming a *CR orbifold.* Chapter 11 aims to a motivation of the need for a theory of CR orbifolds and states some open problems. On the other hand, there is a growing theory of orbifolds, among whose contributors one finds W.L. Baily, [**8**]-[**10**], J.E. Borzellino, [**51**], J.E. Borzellino and B.G. Lorica, [**53**], J.E. Borzellino and S-H. Zhu, [**52**], J.E. Borzellino and V. Brunsden, [**54**], M. Carlotti, [**62**]-[**63**], J. Girbau and M. Nicolau, [**120**], T.D. Jeffres, [**141**], H. Kitahara, [**154**], L-K. Koh, [**172**], T. Shioya, [**220**], and I. Satake himself, [**213**]-[**215**], but to the knowledge of the authors no monograph is available on this subject except for a portion of [**239**], confined to the 3-dimensional case, and of J.E. Borzellino's Ph.D. thesis, [**51**]. There are many differences in style and notations between the above quoted papers and also some inadequacies (for instance [**62**] *postulates* the existence of the monomorphism η while that may be proved, cf. Section 9.3 of this monograph). We choose to expose carefully the basics of the theory of orbifolds in Chapter 9, following mainly the paper [**120**] and hoping to remedy to the mentioned inadequacies and hinting to a further development of differential geometry and analysis on CR orbifolds. Ending these comments, we would like to mention the work by Y-J. Chiang, [**67**], on harmonic maps from a Riemannian orbifold to an ordinary Riemannian manifold (and showing that in the homotopy class of a map of a Riemannian orbifold into a Riemannian manifold of negative sectional curvature there is a harmonic representative). Y-J. Chiang's result is generalized by the work of A. El Kacimi-Alaoui and E.G. Gomez's Theorem 6 in [**148**], p. 121, as $W/\mathrm{SO}(q)$ (where W is the base of the fibration giving rise to the basic foliation associated with the lifted foliation, cf. our section 1.2) is not an orbifold unless the action of $\mathrm{SO}(q)$ on W is locally free. This means that the open problem (of which only the local part is dealt with in Section 11.5) regarding the existence of a parametrix for the Kohn-Rossi operator on a CR orbifold may find its proper and more general setting in a theory of *transversally subelliptic operators* eventually parallelling A. El Kacimi-Alaoui's work [**145**].

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