Introduction

Quand est-ce que vous allez démontrer un vrai théorème?
— Bruno Poizat, printemps 2000

We will be concerned here with the following conjecture.

ALGEBRAICITY CONJECTURE. *An infinite simple group of finite* Morley rank *is algebraic, over an algebraically closed field.*

This conjecture arises in Model Theory, where *Morley rank* is an abstract notion of dimension which generalizes the notion of the dimension of an algebraic variety in some of its usual formulations. The conjecture asserts that any infinite simple group which can be equipped with such a dimension function must be isomorphic, as an abstract group, to a Chevalley group: the group of F-rational points of a simple algebraic group, over some algebraically closed field F. It remains open.

The main result to be proved here can be stated as follows.

MAIN THEOREM. *Let G be a simple group of finite Morley rank. Then G satisfies one of the following two conditions.*

(1) *G is algebraic over an algebraically closed field of characteristic* 2.
(2) *G has finite* 2*-rank.*

The 2*-rank* of G, denoted $m_2(G)$, is the dimension of the largest elementary abelian 2-subgroup of G.

The condition that the 2-rank is finite can be reformulated in more useful but somewhat more technical ways, notably as follows: the Sylow 2-subgroups contain divisible abelian subgroups of finite index and finite 2-rank. Such groups are said to be of "odd type", when the divisible abelian subgroup is nontrivial, and of "degenerate type" when it is trivial. We therefore prefer the following formulation.

MAIN THEOREM. *Let G be a simple group of finite Morley rank, and nonalgebraic. Then G is of odd or degenerate type.*

This is somewhat more than we had set out to do here. We had expected to confine our results to the analysis of *minimal counterexamples* to the Algebraicity Conjecture. The turning point came in [**2**], when it became clear that methods for achieving "absolute" results like the foregoing could be envisioned using many of the techniques already developed for the treatment of minimal cases. This is startling, as it is the analog in our subject of a classification of finite simple groups of characteristic 2 type *without the Feit-Thompson (Odd Order) theorem*. Indeed, groups without involutions fall in the degenerate class—and conversely, simple groups of degenerate type contain no involutions, as the Algebraicity Conjecture predicts (Theorem IV 4.1). About such groups we say nothing, and of course they may occur, in principle, as subgroups or sections of the groups we do study. We work around them.

Much of our approach will be modeled closely on the methods of finite group theory. The Algebraicity Conjecture is analogous to the classification of the finite simple groups as Chevalley groups, possibly twisted, together with the alternating groups and 26 "sporadic" finite simple groups. The methods we use are largely those which were involved in the two proofs of that classification discussed above, combined with certain additional ingredients, namely: (1) the *amalgam method*, which is part of a proposed *third generation* approach to the classification of the finite simple groups, and is very effective in our context; (2) more elementary ideas modeled on the theory of algebraic groups and lacking a finite analog; (3) specific properties of algebraic groups. To this list, a fourth category must be appended, relating to the body of techniques which enables us to work around the presence of degenerate sections. This is a very geometrical theory, based ultimately on dimension computations, and which is developed in Chapter IV. Everything we do there could be done in the category of algebraic groups, but is not—primarily, it seems, because stronger results based on properties of complete varieties are available. In our category, there is no coherent notion of complete variety, and we see no obvious parallel with the methods of Chapter IV, but we observe that the results go in the same general direction. Both model theorists and algebraists may find this chapter of particular interest (though it really has to be seen in action, as in Chapter VI, to be appreciated)—model theorists because the material is model theoretic in character, and group theorists because the line of argument varies considerably from the accustomed lines of group theory, both finite and algebraic, while at the same time having a clear meaning within the algebraic category. The closest model for this kind of analysis is found in the so-called black box group theory (randomized finite group theory), where properties of "most" elements play an important role.

The Main Theorem and some additional results which will be detailed in the final chapter, relating to groups of odd type, impose sharp limitations on the structure of a possible counterexample to the Algebraicity Conjecture, and suggest that such a group is unlikely to contain any involutions at all. When we first began this project, it seemed entirely possible that exceptions to the conjecture do occur in nature, or not far removed from nature; in the finite case one has both the "twisted" Chevalley groups and the sporadic ones to deal with, and possible analogs of both could be envisioned in our case. This possibility now appears to be rapidly receding. On the other hand, one can imagine various model theoretic constructions which would most naturally produce torsion free examples, and our results say nothing about that possibility, except to suggest that the groups so constructed would look more like free groups than like conventional matrix groups.

