

Introduction

Commutative space theory is a common generalization of the theories of compact topological groups, locally compact abelian groups, riemannian symmetric spaces and multiply transitive transformation groups. This is an elegant meeting ground for group theory, harmonic analysis and differential geometry, and it even has some points of contact with number theory and mathematical physics. It is fascinating to see the interplay between these areas, as illustrated by an abundance of interesting examples.

There are two distinct approaches to the theory of commutative spaces: analytic and geometric. The geometric approach, which is the theory of weakly symmetric spaces, is quite beautiful, but slightly less general and is still in a state of rapid development. The analytic approach, which is harmonic analysis of commutative spaces, has reached a certain plateau, so it is an appropriate moment for a monograph with that emphasis. That is what I tried to do here.

Commutative pairs (G, K) (or commutative spaces G/K) can be characterized in several ways. One is that the action of G on $L^2(G/K)$ is multiplicity-free. Another is that the (convolution) algebra $L^1(K \backslash G/K)$ of K -bi-invariant functions on G is commutative. A third, applicable to the case where G is a Lie group, is that the algebra $\mathcal{D}(G, K)$ of G -invariant differential operators on G/K is commutative. The common ground and basic tool is the notion of spherical function. In the Lie group case the spherical functions are the (normalized) joint eigenfunctions of the commutative algebra $\mathcal{D}(G, K)$. The result is a spherical transform, which reduces to the ordinary Fourier transform when $G = \mathbb{R}^n$ and K is trivial, an inversion formula for that transform, and a resulting decomposition of the G -module $L^2(G/K)$ into irreducible representation spaces for G . In many cases this can be made quite explicit. But in many others that has not yet been done.

This monograph is divided into four parts. The first two are introductory and should be accessible to most first year graduate students. The third takes a bit of analytic sophistication but, again, should be reasonably accessible. The fourth describes recent results and is intended for mathematicians beginning their research careers as well as mathematicians interested in seeing just how far one can go with this unified view of algebra, geometry and analysis.

Part 1, “General Theory of Topological Groups”, is meant as an introduction to the subject. It contains a large number of examples, most of which are used in the sequel. These examples include all the standard semisimple linear Lie groups, the Heisenberg groups, and the adèle groups. The high point of Part 1, beside

the examples, is construction of Haar measure and the invariant integral, and the discussion of convolution product and the Lebesgue spaces.

Part 2, “Representation Theory and Compact Groups”, also provides background, but at a slightly higher level. It contains a discussion of the Mackey Little–Group method and its application to Heisenberg groups, and a proof of the Peter–Weyl Theorem. It also contains a discussion of the Cartan highest weight theory with applications to the Borel–Weil Theorem and to recent results on invariant function algebras. Part 2 ends with a discussion of the action of a locally compact group G on $L^2(G/\Gamma)$, where Γ is a co-compact discrete subgroup.

Part 3, “Introduction to Commutative Spaces”, is a fairly complete introduction, describing the theory up to its resurgence. That resurgence began slowly in the 1980’s and became rapid in the 1990’s. After the definitions and a number of examples, we introduce spherical functions in general and positive definite ones in particular, including the unitary representation associated to a positive definite spherical function. The application to harmonic analysis on G/K consists of a discussion of the spherical transform, Bochner’s theorem, the inverse spherical transform, the Plancherel theorem, and uncertainty principles. Part 3 ends with a treatment of harmonic analysis on locally compact abelian groups from the viewpoint of commutative spaces.

Part 4, “Structure and Analysis for Commutative Spaces”, starts with riemannian symmetric space theory as a sort of rôle model, and then goes into recent research on commutative spaces oriented toward similar structural and analytical results. The structure and classification theory for commutative pairs (G, K) , G reductive, includes the information that (G, K) is commutative if and only if it is weakly symmetric, and this is equivalent to the condition that $(G_{\mathbb{C}}, K_{\mathbb{C}})$ is spherical. Except in special cases the problem of determining the spherical functions, for these reductive commutative spaces, remains open. The structure and classification theory for commutative pairs (G, K) , where G is the semidirect product of its nilradical N with the compact group K , is also complete, and in most cases here the theory of square integrable representations of nilpotent Lie groups leads to information on the spherical functions. The structure and classification in general depends on the results for the reductive and the nilmanifold cases; it consists of methods for starting with a short list of pairs (G, K) and constructing all the others. Finally there is a discussion of just which commutative pairs are weakly symmetric.

At this point I should point out two areas that are not treated here. The first, already mentioned, is the general theory of weakly symmetric spaces, and the closely related areas of geodesic orbit spaces and naturally reductive riemannian homogeneous spaces. That beautiful topic, touched momentarily in Section 13.1C, has an extensive literature.

The second area not treated here consists of certain extensions of (at least parts of) the theory of commutative spaces. This includes the extensive but somewhat technical theory of semisimple symmetric spaces, (the pseudo-riemannian analogs of riemannian symmetric spaces of noncompact type), the theory of generalized Gelfand pairs (G, H) , and the study of irreducible unitary representations of G that have an H -fixed distribution vector. It also includes several approaches to

infinite dimensional analogs of Gelfand pairs. That elegant area is extremely active but its level of technicality takes it out of the scope of this book.

Acknowledgments

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Notational Conventions

\mathbb{R} , \mathbb{C} , \mathbb{H} and \mathbb{O} denote the real, complex, quaternionic and octonionic number systems. If \mathbb{F} is one of them, then $x \mapsto x^*$ denotes the conjugation of \mathbb{F} over \mathbb{R} , $\mathbb{F}^{m \times n}$ denotes the space of $m \times n$ matrices over \mathbb{F} , and if $x \in \mathbb{F}^{m \times n}$ then $x^* \in \mathbb{F}^{n \times m}$ is its conjugate transpose. We write $\text{Re } \mathbb{F}^{n \times n}$ for the hermitian ($x = x^*$) elements of $\mathbb{F}^{n \times n}$ and $\text{Re } \mathbb{F}_0^{n \times n}$ for those of trace 0, and we write $\text{Im } \mathbb{F}^{n \times n}$ for the skew-hermitian ($x + x^* = 0$) elements of $\mathbb{F}^{n \times n}$; that corresponds to the case $n = 1$.

In general we use upper case roman letters for groups, and when possible we use the corresponding lower case letters for their elements. If G is a Lie group then \mathfrak{g} denotes its Lie algebra. If \mathfrak{h} is a Lie subalgebra of \mathfrak{g} then (unless it is defined differently) H is the corresponding analytic subgroup of G .