CHAPTER 1

Review of foliation theory

We devote this chapter to recalling the differential geometric notions of foliation theory needed throughout. There are quite a few books on the geometry and topology of foliations, e. g. P. Molino, [**179**], P. Tondeur, [**243**]-[**244**], C. Godbillon, [**121**], H.B. Lawson, [**164**], A. Candel & L. Conlon, [**59**], V.Y. Rovenskii, [**209**], I. Tamura, [**234**], C.C. Moore & C. Schochet, [**185**], G. Hector & U. Hirsch, [**132**], C. Camacho & A. Lins Neto, [**58**], and B.L. Reinhart, [**205**], the first two of which are largely quoted through these notes. Therefore, the material on general foliation theory we include is kept to a minimum, intentionally with the scope to fix notations and conventions, hint to some developements (cf. e. g. the finite dimensionality of the basic chomology of a Riemannian foliation on a compact manifold, [**147**], or the study of harmonic foliated maps, [**148**]) not described in the above quoted texts, and to some unsolved problems.

1.1. Basic notions

This section describes the very basic notions of foliation theory, such as leaves, saturated sets, transverse manifolds, and holonomy. For details, see [**179**].

1.1.1. The model foliation. To start with we think of a foliation of a given manifold M in a quantitative way, as a partition of M whose individual subsets are referred to as *leaves*. There is a remarkable foliation of $M = \mathbb{R}^m$ which turns out to be a local model for foliations of arbitrary manifolds.

Definition 1.1. *Let us consider $\mathbb{R}^m = \mathbb{R}^p \times \mathbb{R}^q$ with the Cartesian coordinates $(x, y) = (x^1, \cdots, x^p, y^1, \cdots, y^q)$. The* model foliation *of dimension p and codimension q is the family of all affine subspaces (the* leaves *of the model foliation) of $\mathbb{R}^m$ parallel to $\mathbb{R}^p$.* □

The foliation of $\mathbb{R}^m$ in Definition 1.1 may be thought of as the foliation by level sets of the map

$$\pi_q : \mathbb{R}^m \to \mathbb{R}^q, \quad \pi_q(x, y) = y.$$

Foliations by level sets of a given function possess remarkable geometric properties and will be given appropriate room later in this book (cf. Section 5.3). The local automorphisms of the model foliation, i.e. the local diffeomorphisms of $\mathbb{R}^m$ mapping leaves to leaves, have a particularly simple structure.

Definition 1.2. Let $\phi : U \to U'$, $\phi = (f^1, \cdots, f^p, g^1, \cdots, g^q)$, be a local diffeomorphism of $\mathbb{R}^m$. Then ϕ is a *local automorphism* of the model foliation if

$$\frac{\partial g^j}{\partial x^\alpha} = 0, \quad 1 \le j \le q, \quad 1 \le \alpha \le p.$$

□

When analyzing the local structure of a given foliation, one is interested in the connected components of the intersection of a leaf with some distinguished open set, the so called *plaques* of the foliation. For the model foliation these have a simple aspect, yet a leaf may intersect the same open set along several plaques. Let $\pi_q : \mathbb{R}^m \to \mathbb{R}^q$ be the natural projection, as above. Let $U \subseteq \mathbb{R}^m$ be an open subset and $\xi \in U$. Set $y = \pi_q(\xi)$.

DEFINITION 1.3. The *plaque* of ξ in U is the connected component of $U \cap \pi_q^{-1}(y)$ containing ξ. □

If $\phi : U \to U'$, $\phi = (f, g)$, is a local automorphism of the model foliation and $\xi \in U$ then it is an easy exercise that

$$\phi(U \cap \pi_q^{-1}(y)) \subseteq U' \cap \pi_q^{-1}(g(y))$$

i.e. ϕ maps the plaque of ξ in U onto the plaque of $\phi(\xi)$ in U'.

Let $\Gamma_{m,q}$ be the set of all local automorphisms of the model foliation of $\mathbb{R}^m$. This is a pseudogroup of transformations of $\mathbb{R}^m$. For arbitrary manifolds M a foliation will be specified on M by indicating an atlas of M whose transition functions belong to $\Gamma_{m,q}$.

DEFINITION 1.4. Let M be a real m-dimensional C^∞ manifold. A *foliated atlas* of codimension q on M is a C^∞ atlas of M whose transition functions are local automorphisms of the model foliation of codimension q of $\mathbb{R}^m$. A *foliation* of *codimension* q of M is a maximal foliated atlas $\mathcal{F}$ of codimension q of M and a pair $(M, \mathcal{F})$ is a *foliated manifold.* □

DEFINITION 1.5. *Let* $(M, \mathcal{F})$ *be a foliated manifold. Local coordinate systems* $(U, \varphi) \in \mathcal{F}$ *are referred to as* foliated charts *and* U *is a* distinguished *open set. Also* $\varphi = (x^1, \cdots, x^p, y^1, \cdots, y^q)$ *are* distinguished coordinates *and* $(y^1, \cdots, y^q)$ *are* transverse coordinates. □

DEFINITION 1.6. *Let* (U, φ) *be a foliated chart of* $(M, \mathcal{F})$. *If* $x \in U$ *let* α *be the plaque of* $\varphi(x)$ *in* $\varphi(U)$ *with respect to the model foliation. Then* $\varphi^{-1}(\alpha)$ *is the* plaque *of* x *in* U. □

Clearly given a foliated chart (U, φ) with $\varphi = (x^1, \cdots, x^p, y^1, \cdots, y^q)$ and given a plaque $\varphi^{-1}(\alpha) \subset U$ there are constants $(c^1, \cdots, c^q) \in \mathbb{R}^q$ such that $\varphi^{-1}(\alpha)$ is a connected component of $\{x \in U : y^1(x) = c^1, \cdots, y^q(x) = c^q\}$.

Let $x_0 \in M$ and let (U, φ) be a foliated chart such that $x_0 \in U$. Let P_{x_0} be the span of $\{(\partial/\partial x^\alpha)(x_0) : 1 \leq \alpha \leq p\}$. Clearly the definition of P_{x_0} does not depend upon the choice of distinguished local coordinates at x_0 and the assignment $x \in M \mapsto P_x$ is a C^∞ distribution of rank p on M.

DEFINITION 1.7. P_x is the *tangent space* to $\mathcal{F}$ at x and is also denoted by $T(\mathcal{F})_x$. The distribution $T(\mathcal{F})$ is the *tangent bundle* of $\mathcal{F}$. □

Each $X \in T(\mathcal{F})$ is locally a linear combination of the $\partial/\partial x^\alpha$'s hence $T(\mathcal{F})$ is involutive i.e. $[X, Y] \in T(\mathcal{F})$ for any $X, Y \in T(\mathcal{F})$. The converse is a difficult yet classical result, the Frobenius theorem: if P is an involutive p-dimensional C^∞ distribution on a m-dimensional manifold M then P is *integrable* i.e. there is a codimension $q = m - p$ foliation $\mathcal{F}$ of M such that $P = T(\mathcal{F})$. An integrable subbundle of $T(M)$ is often referred to as a foliation of M, as well.