The final chapter is where we expose a detailed summary of concrete applications of the main theorem of this book. The analysis of permutation groups of finite Morley rank outlined in that chapter illustrates, in a way reminiscent of the applications of the classification of the finite simple groups, how the main result of this book can be put into action to obtain results not directly related to classification issues. It is worth noting that the proofs of some results (e.g. generic equations) in Chapter IV, in their first incarnations, used the classification of simple groups of even type. It later turned out that the full classification was not necessary for these results.

When we set out on this project, we looked forward to the possibility of extracting from it, as a byproduct, a "skeletal" version of the classification of the finite simple groups, showing roughly what the core of that proof would look like in the absence of such complications as sporadic groups, very small base fields, and wreath products. In other words, we aimed to give a *reading* of the very long classification proof of the finite simple groups that imparts some particular structure to it, while providing a rigorous proof in a different context. What we do here, supplemented by the other material to be described below, could be taken as such a reading, but that is not how we see it after the fact. Rather, what emerges from this analysis is that the methods used to prove the classification of the finite simple groups are *more than adequate* to the task, and there is an embarrassment of riches. At various points, and indeed at the level of global strategy, one is confronted with several approaches, all apparently adequate, though differing in their efficiency. The theory in the finite case, and the fragment given here, sufficient for our purposes, can be read as involving a number of large and not very intimately connected theories, which have been developed simultaneously, and in some cases, it seems, only as far as a particular approach to the classification requires. We have made a selection from among these theories, which works particularly efficiently in the case of groups of finite Morley rank, but which might not represent a particularly efficient, or even viable, way of handling the finite case. Most strikingly, the theme of "standard components", which plays a large role in the finite case, almost disappears from view in our work, simply because at a key point more efficient methods appear on the scene. We welcomed this—we had no desire to pursue standard components, and a lingering suspicion that a lifetime (or three) might not be sufficient, though it is in fact likely the theory would collapse to reasonable proportions, adapted to the finite Morley rank context. Had we taken the conventional route, what we do here would look very much like the two proofs known in the finite case—whether it would be more difficult than the one we give remains unclear, but it would certainly be longer! We will suggest at the end that we may be following a different line of proof which makes sense in the finite case, not necessarily as a classification of all finite simple groups, but as an independent approach to a narrower subclass, including the Chevalley groups in characteristic 2. There is an analog with work of Timmesfeld in the finite case; while what we do here is not strictly parallel to that, the relationship seems real.

The difference between our problem and the finite problem seems to have less to do with sporadic groups than with small fields. Indeed, we make considerable use of tori, which over the field $\mathbb{F}_2$ reduce to the identity. We do have some trouble laying our hands on nontrivial tori sometimes, but in the end they can be produced when needed.

One point which does work out largely as we anticipated is the following: the theories that we do develop are applied here in much the same way that they are used in finite group theory, but with considerably less "background noise", and as a result the connection between methods adapted from the finite case, and the situation in algebraic groups, becomes more transparent. However, even here there is a nuance. The starting point for our main analysis (in Part C) is the classical theory of groups with strongly embedded subgroups, and its neo-classical revival, groups with weakly embedded subgroups. If one consults the original papers [**1, 124**] which deal with the K^*-case, one finds lines of argument which are certainly different from those used at the corresponding point in the theory of finite simple groups, but which nonetheless have very much the flavor of finite group theory, and in particular rely heavily on the theory of solvable groups, which runs in important respects closely parallel to the theory of finite solvable groups. All of the latter goes away when one drops the inductive hypothesis (K^*) and it is here that our Chapter IV comes into its own. As a result, this particular piece of the theory blows up considerably, and the chapter is a long one.

At the opposite extreme, our Chapter IX is a direct adaptation of work of Stellmacher to the finite Morley rank context. The subject would be rather dull if this chapter were typical. But the bulk of the developments have a different character: the main results achieved are closely parallel to results in the theory of finite simple groups, and the methods used owe much to the theory of finite group theory—but not to the proofs of the corresponding results! The dominant theme in these more typical parts of the theory is the adaptation to the context of connected groups of the fundamental notions of finite group theory, which in many cases brings them much closer to the notions of algebraic group theory which inspired them.

