CHAPTER I

Tools

Oh me dolente! come mi riscossi
quando mi prese dicendomi: "Forse
tu non pensavi ch'io loico fossi!".
— *l'Inferno,* Canto XXVII

Introduction

This chapter contains material relating to the general theory of groups of finite Morley rank. Much of this material may be found in [**52**], and some of it in considerably more general form in [**186**]. For the fundamental principles, particularly those involving a certain amount of model theory, the source [**156**] is excellent. We avoid anything involving particular classes of groups, such as Chevalley groups. Everything we need in that line of a general character will be given in the following chapter, with additional specialized topics in the third.

Our treatment is compact but reasonably full. We begin with a few points from abstract group theory, notably the basic commutator laws and the classical Schur-Zassenhaus splitting which we transfer later in the chapter to the context of finite Morley rank. We then lay out the rank axioms on which everything we do will depend, and derive the theory of connected groups.

In §4 we take up the theory of fields of finite Morley rank. Recent advances in this direction play a major role in our work. We give Macintyre's theorem, that infinite fields of finite rank are algebraically closed, Zilber's results on the interpretation of fields in groups, results of Wagner and Newelski limiting the nongeneric definable subsets of fields of finite rank, and a more recent result of Wagner on fields of finite Morley rank in positive characteristic: torsion is dense in any definable subgroup of a torus. In the last part of this book we will be in a context in which all fields should have characteristic 2, and this will be a critical property; so we encapsulate it in the term "good torus" which is introduced in this section.

The next section deals with the theory of nilpotent groups of finite Morley rank: their structure, the existence in general of the Fitting subgroup, two notions of Frattini subgroup (both useful, but quite distinct), and other staples of general group theory like the normalizer condition. In the presence of connectedness, a number of these points take on noticeably strengthened forms.

A subject which is closely related to the theory of nilpotent groups is the Sylow theory, which also exists in two forms, connected and general, but only for the prime 2. It is most convenient to work with the connected theory as the main variant, but it occasionally misses something relevant, in which case one passes to the general theory, but for the most part after first profiting from the connected version. We write "Sylow°" and "Sylow" for the two theories respectively, so the reader should expect to encounter "Sylow°" rather than "Sylow" more or less throughout (and heuristically one does little damage by ignoring the distinction).

Up to a point, the reader knows what to expect from Sylow theory. However, one remarkable, and relatively elementary point, is that the structure of a Sylow° 2-subgroup of a group of finite Morley rank is remarkably limited. This can be explained by invoking the theory of algebraic groups, a topic we leave for the next chapter. The structures of Sylow° 2-subgroups in algebraic groups are remarkably limited, and depend mainly on whether the characteristic is 2 or not (in this context, 0 is odd, or in any case not equal to 2). The structure of a Sylow° 2-subgroup of a group of finite Morley rank in general is close to a direct product of the two types of Sylow° 2-subgroups occurring in algebraic groups. Since a product of algebraic groups over various fields is an example, possibly typical, of a group of finite Morley rank, this is not a completely unexpected result.

This structural result plays a fundamental role in our approach to the subject. It gives us a way of distinguishing groups of "characteristic 2" type from the others, at the outset. Something similar is done in finite group theory, but in a more complicated way, using the structure of centralizers of involutions instead, which complicates the treatment of small groups (groups of low Lie rank). On the other hand, this is inevitable in the finite case, as there are various isomorphisms between quasisimple groups over small fields of different characteristics, and the characteristic of a very small group may depend on the group in which it is located. In our case, we have a clearcut distinction at the outset, unless the 2-Sylow° subgroup is trivial; this would mean that the ordinary 2-Sylow subgroup is finite, and should not occur in a connected group of finite Morley rank. This last point would be the analog of the Feit-Thompson theorem in the finite Morley rank context.

In §7 we take up Bender's generalized Fitting subgroup $F^*(G)$. This theory plays a central role in the classification of the finite simple groups. It will be less visible in our treatment, but only because we stay on the "characteristic 2" side of the theory. On the other side, this notion plays an absolutely central role, and indeed the same role as in the finite case. The theory in the finite Morley rank context is exactly parallel to the finite theory, once one takes into account the slight variations associated with taking connected components of everything, which we tend to do on every possible occasion.