When his branch is yet tender, and putteth
forth *leaves*, ye know that summer is nigh.
Mathew 24-23

1.1.2. Leaves. Let $(M, \mathcal{F})$ be a foliated manifold, of codimension q. Let $\Omega(x_0, x)$ be the set of all piecewise C^1 differentiable curves in M joining the points x_0 and x. Let P be the tangent bundle of $\mathcal{F}$.

DEFINITION 1.8. The *leaf* through x_0 is the set L_{x_0} consisting of all points $x \in M$ which may be joined to x_0 by a curve $\gamma \in \Omega(x_0, x)$ tangent to P i.e. $(d\gamma/dt)(t) \in P_{\gamma(t)}$ for each value of the parameter t for which $(d\gamma/dt)(t)$ is defined. □

PROPOSITION 1.9. *If $x \in L_{x_0}$ and α_x is the plaque through x in a distinguished open set then α_x is contained in L_{x_0}.*

Proof. Let $\Omega_P(x, z)$ be the set of all piecewise C^1 curves $\gamma : [0, 1] \to M$ joining the points $x, z \in M$ (i.e. $\gamma(0) = x$ and $\gamma(1) = z$) and such that $(d\gamma/dt)(t) \in P_{\gamma(t)}$ for all values of the parameter t for which $(d\gamma/dt)(t)$ is defined. Then the leaf L_{x_0} of $\mathcal{F}$ passing through x_0 is given by $L_{x_0} = \{x \in M : \Omega_P(x_0, x) \neq \emptyset\}$. Let $x \in L_{x_0}$ and $(U, \varphi) \in \mathcal{F}$ such that $x \in U$. We set $p := \varphi(x) \in \mathbb{R}^m$ and $p = (\xi, \eta) \in \mathbb{R}^p \times \mathbb{R}^q$. Let C be the connected component of p in $\Omega \cap (\mathbb{R}^p \times \{\eta\})$ where $\Omega = \varphi(U) \subseteq \mathbb{R}^m$. Then the plaque α_x in U through x is the set $\alpha_x = \varphi^{-1}(C)$. Let $y \in \alpha_x$ and consider the curve $a : [0, 1] \to \mathbb{R}^m$ given by $a(t) = (1 - t)p + tq$ for any $0 \leq t \leq 1$, where $q = \varphi(y)$. We may arrange from the very beginning that Ω is a convex set so that a is a curve in $\Omega \cap (\mathbb{R}^p \times \{\eta\})$. Moreover $a([0, 1])$ is a connected set containing the point p hence $a([0, 1]) \subseteq C$. Therefore the curve

$$\gamma : [0, 1] \to U, \quad \gamma(t) = \varphi^{-1}(a(t)), \quad 0 \leq t \leq 1,$$

is well defined and $\gamma(t) \in \alpha_x$ for any $0 \leq t \leq 1$. If the distinguished local coordinates are $\varphi = (x^1, \cdots, x^p, y^1, \cdots, y^q)$ then

$$\frac{d\gamma}{dt}(t) = (x^\alpha(q) - \xi^\alpha) \left. \frac{\partial}{\partial x^\alpha} \right|_{\gamma(t)} \in P_{\gamma(t)}$$

for all t (where the tangent to γ is defined). Consequently $\gamma \in \Omega_P(x, y)$ and we are done.

The plaques in L_{x_0} form a base of open sets for the *leaf topology* on L_{x_0}. In this topology L_{x_0} is arcwise connected. Also L_{x_0} admits a structure of p-dimensional C^∞ manifold whose underlying topology is the leaf topology. Traces of open sets in M on a leaf L are open in the leaf topology.

DEFINITION 1.10. Let S be a submanifold of M i.e. the inclusion $i : S \hookrightarrow M$ is an immersion. Then S is *weakly embedded* if for any C^∞ manifold N and any C^∞ map $f : N \to M$ with $f(N) \subset S$ the corestriction $f : N \to S$ is a C^∞ map. □

Each leaf of a foliation $\mathcal{F}$ of M is a weakly embedded submanifold of M. An important example of foliation is that of a foliation defined by a surjective C^∞ submersion $f : M \to N$. One checks easily that the *vertical distribution* $P = \mathrm{Ker}(df)$ is involutive and hence integrable. It determines a foliation $\mathcal{F}$ of M whose leaves are the connected components of the fibres of f. The leaves of $\mathcal{F}$ are closed embedded submanifods of M. More room will be dedicated to the argument in section 1.1.5.

1.1.3. Foliated maps. Let $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ be two foliated manifolds and $P = T(\mathcal{F})$, $P' = T(\mathcal{F}')$ the tangent bundles of $\mathcal{F}$ and $\mathcal{F}'$, respectively.

DEFINITION 1.11. A C^∞ map $f : M \to M'$ is a *foliated map* if $(d_x f)P_x \subseteq P'_{f(x)}$ for any $x \in M$. An *automorphism* of $(M, \mathcal{F})$ is a C^∞ diffeomorphism of M and a foliated map. □

Let $\mathrm{Aut}(M, \mathcal{F})$ be the group of all (global) automorphisms of $(M, \mathcal{F})$. Each $\phi \in \mathrm{Aut}(M, \mathcal{F})$ maps locally leaves to leaves.

1.1.4. Saturated sets. Let $(M, \mathcal{F})$ be a foliated manifold.

DEFINITION 1.12. A subset $A \subseteq M$ is a *saturated set* if A is a union of leaves of $\mathcal{F}$. □

DEFINITION 1.13. Two points in M are *equivalent* if they belong to the same leaf of $\mathcal{F}$. □

One checks easily that the relation in Definition 1.13 is an equivalence relation on M and that the corresponding equivalence classes are the leaves of $\mathcal{F}$ themselves. The corresponding quotient space is denoted by $M/\mathcal{F}$ and is thought of as carrying the quotient topology.

DEFINITION 1.14. $M/\mathcal{F}$ is the *leaf space* of $(M, \mathcal{F})$. □

In general $M/\mathcal{F}$ may fail to admit a manifold structure whose underlying topology is the quotient topology. Let $\pi : M \to M/\mathcal{F}$ be the natural projection.

DEFINITION 1.15. The set $\pi^{-1}(\pi(A))$ is the *saturation* of $A \subseteq M$. □

For an arbitrary subset $A \subseteq M$ its saturation $\pi^{-1}(\pi(A))$ is a saturated set containing A. The closure and interior of a saturated set are saturated sets. The saturation of an open set is open.

DEFINITION 1.16. The *saturated topology* of M is the topology whose open sets are the saturated open sets. □

Let us mention that, besides from the saturated topology and the topology underlying the C^∞ manifold structure, a foliated manifold M admits a third topology, the so called *leaf topology* for which the plaques in M are a base of open sets.