In any case, the pursuit of this classification problem has led those involved to develop a set of theories for groups of finite Morley rank which provide useful extensions of the theories developed in the finite case, and the specific requirements and challenges of the classification project have suggested some lines of development which were not immediately obvious; we mention particularly work by Corredor, Frécon, Poizat, and Wagner in this connection. About half of the present volume is devoted to the treatment of general topics of this kind, continuing in the vein of [**52**], and the other half to its applications to classification theorems in the simple case.

Our Main Theorem contains roughly half, or perhaps somewhat more, of what is currently known about the Algebraicity Conjecture (at least, as far as 2-local structure is concerned). To explain the present state of affairs more fully requires a little more background.

There is a Sylow theory for groups of finite Morley rank, for the prime 2. In addition to the conjugacy of the Sylow 2-subgroups, there is a very particular structure theory, considerably more reminiscent of the situation in algebraic groups than the situation in finite groups, which is summarized by the following cryptic formula.

$$S^\circ = U * T \tag{$*$}$$

Using the language of algebraic groups, this formula may be read as follows: *"The connected component of a Sylow 2-subgroup is a central product of a unipotent 2-group and the 2-torsion from a split torus."* For a precise interpretation of the statement in our more general context, see §I 6. The point to bear in mind is that if we actually were dealing with an algebraic group, this result would hold in a considerably sharper form, depending on the characteristic of the base field:

$$S^\circ = U \text{ in characteristic 2; } S^\circ = T \text{ in all other characteristics.}$$

In particular the Algebraicity Conjecture predicts that this strong form should hold for simple groups of finite Morley rank, and the Main Theorem can be reformulated more lucidly as stating that this is, in large measure, the case. According to formula ($*$), there are four possible structures for S°, depending on which of the factors U and T are present, and they correspond in some sense to hypotheses on the characteristic of the, as yet, unidentified base field: If $U \neq 1$ and $T = 1$, we say the group has *even type*; if $U = 1$ and $T \neq 1$, we speak of *odd type*, thereby inadvertently taking 0 to be odd; when U and T are both nontrivial we speak of *mixed type*, and finally when both are absent—which means the full Sylow 2-subgroup is finite, and possibly trivial—we speak of *degenerate type*. It will be seen that this terminology is consistent with the abbreviated account with which we began.

The Algebraicity Conjecture therefore breaks up naturally into four cases; in mixed and degenerate types we seek a contradiction, and in odd and even types we seek an identification of the group as an algebraic group (or, to put the matter both more concretely and more accurately, as a Chevalley group) over a field of appropriate characteristic. The Main Theorem can then be put in a third and very natural form as follows.

Main Theorem, Version II.

(1) *There are no simple groups of finite Morley rank of mixed type.*

(2) *A simple group of finite Morley rank of even type is isomorphic to a Chevalley group over a field of characteristic* 2*.*

In view of the formula $(*)$, this is equivalent to the previous versions, and it is in this form that we will prove it. One can see now the sense in which we deal with “half” of the problem; but actually the deepest problem lies in the degenerate case. Since we know that there are no involutions in this case, 2-local analysis ends there, but the problem remains. In odd type there is now a substantial theory, which we omit.

The state of knowledge in odd type was covered until recently by the thesis of Jeff Burdges [**60**]. In odd type one has the following, which is limited to the inductive framework of K^*-groups, where a K^*-group is a group of finite Morley rank all of whose proper definable infinite simple sections are Chevalley groups, or in practical terms, as we suggested earlier, a group which is a putative minimal counterexample to the Algebraicity Conjecture.

Odd Type. *A simple K^*-group G of finite Morley rank and odd type satisfies one of the following conditions, where S° is the connected component of a Sylow* 2*-subgroup.*

(1) *G is algebraic.*
(2) *$m_2(S^\circ) \leq 2$.*

Can this approach actually prove the Algebraicity Conjecture in full? This seems very unlikely, for reasons well known to model theorists. The critical case is that in which there are no involutions, the most degenerate case in our taxonomy. Here the methods of the present text are not helpful, though the methods used in odd type have a certain force even in the absence of involutions, and we hope that further exploration of the degenerate case will lead to the further development of such methods. The focus of attention in the degenerate case is on Borel subgroups (maximal connected solvable subgroups) and the pattern of their intersections; they may, however, intersect trivially, at which point group theoretic analysis appears to come to a final halt. In any case, we are not yet this far.

The conjecture antipodal to the Algebraicity Conjecture runs as follows.

Anti-Algebraicity Conjecture. *There is a simple torsion free group of finite Morley rank.*

The conventional wisdom at present is Manichaean: one of the two extremes ought to be correct. Beyond that, there seem to be few strong opinions as to how the matter should stand, though model theory has certainly clarified the issues involved over time. In particular, our work here relies crucially on some clarification of the model theoretic issues by Frank Wagner, as will be seen in Chapter IV at a preparatory level, and in Chapter VI in the context of a concrete application.

One striking difference between our subject and the theory of finite groups is our ability to prove a general result on groups of even type without first disposing of the case of degenerate groups. This would be analogous to disposing of characteristic 2 type finite simple groups without first proving the Feit-Thompson theorem. Evidently, the two situations differ substantially.

The proof of the Main Theorem evolved gradually, as we have mentioned. At first, we dealt with K^*-groups, that is, with minimal potential counterexamples, under the additional assumption (called *tameness*) of the noninvolvement of "bad fields" (cf. §I 4), though with the intention of reexamining the latter hypothesis at a later stage. After Jaligot's thesis [**125**] (cf. also [**123, 124**]), it became clear that the time had already come to proceed in the mixed and even type cases without reliance on this simplifying hypothesis (and to a large extent Burdges' thesis [**60**] has performed a comparable service for odd type). At this stage the K^*-hypothesis remained an integral part of the project. The program aiming at the full classification by adjusting the inductive framework was initiated in [**2**]. In this connection, methods derived from Wagner's work on the model theory of fields of finite rank have been essential.

Finally, one should not lose sight of two trivial but important points:

- The class of algebraic groups over algebraically closed fields of characteristic 2 is already a rich class, in the sense that the classification of Dynkin diagrams is an interesting, though relatively direct, classification, with its own "sporadic" (non-classical) members.
- At a deeper level, there are many nonalgebraic simple groups of finite Morley rank, because there are many fields of finite Morley rank with pathological structure, furnished by the Hrushovski construction—and all of this structure is visible in the associated groups. This is an important point, and more than once we have been confronted with the fact that we do not actually know the properties of "algebraic groups" when they are endowed with a finite Morley rank different from the usual dimension theory. Strictly speaking, algebraic groups (in this broad sense) are not even known to be K-groups!; though this does hold in positive characteristic, via work of Poizat.

Our main theorem says that in the presence of a fairly strong dimension concept—and nothing further—the underlying *group structure* is governed by the same finite combinatorics as in the algebraic case (Coxeter groups), at least in the case which corresponds to characteristic 2 in the algebraic setting; furthermore, this holds regardless of what pathology is allowed a priori in definable sections. We do not actually show that our groups are algebraic; we show that, like simple algebraic groups, they are Chevalley groups, which from our point of view means that they are amalgams of copies of SL_2 governed by the "recipe" encoded in a Dynkin diagram.

From this point of view, the reader should not be surprised to see considerable space devoted to the "tiny" group SL_2: all of Chapters VI and

VII, and much of Chapters V and VIII. On the other hand, the finite group theorist may be surprised to see that so little space is taken up with the remaining groups. By the standards of finite group theory, our inductive analysis is instantaneous.

The proof of the classification of the finite simple groups has given rise to a polemic between some who feel that the complexity of the proof *must* be due more to a poor choice of methods than to the nature of the problem, and those who feel that this is not at all the case—including, of course, most of those who have worked on the proof. This is not a polemic into which we feel a need to enter. We find the methods used extremely attractive. We also feel largely fortunate that we are not obliged to follow them too closely, and at the same time a bit unfortunate that we have no access to character theory or transfer methods—either one would be enormously helpful. Possibly our present work can make a modest contribution to the discussion underlying the polemic, by giving a demonstration of the flavor of a substantial portion of the finitistic methods in a context which lies somewhere in between the conceptual theory of algebraic groups and the more combinatorial theory of finite simple groups, and whose complexity in the primitive measure theoretic sense of length (or volume) is in the vicinity of the geometric mean of the two.