1.1.5. Simple foliations. Let $(M, \mathcal{F})$ be a foliated manifold and $P = T(\mathcal{F})$ its tangent bundle. We adopt the following

DEFINITION 1.17. The foliation $\mathcal{F}$ is said to be *simple* if there is a surjective submersion f of M onto some C^∞ manifold N such that each fibre of f is connected and $T(\mathcal{F})_x = \mathrm{Ker}(d_x f)$ for any $x \in M$ i.e. $T(\mathcal{F})$ is the vertical bundle associated to f. □

For a simple foliation $\mathcal{F}$ there is a natural identification of the leaf space $M/\mathcal{F}$ with N. Hence the leaf space of a simple foliation admits a C^∞ manifold structure whose underlying topology is the quotient topology. Let us look at the following

EXAMPLE 1.18. Let $\mathbb{C}^{n+1}$ with the complex coordinates $(z^1, \cdots, z^n, w)$, $w = u + iv$, and $\alpha : \mathbb{R} \to \mathbb{R}$ a C^∞ function such that $\alpha(0) = 0$ and $\alpha'(t) < 0$, for any $t \in \mathbb{R}$. Define $f : \mathbb{C}^{n+1} \to \mathbb{R}$ by

$$f(z^1, \cdots, z^n, w) = \alpha(|z^1|^2 + \cdots + |z^n|^2 - v)e^u.$$

Then f is a submersion so that it defines a simple foliation $\mathcal{F}$ of $\mathbb{C}^{n+1}$ whose leaves are the level sets of f. Note that

$$f^{-1}(c) = \begin{cases} \{(z,w) \in \Omega_{n+1} : u = \log(c/\alpha(\rho))\} & \text{if } c > 0 \\ \partial\Omega_{n+1} \approx \mathbb{H}_n & \text{if } c = 0 \\ \{(z,w) \in \mathbb{C}^{n+1} \setminus \overline{\Omega}_{n+1} : u = \log(c/\alpha(\rho))\} & \text{if } c < 0 \end{cases}$$

where $\rho = \sum_{\alpha=1}^{n} |z^\alpha|^2 - v$ and $\Omega_{n+1} \subset \mathbb{C}^{n+1}$ is the Siegel domain $\Omega_{n+1} = \{(z,w) : \rho < 0\}$. Thus $\mathcal{F}$ is a foliation of $\mathbb{C}^{n+1}$ by real hypersurfaces one of whose leaves is the Heisenberg group $\mathbb{H}_n$ (cf. Chapter 2). □

1.1.6. Transverse submanifolds. Let $(M, \mathcal{F})$ be a foliated manifold. Let $\nu(\mathcal{F}) = T(M)/T(\mathcal{F})$ be the *normal*, or *transverse*, bundle. In general there isn't a natural choice of complement E to $T(\mathcal{F})$ in $T(M)$ such that $E \approx \nu(\mathcal{F})$ (a vector bundle isomorphism). Of course the possibility of choosing an integrable distribution E on M such that $T_x(M) = T_x(\mathcal{F})_x \oplus E_x$ for any $x \in M$ is even scarcer. However it turns out that such a decomposition is feasible along certain submanifolds of M.

DEFINITION 1.19. A submanifold T of a foliated manifold $(M, \mathcal{F})$ is a *transverse submanifold* if $T_x(M) = T(\mathcal{F})_x \oplus T_x(T)$ for any $x \in T$. A *total transversal* is a transverse submanifold which meets all leaves of $\mathcal{F}$. □

Let T be a transverse submanifold of $(M, \mathcal{F})$. If $U \subseteq M$ is an open subset and $\mathcal{F}_U$ the foliation induced by $\mathcal{F}$ on U (i.e. $T(\mathcal{F}_U)$ is the portion of $T(\mathcal{F})$ over U) then $T \cap U$ is a transverse submanifold of $(U, \mathcal{F}_U)$.

DEFINITION 1.20. An open set $U \subseteq M$ is said to be *simple* if i) the leaf space $U/\mathcal{F}_U$ has a C^∞ manifold structure whose underlying topology is the quotient topology, ii) the quotient map $\pi_U : U \to U/\mathcal{F}_U$ is a C^∞ submersion, and iii) $\mathcal{F}_U$ is the simple foliation associated with π_U. □

REMARK 1.21. If $V \subseteq M$ is a simple open set then for any point $x_0 \in V$ there is an open subset $U \subseteq V$ such that $x_0 \in U$ which is both simple and distinguished. Indeed let $(U, \varphi) \in \mathcal{F}$ such that $x_0 \in U \subseteq V$. Let L be a leaf of $\mathcal{F}$ passing through $x \in U$. Let $p := \pi_V(x)$ where $\pi : V \to V/\mathcal{F}_V$ is the C^∞ submersion locally defining $\mathcal{F}$, according to Definition 1.20. Let π_U be the restriction of π_V to U. Then

$$\pi_U^{-1}(p) = U \cap \pi_V^{-1}(p) = U \cap (V \cap L) = U \cap L$$

i.e. U is simple, too. □

The following local reformulation[1] of the Frobenius theorem is also useful in practice

THEOREM 1.22. *Let $U \subseteq \mathbb{R}^n$ be an open subset and $\eta_1, \cdots, \eta_q \in \Omega^1(U)$ real-valued linearly independent differential* 1*-forms. Let $x_0 \in U$. The following statements are equivalent*

i) *There is an open neighborhood $W \subseteq U$ of x_0 and there exist real-valued* 1*-forms $\alpha_{ij} \in \Omega^1(W)$ such that*

$$d\eta_i = \sum_{j=1}^{q} \alpha_{ij} \wedge \eta_j \,, \quad 1 \leq i \leq q. \tag{1.1}$$

[1]Resembling superficially to the deeper complex Frobenius-Nirenberg theorem (cf. [**194**]).

ii) *There is an open neighborhood* $W \subseteq U$ *of* x_0 *and there exist smooth functions* $\sigma_i, \beta_{ij} \in C^\infty(W, \mathbb{R})$, $1 \le i, j \le q$, *such that* $\det[\beta_{ij}] \neq 0$ *everywhere in* W *and*

$$\eta_i = \sum_{j=1}^{q} \beta_{ij} d\sigma_j \, , \quad 1 \le i \le q. \tag{1.2}$$

Let us show that Theorem 1.22 follows from the Frobenius theorem as stated above. Let us assume that (i) in Theorem 1.22 holds and set

$$P_x = \{v \in T_x(\mathbb{R}^n) : \eta_{i,x}(v) = 0, \quad 1 \le i \le q\}, \quad x \in U.$$

Then $x \in U \mapsto P_x$ is a $(n-q)$-dimensional smooth distribution on U. As a consequence of (i) the distribution P is involutive. Indeed, for any $X, Y \in P$

$$\eta_i([X,Y]) = -2(d\eta_i)(X,Y) = -2\sum_{j=1}^{q}(\alpha_{ij} \wedge \eta_j)(X,Y) =$$

$$= -\sum_{j=1}^{q}\{\alpha_{ij}(X)\eta_j(Y) - \alpha_{ij}(Y)\eta_j(X)\} = 0$$

i.e. $[X,Y] \in P$. By the Frobenius theorem P must be integrable. Let then $\mathcal{F}$ be a codimension q foliation such that $P = T(\mathcal{F})$. As $\mathcal{F}$ may be described locally by C^∞ submersions (cf. Definition 1.20 and Remark 1.21 above) given $x_0 \in U$ there is an open neighborhood $W \subseteq U$ of x_0 and there exists a C^∞ submersion $\sigma = (\sigma_1, \cdots, \sigma_q) : W \to \mathbb{R}^q$ such that

$$T(\mathcal{F})_x = \bigcap_{i=1}^{q} \mathrm{Ker}(d\sigma_i)_x \, , \quad x \in W.$$

Then it is a matter of linear algebra that $\eta_{i,x} = \sum_{j=1}^{q} \beta_{ij}(x)\,(d\sigma_j)_x$ for some $\beta_{ij}(x) \in \mathbb{R}$ (depending differentiably of x and such that $\det[\beta_{ij}(x)] \neq 0$), which is (ii) in Theorem 1.22. Finally we may take the exterior differential of (1.2) to obtain (1.1) with $\alpha_{ij} = \sum_{k=1}^{q} \gamma_{kj} d\beta_{ik}$ and $[\gamma_{ij}] = [\beta_{ij}]^{-1}$. □

We emphasize that the statements in Theorem 1.22 are really about the differential ideal $\mathcal{I}^\bullet_\eta(W)$ of $\Omega^\bullet(W)$ (the de Rham algebra of all real-valued differential polynomials on W) spanned by $\eta_1, \cdots, \eta_q$ that is

$$\mathcal{I}^\bullet_\eta = \{\sum_{i=1}^{q} \eta_i \wedge \omega_i : \omega_i \in \Omega^\bullet(W), \quad 1 \le i \le q\}.$$

Indeed (i) in Theorem 1.22 is equivalent to $d\mathcal{I}^\bullet_\eta(W) \subseteq \mathcal{I}^\bullet_\eta(W)$, while (ii) is equivalent to the statement that $\{d\sigma_1, \cdots, d\sigma_q\}$ spans $\mathcal{I}^\bullet_\eta(W)$.

If $U \subseteq M$ is a simple open set and $T \subset M$ is a transversal such that $T \cap U \neq \emptyset$ then the differential

$$d_x\pi_U : T_x(T \cap U) \to T_{\pi_U(x)}(U/\mathcal{F}_U)$$

is a linear isomorphism for any $x \in T \cap U$. Hence π_U is a local diffeomorphism of $T \cap U$ into $U/\mathcal{F}_U$. Consequently, the saturation $\pi_U^{-1}(\pi_U(T \cap U))$ of $T \cap U$ is an open set in U hence the saturation $\pi^{-1}(\pi(T))$ of T in M is an open set.

For any $x \in M$ there is a transverse submanifold T through x. This may be constructed locally, in a foliated local chart (U, φ) at x by merely pulling back

$(\{\xi\} \times \mathbb{R}^q) \cap \varphi(U)$ via φ, where $\varphi(x) = (\xi, \eta) \in \mathbb{R}^p \times \mathbb{R}^q$. It is noteworthy that the resulting transverse submanifold T meets each plaque of U in but one point i.e. T is a (connected) total transversal of $(U, \mathcal{F}_U)$. In general, this does not imply that T meets each leaf L of $\mathcal{F}$ in a single point (as two plaques of U may lie on the same leaf) unless L is *proper* i.e. the leaf topology and the induced topology of L actually coincide. It should be emphasized that given a proper leaf L and a point $x \in L$ there is a transversal T at x such that $L \cap T = \{x\}$ (in general a proper leaf may meet a transversal at infinitely many points). Every closed leaf is proper.

1.1.7. Holonomy. Let $(M, \mathcal{F})$ be a foliated manifold. Let $L \in M/\mathcal{F}$ be a fixed leaf and T, T' two transversals passing through the points $x, x' \in L$. Let $\gamma : [0,1] \to L$ be a continuous curve joining x and x' i.e. $\gamma(0) = x$ and $\gamma(1) = x'$. Next, let us consider a partition $t_0 = 0 < t_1 < \cdots < t_{k-1} < t_k = 1$ of the interval $[0,1]$ such that each curve segment $\gamma([t_{i-1}, t_i])$ is contained in a simple open set U_i, for $1 \le i \le k$. In other words, we cover the curve γ by a finite set $\{U_1, \cdots, U_k\}$ of simple open sets. Of course, this is possible because $\gamma([0,1])$ is compact and $M \supset \gamma([0,1])$ admits a covering consisting of simple open sets. By Remark 1.21 one may assume w.l.o.g. that the open sets U_i are both simple and distinguished. Let $f_i : U_i \to N_i := U_i/\mathcal{F}_{U_i}$ be the local defining submersions and $\varphi_i : U_i \to \Omega_i := \varphi_i(U_i) \subseteq \mathbb{R}^m$ the corresponding distinguished local charts. Let us set $x_i := \gamma(t_i)$ and $p_i = f_i(x_i) \in N_i$ for $1 \le i \le k$ so that $L \cap U_i = f_i^{-1}(p_i)$. Let α_i be the plaque in U_i passing through the point x_i i.e. $\alpha_i = \varphi^{-1}(C_i)$ and C_i is the connected component of $\varphi_i(x_i) = (\xi_i, \eta_i) \in \mathbb{R}^p \times \mathbb{R}^q$ in $\Omega_i \cap (\mathbb{R}^p \times \{\eta_i\})$. By Proposition 1.9 one has $x_i \in \alpha_i \subseteq L$. Moreover $K_i := \gamma([t_{i-1}, t_i])$ is a connected set containing the point x_i hence $K_i \subseteq \alpha_i$. Let now T_i be just any transverse submanifold of $(U_i, \mathcal{F}_{U_i})$ passing through x_i, for $1 \le i \le k-1$. For instance one may consider

$$T_i := \varphi_i^{-1}(\Omega_i \cap (\{\xi_i\} \times \mathbb{R}^q)).$$

Also we set $T_0 = T$ and $T_k = T'$. Both $d_{x_{i-1}} f_i : T_{x_{i-1}}(T_{i-1}) \to T_{p_i}(N_i)$ and $d_{x_i} f_i : T_{x_i}(T_i) \to T_{p_i}(N_i)$ are $\mathbb{R}$-linear isomorphisms hence there are open neighborhoods of x_{i-1} in T_{i-1}, respectively of x_i in T_i

$$x_{i-1} \in \mu_{i-1} \subseteq T_{i-1}, \quad x_i \in \mu_i \subseteq T_i,$$

such that $f_i : \mu_{i-1} \to N_i$ and $f_i : \mu_i \to N_i$ are diffeomorphisms on their images. Then $A_i := f_i(\mu_{i-1}) \cap f_i(\mu_i)$ is an open neighborhood of p_i in N_i. Let F_{i-1} and F_i be the restrictions of f_i to μ_{i-1} and μ_i, respectively. We consider the open sets

$$\nu_{i-1} = F_{i-1}^{-1}(A_i) \subseteq T_{i-1}, \quad \nu_i = F_i^{-1}(A_i) \subseteq T_i,$$

and the diffeomorphism

$$\phi_i := (F_i : A_i \to \nu_i)^{-1} \circ (F_{i-1} : \nu_{i-1} \to A_i) : \nu_{i-1} \to \nu_i.$$

DEFINITION 1.23. The diffeomorphism $\phi_i : \nu_{i-1} \approx \nu_i$ is said to be obtained by *sliding along the plaques* in U_i. □

Moreover let us set $\nu = \nu_0$ and $\nu' = \nu_k$. We may consider the diffeomorphism

$$\phi_k \circ \phi_{k-1} \circ \cdots \circ \phi_1 : \nu \to \nu'. \tag{1.3}$$

and denote its germ at x by h_γ. It may be shown that the definition of the diffeomorphism (1.3) doesn't depend upon the choice of the transversal submanifolds $T_1, \cdots, T_{k-1}$.