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Among the programs which have supported this work and advanced it in significant ways, we would like to mention the conference on Finite and Locally Finite Groups held in Istanbul in 1994, where the first concrete classification results were presented, the MSRI model theory program in 1997-1998 and specifically the support offered in January 1998, the Euroconference on Groups of finite Morley rank at the University of Crete's charming and tranquil Anogia Academic Village (June 1998), where finite group theorists and model theorists mixed with particular fluency, the conference Groupes, Géométrie et Logique at CIRM, Luminy (September, 2004), and the Newton Institute's program in model theory and its applications, for the month devoted to groups of finite Morley rank (March, 2005).

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Haso Arkadaşım, bir parça sohbet edebiliriz artık. Sait Faik'in Simitle Çay öyküsünü kendi elyazımla yazıp gönderdiğim mektubun üzerinden on beş yıl geçti. Sanırım aynı mektupta da yazmıştım, amacım başkalarından hırsızlamadığım bir öyküyü gönderebilmek oldu hep. Bu kitap, bu amacıma en yakın olduğum nokta. Yazılmasına katkım olmadı. İş bölümü(!) böyle gerektirdi. Ama, içinde anlatılan öyküye bir katkım oldu. Ne ilginçtir ki, katkımın matematiğe olan herhangi bir yetenekle ilgisi yok. Yalnızca inat... Bolca da şans... Pilavdan dönmedim, kaşığı kırdırmadım.

In a historical vein

The history of the subject is bound up not only with the history of the theory of finite simple groups, but much of the history of pure model theory, which underwent a revolution beginning in the late sixties, and even (or perhaps, particularly) for those who lived through much of the latter, is not easy to reconstruct in a balanced way. We offer just a few scattered remarks, first from the second author:

> Vladimir Nikanorovich Remeslennikov in 1982 drew my attention to Gregory Cherlin's paper [**68**] on groups of finite Morley rank and conjectured that some ideas from my work [on periodic linear groups] could be used in this then new area of algebra. A year later Simon Thomas sent to me the manuscripts of his work on locally finite groups of finite Morley rank. Besides many interesting results and observations his manuscripts contained also

> an exposition of Boris Zilber's fundamental results on $\aleph_1$-categorical structures which were made known to many western model theorists in Wilfrid Hodges' translation of Zilber's paper [**196**] but which, because of the regrettably restricted form of publication of the Russian original, remained unknown to me.

The third author came to the subject by a rather different route, the common point of origin being the work of Zilber, which seems to have become more rapidly known in the West than in his country of origin, and in the original Russian. This was the subject of considerable interest (notably in Paris and Jerusalem) in the summer and fall of 1980, where as a result of their relationships with a notorious open problem in pure model theory, the broader conjectures of Zilber began to reach a wide audience. At the outset, work on the algebraic content of stability theory was stimulated in the West by Macintyre's work on $\aleph_1$-categorical fields [**137**], and in the East by a suggestion of Taitslin. For the third author, coming into model theory via the Robinson school, the question of the algebraic content of stability theory was both natural and inevitable. The notions of $\aleph_1$-categoricity and model completeness, characteristic of the two main schools of model theory at that time, had both arisen from considerations of the double-edged question: What is so special about the theory of algebraically closed fields, and is anything in fact special about this theory? The text [**128**] also arrived at a timely moment; in particular, this text made use of a notion of connectivity close to the one adopted in the present text, and for similar reasons.

The first author came to the subject from Mecidiyeköy, İstanbul.

The complex and provocative Poizat has played a complex and provocative role in the development of this theory. In particular, his early intervention brought the more "algebraic" formulation of the rank notion into its proper form, and generally he has been very attentive to foundational issues, some of great practical importance.

The complex and vigorous Nesin has played a complex and vigorous role in the development of this theory, entering at an early phase and, with his collaborators, treating a number of key configurations. In a historical vein, we remark that a period of forced confinement gave him the necessary leisure to familiarize himself with the contents of [**183**]; whether he wishes to convey his thanks for this we do not know, but it may be doubted. ("Si j'avais quelque chose à adresser aux grands de ce monde, je jure, ça ne serait pas des remerciements!" - Galois)

Model theorists will be aware that the subject has also grown in other directions—geometrical stability theory, applications to number theory in the hands of Hrushovski and several others, and that in these developments the structure of abelian groups turns out to be central, and not at all as trivial as might appear at first glance. We have also found close attention to the structure of abelian subgroups and their definable subgroups valuable, notably in connection with the theory of "good tori".