Definition 1.24. h_γ is called *sliding along the leaves along* γ. □

It may be shown that the definition of h_γ doesn't depend upon the choice of the simple sets U_i covering γ but only on γ itself. Also, by considering a continuous deformation $\gamma_s(t)$ of γ with end points fixed such that the previously chosen chain $\{U_1, \cdots, U_k\}$ covers $t \mapsto \gamma_s(t)$ for each value of the parameter s, one may show that h_γ depends only on the homotopy class of γ.

Let $\gamma : [0,1] \to L$ be a loop at x i.e. $\gamma(0) = \gamma(1) = x$. Let T be a transversal at x. Then h_γ built above (with $x' = x$ and $T' = T$) is a germ at x of a local diffeomorphism of T which maps x to itself. One may check that given another loop τ in L at x one has $h_{\gamma\cdot\tau} = h_\gamma \circ h_\tau$ (where $\gamma \cdot \tau$ is the juxtaposition of the two loops) so that there is a natural group homomorphism

$$h_x : \pi_1(L, x) \to \mathrm{Diff}_x(T), \quad [\gamma] \mapsto h_\gamma\,.$$

Here $\pi_1(L, x)$ is the fundamental group of L with base point x and $[\gamma] \in \pi_1(L, x)$ is the homotopy class of the loop γ at x. Also $\mathrm{Diff}_x(T)$ denotes the group of germs at x of local diffeomorphisms of T in itself.

Definition 1.25. h_x *is the* holonomy representation *of* L *at* x *and its image* $h_x(\pi_1(L, x))$ *is the* holonomy group *of* L *at* x. □

1.2. Transverse geometry

In this section we discuss basic forms and the corresponding basic cohomology of a foliated manifold, cf. [**179**]. Also we recall the facts we need from the geometry of transverse G-structures on foliated manifolds, cf. L. Conlon, [**70**], and P. Molino, [**180**]. See also F. Kamber & P. Tondeur, [**149**]-[**150**]. Moreover, we state the first structure theorem on Riemannian foliations (cf. P. Molino, [**182**]) and review some of its applications.

1.2.1. Basic functions, foliate vector fields, and basic forms.

Let $(M, \mathcal{F})$ be a foliated manifold and $P = T(\mathcal{F})$ the tangent bundle of the foliation.

Definition 1.26. A C^∞ function f on M is *basic* if $X(f) = 0$ for any $X \in P$. □

Let $\Omega^0_B(\mathcal{F})$ be the ring of all real valued basic functions, a subring of $\Omega^0(M) = C^\infty(M)$. A function $f \in \Omega^0(M)$ is basic if and only if its local expression with respect to any foliated chart (U, x^α, y^i) is a function of the variables y^i only.

Definition 1.27. A tangent vector field $Y \in \mathcal{X}(M)$ is *foliate* if $[X, Y] \in P$ for any $X \in P$. □

A vector field $Y \in \mathcal{X}(M)$ is foliate if and only if the components Y^{j+p} of Y with respect to an arbitrary foliated chart (U, x^α, y^i)

$$Y = Y^\alpha \frac{\partial}{\partial x^\alpha} + Y^{j+p} \frac{\partial}{\partial y^j}$$

depend on the variables y^i only. Let $V(\mathcal{F})$ be the Lie algebra of all foliate vector fields. $V(\mathcal{F})$ is a Lie subalgebra of $\mathcal{X}(M)$.

Let $Q = \nu(\mathcal{F}) = T(M)/P$ be the normal bundle of $(M, \mathcal{F})$ and let $\Pi : T(M) \to Q$ be the natural bundle map.

DEFINITION 1.28. The elements of $\ell(\mathcal{F}) = \Pi\, V(\mathcal{F}) = \{\Pi(Y) : Y \in V(\mathcal{F})\}$ are the *transverse vector fields* of $(M, \mathcal{F})$. □

Clearly

$$0 \to \Gamma^\infty(P) \hookrightarrow V(\mathcal{F}) \xrightarrow{\Pi} \ell(\mathcal{F}) \to 0$$

is a short exact sequence of Lie algebras and Lie algebra homomorphisms.

DEFINITION 1.29. A C^∞ differential k-form η on M is *basic* if $X \rfloor \eta = 0$ and $X \rfloor d\eta = 0$ for any $X \in P$. □

Let $\Omega^k_B(\mathcal{F})$ be the space of all basic k-forms, a $\Omega^0_B(\mathcal{F})$-submodule of $\Omega^k(M) = \Gamma^\infty(\Lambda^k T^* M)$. Note that the differential of a basic form is again a basic form, hence d induces a differential operator

$$d_B : \Omega^k_B(\mathcal{F}) \to \Omega^{k+1}_B(\mathcal{F}).$$

DEFINITION 1.30. The *basic cohomology* of $(M, \mathcal{F})$ is the cohomology of the complex $\{\Omega^\bullet_B(\mathcal{F}), d_B\}$ i.e.

$$H^k_B(\mathcal{F}) = \frac{Ker\{d_B : \Omega^k_B(\mathcal{F}) \to \cdot\}}{d_B \Omega^{k-1}_B(\mathcal{F})}, \quad k \geq 0.$$

Here $\Omega^{-1}_B(\mathcal{F}) = (0)$. □

Roughly speaking $H^\bullet_B(\mathcal{F})$ is the 'de Rham cohomology' of the leaf space $M/\mathcal{F}$. Therefore, it is a natural question whether M compact implies that $H^\bullet_B(\mathcal{F})$ is finite dimensional for every $\mathcal{F}$. For instance, if $\mathcal{F}$ is the simple foliation defined by the surjective submersion $f : M \to N$ then $H^\bullet_B(\mathcal{F}) \approx H^\bullet(N, \mathbb{R})$ (an algebra isomorphism); if M is compact then N is compact and $H^\bullet_B(\mathcal{F})$ is finite dimensional.

Note that in general $H^1_B(\mathcal{F})$ injects into $H^1(M, \mathbb{R})$ (for if $\eta \in \Omega^1_B(\mathcal{F})$ and $\eta = df$ then $f \in \Omega^0_B(\mathcal{F})$). Hence if we assume that M is compact then $H^1_B(\mathcal{F})$ is finite dimensional. G.W. Schwarz, [**217**], has built examples of n-dimensional foliations $\mathcal{F}$ on compact $(n+3)$-dimensional manifolds M so that $\dim H^k_B(\mathcal{F}) = \infty$, $2 \leq k \leq n+2$. See also [**179**] (the appendix authored by V. Sergiescu) for an example (of a foliation with infinite dimensional basic cohomology) due to E. Ghys.

1.2.2. Transverse G-structures. Let $\mathcal{F}$ be a codimension q foliation of a C^∞ manifold M and $Q = T(M)/T(\mathcal{F})$. Let $\Pi : T(M) \to Q$ be the projection.

DEFINITION 1.31. A $\mathbb{R}$-linear isomorphism $z : \mathbb{R}^q \to Q_x$, $x \in M$, is a *transverse frame* on M at x. □

Let $B^1_T(M, \mathcal{F})_x$ be the space of all transverse frames at x. Then $B^1_T(M, \mathcal{F})$ is a principal $\mathrm{GL}(q, \mathbb{R})$-bundle over M.

DEFINITION 1.32. The *canonical* 1*-form* $\theta_T \in \Gamma^\infty(T^*(B^1_T(M, \mathcal{F})) \otimes \mathbb{R}^q)$ is given by the commutative diagram

$$\begin{array}{ccc} T_z(B^1_T(M,\mathcal{F})) & \xrightarrow{(\theta_T)_z} & \mathbb{R}^q \\ d_z p^1_T \downarrow & & \uparrow z^{-1} \\ T_x(M) & \xrightarrow{\Pi_x} & Q_x \end{array}$$

for any $z \in B^1_T(M, \mathcal{F})_x$, where $p^1_T : B^1_T(M, \mathcal{F}) \to M$ is the natural projection. □

The p-dimensional distribution P_T^1 on $B_T^1(M,\mathcal{F})$ given by

$$(P_T^1)_z = \{X \in T_z(B_T^1(M,\mathcal{F})) : X \rfloor (\theta_T^1)_z = 0\ ,\ X \rfloor (d\theta_T^1)_z = 0\}$$

is integrable hence gives rise to a foliation $\mathcal{F}_T^1$ of $B_T^1(M,\mathcal{F})$ such that $P_T^1 = T(\mathcal{F}_T^1)$.

DEFINITION 1.33. $\mathcal{F}_T^1$ is called the *lifted foliation.* □

It is invariant by right translations and each leaf of $\mathcal{F}_T^1$ is a Galois covering[2] of a leaf of $\mathcal{F}$.

DEFINITION 1.34. A connection 1-form $\omega \in \Gamma^\infty(T^*(B_T^1(M,\mathcal{F})) \otimes \mathbf{gl}(q,\mathbb{R}))$ is a *transverse connection* if $P_T^1 \subset \mathrm{Ker}(\omega)$. □

By a partition of unity argument, transverse connections always exist.

DEFINITION 1.35. A connection 1-form ω in $B_T^1(M,\mathcal{F})$ is *projectable* if $\omega \in \Omega_B^1(\mathcal{F}_T^1)$. □

Clearly projectable connections are transverse. Let $G \subset \mathrm{GL}(q,\mathbb{R})$ be a Lie subgroup and $B_G(M) \to M$ a principal G-subbundle of $B_T^1(M,\mathcal{F})$.

DEFINITION 1.36. $B_G(M)$ is a *transverse G-structure* on $(M,\mathcal{F})$ if $B_G(M)$ is a saturated subset of the foliated manifold $(B_T^1(M,\mathcal{F}),\mathcal{F}_T^1)$. □

The notion is due to L. Conlon, [**70**]. A principal G-subbundle $B_G(M)$ of $B_T^1(M,\mathcal{F})$ is a transverse G-structure if and only if there is a transverse connection ω on $B_T^1(M,\mathcal{F})$ which is adapted to $B_G(M)$ i.e. if $\mathbf{g}$ is the Lie algebra of G then the pullback of ω to $B_G(M)$ is $\mathbf{g}$-valued.

DEFINITION 1.37. If $G = \mathrm{GL}^+(q,\mathbb{R})$ then a transverse G-structure on $(M,\mathcal{F})$ is a *transverse orientation.* If $G = \{e\}$ a transverse G-structure is a *transverse parallelism.* If $G = \mathrm{O}(q)$ then a transverse G-structure is a *transverse Riemannian structure* and a foliation carrying such a structure is a *Riemannian foliation.* Moreover, if $q = 2r$ then a transverse $\mathrm{GL}(r,\mathbb{C})$-structure is a *transverse holomorphic structure* and a foliation with such a structure is a *transversally holomorphic* foliation. □

Therefore, one may discuss first order geometric structures on $Q = \nu(\mathcal{F})$ in the unifying language of transverse G-structures, very much in the spirit of the theory of ordinary G-structures on C^∞ manifolds, cf. e.g. S. Sternberg, [**230**], M. Crampin, [**74**], P. Molino, [**183**]. However, a general theory of equivalence (structure functions, prolongations, etc.) of transverse G-structures has not been fully developed as yet. The existing results in this sense belong to R. Wolak, [**252**].

1.2.3. $d_{\mathcal{F}}$-Cohomology, the Atiyah class. Let $(M,\mathcal{F})$ be a foliated manifold.

DEFINITION 1.38. A principal G-bundle $p : E \to M$ is a *foliated principal bundle* over $(M,\mathcal{F})$ if E carries a foliation $\mathcal{F}_E$ such that 1) $T(\mathcal{F}_E)$ is invariant by right translations, 2) $T(\mathcal{F}_E) \cap \mathrm{Ker}(dp) = (0)$, and 3) $(dp)T(\mathcal{F}_E) = T(\mathcal{F})$. Given an open subset $U \subseteq M$, a section $s : U \to E$ is *foliate* if $(d_x s)T(\mathcal{F})_x \subseteq T(\mathcal{F}_E)_{s(x)}$ for any $x \in U$. □

[2]That is if $S \in B_T^1(M,\mathcal{F})/\mathcal{F}_T^1$ then $L := p_T^1(S) \in M/\mathcal{F}$ and $p_T^1 : S \to L$ is an étale mapping and a locally trivial fibration. Cf. [**179**], p. 47.

Given a point $x_0 \in M$ there is a neighborhood U of x_0 such that foliate sections $s : U \to E$ exist. Let (E, p, M) be a foliated principal G-bundle over $(M, \mathcal{F})$ and $\omega \in \Gamma^\infty(T^*(E) \otimes \mathbf{g})$ a connection 1-form on E.

DEFINITION 1.39. ω is *adapted* to $\mathcal{F}_E$ if $T(\mathcal{F}_E) \subset \mathrm{Ker}(\omega)$. $\square$

Connections adapted to $\mathcal{F}_E$ always exist.

DEFINITION 1.40. ω is *projectable* if $\omega \in \Omega^1_B(\mathcal{F}_E)$. $\square$

Clearly projectable connections are adapted. Let $P = T(\mathcal{F})$ and let P' be a direct summand to P in $T(M)$ i.e. $T(M) = P \oplus P'$. Let $\{X_1, \cdots, X_p\}$ be a local frame of P and $\{Y_1, \cdots, Y_q\}$ a local frame of P' defined on the same open set. Moreover let $\{\theta^1, \cdots, \theta^p, \eta^1, \cdots, \eta^q\}$ be dual to $\{X_\alpha, Y_j : 1 \le \alpha \le p\, , \, 1 \le j \le q\}$.

DEFINITION 1.41. A *pure form* of type (r, s) on $(M, \mathcal{F})$ is a k-form $\alpha \in \Omega^k(M)$ whose restriction to U admits a representation of the form

$$\alpha = \sum_{\substack{1 \le \beta_1 < \cdots < \beta_r \le p \\ 1 \le j_1 < \cdots < j_s \le q}} \alpha_{\beta_1 \cdots \beta_r j_1 \cdots j_s} \, \theta^{\beta_1} \wedge \cdots \wedge \theta^{\beta_r} \wedge \eta^{j_1} \wedge \cdots \wedge \eta^{j_s}$$

for some local C^∞ functions $\alpha_{\beta_1 \cdots \beta_r j_1 \cdots j_s}$ on M, where $r + s = k$. $\square$

Any differential form on M admits a decomposition in terms of pure forms. If $\alpha \in \Omega^k(M)$ then $\alpha_{r,s}$ will denote its pure component of type (r, s).

Let (U, x^α, y^j) be a foliated chart of $(M, \mathcal{F})$ and let Y_j be the P'-component of $\partial/\partial y^j$. Then $\{Y_1, \cdots, Y_q\}$ is a local frame of P' on U and we may take $\eta_\alpha = dy^j$, $1 \le j \le q$. With this choice of local coframes, it follows that the differential $d\alpha$ of any pure form α of type (r, s) is a sum of pure components of type $(r-1, s+2)$, $(r, s+1)$, and $(r+1, s)$. Let $d_{\mathcal{F}}\alpha$ be the pure component of type $(r+1, s)$ of $d\alpha$. Let $\Omega^{r,s}(\mathcal{F})$ be the $\Omega^0(M)$-module of pure forms of type (r, s). We built the operators

$$d_{\mathcal{F}} : \Omega^{r,s}(\mathcal{F}) \to \Omega^{r+1,s}(\mathcal{F}) \tag{1.4}$$

such that $d^2_{\mathcal{F}} = 0$ i.e. (1.4) is a cochain complex.

DEFINITION 1.42. The cohomology

$$H^{r,s}_{\mathcal{F}}(M) = \frac{\mathrm{Ker}\{d_{\mathcal{F}} : \Omega^{r,s}(\mathcal{F}) \to \cdot\}}{d_{\mathcal{F}}\Omega^{r-1,s}(\mathcal{F})}$$

of the cochain complex (1.4) is the *$d_{\mathcal{F}}$-cohomology* of $(M, \mathcal{F})$. $\square$

See also C. Roger, [**207**], where the cohomology groups $H^{r,s}_{\mathcal{F}}(M)$ are studied in relation to infinitesimal deformations of foliations, classifying spaces for foliations, and characteristic classes. Let $(E, M, p, G, \mathcal{F}_E)$ be a foliated principal bundle over $(M, \mathcal{F})$ and $\rho : G \to \mathrm{End}_{\mathbb{R}}(V)$ a representation of G in a finite dimensional linear space V. We shall need the following notion

DEFINITION 1.43. A k-form $\alpha \in \Gamma^\infty(\Lambda^k T^*(E) \otimes V)$ is *tensorial of type $\rho(G)$* if 1) $\mathrm{Ker}(dp) \rfloor \alpha = 0$, and 2) $R^*_g \alpha = \rho(g^{-1})\alpha$ for any $g \in G$. A tensorial form α of type $\rho(G)$ is *pure of type* (r, s) if $s^*\alpha$ is a pure form of type (r, s) for any local section $s : U \to E$. $\square$

A tensorial form α on E is completely determined by the local forms $\alpha_i = s_i^*\alpha$, where $\{s_i\}_{i\in I}$ is a family of local sections $s_i : U_i \to E$ such that $\{U_i\}_{i\in I}$ is an open cover of M. For any $i, j \in I$ with $U_{ij} = U_i \cap U_j \neq \emptyset$ there is $\gamma_{ij} : U_i \cap U_j \to G$ such that $s_j = s_i\gamma_{ij}$ on U_{ij}. Consequently $\alpha_j = \rho(\gamma_{ij}^{-1})\alpha_i$. Viceversa, let $\gamma_i : p^{-1}(U_i) \to U_i \times G$ be a local trivialization atlas of E with the corresponding transition functions $\gamma_{ij} : U_{ij} \to G$. Any family $\{\alpha_i\}_{i\in I}$ of local forms $\alpha_i \in \Gamma^\infty(U_i, \Lambda^k T^*(M) \otimes V)$ satisfying $\alpha_j = \rho(\gamma_{ij}^{-1})\alpha_i$ on U_{ij} determines a tensorial form of type $\rho(G)$ on E.

As E is a foliated principal bundle, one may choose a family $\{s_i\}_{i\in I}$ of foliate local sections such that γ_{ij} are G-valued basic functions. Let α be a tensorial form of type $\rho(G)$, which is also pure of type (r, s). As $d_{\mathcal{F}} f = 0$ for any $f \in \Omega^0_B(\mathcal{F})$ it follows that

$$d_{\mathcal{F}}\alpha_j = \rho(\gamma_{ij}^{-1}) d_{\mathcal{F}}\alpha_i$$

hence the local forms $\{d_{\mathcal{F}}\alpha_i\}_{i\in I}$ determine a tensorial form of type $\rho(G)$ on E which is also a pure form of type $(r + 1, s)$. We built an operator

$$d_{\mathcal{F}} : \Omega^{r,s}_\rho(E, V) \to \Omega^{r+1,s}_\rho(E, V)$$

where $\Omega^{r,s}_\rho(E, V)$ is the space of pure tensorial forms of type (r, s). Once again $d^2_{\mathcal{F}} = 0$.

Definition 1.44. The cohomology

$$H^{r,s}_{\rho,\mathcal{F}}(E, V) = \frac{\text{Ker}\{d_{\mathcal{F}} : \Omega^{r,s}_\rho(E, V) \to \cdot\}}{d_{\mathcal{F}}\Omega^{r-1,s}_\rho(E, V)}$$

is the *$d_{\mathcal{F}}$-cohomology* of (E, V). □

Let ω be an adapted connection in the foliated principal bundle $(E, p, M, G, \mathcal{F}_E)$ and Ω the curvature 2-form of ω. Then Ω is a tensorial form on E of type $\text{ad}(G)$, where $\text{ad} : G \to \text{End}_{\mathbb{R}}(\mathbf{g})$ is the adjoint representation of G in its Lie algebra. An argument based on the second structure equation (cf. e.g. [**155**], vol. I, p. 7) shows that Ω is a sum of a pure form $\Omega_{1,1}$ of type $(1, 1)$ and a pure form $\Omega_{0,2}$ of type $(0, 2)$, i.e. $\Omega = \Omega_{1,1} + \Omega_{0,2}$. In particular $\Omega_{1,1} \in \Omega^{1,1}_{\text{ad}}(E, \mathbf{g})$ and $d_{\mathcal{F}}\Omega_{1,1} = 0$. The corresponding cohomology class $[\Omega_{1,1}] \in H^{1,1}_{\text{ad},\mathcal{F}}(E, \mathbf{g})$ does not depend upon the choice of an adapted connection ω in E.

Definition 1.45. $[\Omega_{1,1}]$ is the *Atiyah class* of $(E, \mathcal{F}^1_E)$. □

The notion is due to P. Molino, [**181**]. The Atiyah class of $(E, \mathcal{F}_E)$ vanishes if and only if there exists a projectable connection on $(E, \mathcal{F}_E)$. Note that $(B^1_T(M, \mathcal{F}), \mathcal{F}^1_T)$ is a foliated $GL(q, \mathbb{R})$-principal bundle over $(M, \mathcal{F})$.

Definition 1.46. The Atiyah class

$$A(M, \mathcal{F}) \in H^{1,1}_{\text{ad},\mathcal{F}}(B^1_T(M, \mathcal{F}), \mathbf{gl}(q, \mathbb{R}))$$

of the foliated bundle $(B^1_T(M, \mathcal{F}), \mathcal{F}^1_T)$ is the *Atiyah class* of $(M, \mathcal{F})$. □

1.2.4. Riemannian foliations. Let M be a C^∞ manifold and $\mathcal{F}$ a codimension q foliation of M.

Definition 1.47. A Riemannian bundle metric g_Q in $Q = \nu(\mathcal{F})$ is *holonomy invariant* if $\mathcal{L}_X g_Q = 0$ for any $X \in T(\mathcal{F})$. □

Any holonomy invariant metric g_Q in Q gives rise naturally to a transverse $\mathrm{O}(q)$-structure $O(Q) \to M$ hence $(\mathcal{F}, g_Q)$ is a Riemannian foliation of M.

Let g be a Riemannian metric on M and let $P^\perp$ be the orthogonal complement of $P = T(\mathcal{F})$ in $T(M)$. There is a natural bundle isomorphism $\sigma_g : Q \to P^\perp$. Let us set

$$g_Q(r, s) = g(\sigma_g(r), \sigma_g(s))$$

for any $r, s \in Q$.

DEFINITION 1.48. The Riemannian metric g is said to be *bundle-like* if g_Q is holonomy invariant. □

By a geometric interpretation due to B. Reinhart, [**204**], g is bundle-like if and only if any geodesic of (M, g) orthogonal to some leaf of $\mathcal{F}$ is orthogonal on every other leaf it meets.

Riemannian foliations of compact manifolds are described by the following *first structure theorem*, cf. [**179**], p. 155-156. Let $(\mathcal{F}, g_Q)$ be a Riemannian foliation on a compact connected manifold M. Then i) the closures of the leaves of $\mathcal{F}$ form a partition $\overline{\mathcal{F}}$ of M by compact embedded submanifolds which are integral manifolds of an involutive distribution of variable dimension, ii) the closure of the leaves of $\mathcal{F}$ are the projections on M of the closures of the leaves of the lifted foliation $\mathcal{F}_T^1$ on $O(Q)$; the space $M/\overline{\mathcal{F}}$ of closures of leaves of $\mathcal{F}$ may be identified with the quotient space $W/\mathrm{O}(q)$, where W is the basic manifold of $(O(Q), \mathcal{F}_T^1)$, and iii) when restricted to the closure of a leaf, $\mathcal{F}$ induces a transversally homogeneous foliation (in the sense of R.A. Blumenthal, [**46**]). The key fact in the proof of the first structure theorem is that $\mathcal{F}_T^1$ is transversally parallelizable (hence the *basic foliation* associated with $\mathcal{F}_T^1$, in the sense of [**179**], p. 105, is simple and defined by a locally trivial fibration $O(Q) \to W$). Cf. also [**182**].

The first structure theorem has several important applications. For instance, it plays a key role in the proof of the A. El Kacimi-Alaoui & V. Sergiescu & G. Hector result (cf. [**147**]) that for any codimension q Riemannian foliation $\mathcal{F}$ of a closed (i.e. compact, without boundary) manifold M the basic cohomology groups $H_B^\bullet(\mathcal{F})$ are finite dimensional; moreover $H_B^q(\mathcal{F})$ is either 0 or $\mathbb{R}$. See also A. El Kacimi-Alaoui & G. Hector, [**146**], and V. Sergiescu, [**218**].

As an other application, A. El Kacimi-Alaoui & G. Gomez, [**148**], have established the following result. Let M be a compact manifold and $\mathcal{F}$ a Riemannian foliation of M of codimension $q \geq 2$. Let N be a compact orientable Riemannian manifold and $\varphi : M \to N$ a foliated map (N is thought of as endowed with the trivial foliation by points). If N has negative sectional curvature then there is a harmonic foliated map homotopic to φ. We recall that given a foliated map $\varphi : M \to N$ of a foliated manifold $(M, \mathcal{F})$ into an ordinary manifold N (i.e. $(d_x\varphi)P_x = (0)$, for any $x \in M$) one may define a section

$$d_T\varphi \in \Gamma^\infty(Q^* \otimes \varphi^* T(N))$$

by setting $(d_T\varphi)s = (d\varphi)Y$ for some $Y \in T(M)$ with $\pi Y = s$. Then, if $\mathcal{F}$ and N are Riemannian, as above, we set

$$e_T(\varphi) = \frac{1}{2}\|d_T\varphi\|^2$$

where the norm in $Q^* \otimes \varphi^{-1}TN$ is induced by g_Q and by the Riemannian structure of N. As $e_T(\varphi) \in \Omega_B^0(\mathcal{F})$, i.e. $e_T(\varphi)$ is constant along the leaves of $\mathcal{F}$, it follows that $e_T(\varphi)$ is constant along the closures of the leaves of $\mathcal{F}$. Hence $e_T(\varphi)$ may be lifted to a $\mathrm{O}(q)$-invariant function on $O(Q)$, which in turn induces a $\mathrm{O}(q)$-invariant function $\overline{e}_T(\varphi) : W \to \mathbb{R}$. The *transverse energy* $E_T(\varphi)$ of φ is given by

$$E_T(\varphi) = \int_W \overline{e}_T(\varphi) dw$$

where dw is the volume form of the metric on W induced by the metric on $O(Q)$. Finally $\varphi : M \to N$ is *transversally harmonic* if φ is an extremal of E_T for all variations through foliated maps. Holomorphic maps of generalized Hopf, or Vaisman, manifolds (in the sense of [**88**], p. 33) are transversally harmonic (with respect to the transversally Kählerian foliations determined by the Lee and anti-Lee fields), cf. [**23**].