In the following section we give some of the theory of solvable groups of finite Morley rank, notably the Hall and Carter theories. The Carter theory is more important here than in the finite case; indeed, it is reminiscent of the theory of maximal tori in the algebraic case. We also discuss the solvable radical, along with the socle and the p-unipotent radicals $U_p(H)$, which naturally accompany the solvable radical. This section also includes some results on "lifting" centralizers which are extremely useful in practice, and are well known in the finite case. There is also an early form of Schur-Zassenhaus, in a minimal case.

In the following section we come back to the Schur-Zassenhaus theory in general, one of the leading themes of [**52**].

The next section collects some useful information about automorphisms, in a general setting. There are four topics: (1) the actions of automorphisms of finite order, notably order 2 or order p where the group being acted on contains no p-elements; (2) action of a group of even type on a degenerate type group; (3) automorphisms of p-tori; (4) "continuously characteristic" subgroups. With the exception of (2) these provide useful general principles of an elementary nature. However, (2) is the motor for much of the present work. It turns out that the action of a connected 2-group on a degenerate type group must be trivial, or in other words: if a connected 2-group normalizes a degenerate type group, then the two groups commute. In practice this is what allows us to prove a classification theorem for groups of even type without first proving a Feit-Thompson theorem; this result uncouples any degenerate type sections of the groups in question from the more interesting parts of the group—most easily in proper subgroups, where induction applies.

In §11 we take up some matters connected with modules, that is definable actions of groups on abelian groups. For the irreducible case, the main points were dealt with under the interpretability of fields, and for the most part we record some generalities here concerning Clifford theory and composition series. We also point out one situation in which a group action must involve a good torus. The subject of modules, or representation theory, is certainly an important one in our subject, and we will return to it in various more specialized contexts, but in a general setting it is largely exhausted by Zilber's results on interpretability of fields and subsequent elaborations by Wagner and Poizat already given in §4.

Our last two sections are more technical. We first discuss the Thompson $A \times B$-lemma, which goes over into our context very naturally, and can be used to "kill cores" (and a bit more than cores, actually) in 2-local subgroups in an inductive even type setting. As it happens, our approach uses fewer 2-local subgroups than the standard approaches of finite group theory, and any cores we need killed are more or less dead on arrival anyway, but in any case we give this theory, and show later how it may be applied to simplify the situation.

The other point concerns the theory of complex reflection groups, which is our main route toward the identification of Coxeter groups, just prior to final identification of a generic simple group of even type. We will use various representations of our "Weyl group" on the torsion subgroup of a maximal torus to build a representation in characteristic zero (an ultraproduct). This can be viewed as a complex representation, and retain enough of the character of the natural reflection representations (in finite characteristic) of the Weyl group to be viewed as irreducible complex reflection groups. Fortunately, these have been classified explicitly, and this material is given here, with specific information needed later on to eliminate the non-Coxeter "interlopers".

Overview

We review the main points of the chapter here for the general reader unfamiliar with the foundations of the subject (as given in [**52**]) and anxious to move along rapidly.

The section on general group theory (§1) consists largely of points we will call on occasionally in the sequel, and which may not be familiar in the precise form we require. Our development actually begins in §2 with a discussion of *rank* as a "dimension function" on definable sets. This notion of rank coincides with Morley rank on the class of groups, but not in general. We do not give a separate definition of Morley rank; our rank notion is adequate not only in groups, but in any structures which can be interpreted into groups. One should be a little cautious though in looking beyond groups. We will deal later with the theory of buildings, for example, and it takes some work to show that our rank notion can again be called "Morley rank" in that context—none of which affects any of our applications, but does raise some doubts about our terminology, for those who take the model theory seriously.

Rank behaves for practical purposes like an estimate on the logarithm of the cardinality, a point that can be rendered rigorous in some contexts (e.g. for families of groups defined, uniformly, over finite fields), and also behaves like the Zariski dimension (which by the Lang-Weil theorem agrees with the former in the large finite case).

Thus the rank of a Cartesian product is the sum of the ranks. Less obvious, by far, is the following property: in any uniformly definable family of definable sets, the sizes of the finite sets are bounded. This is connected with the notion of rank as the sets of rank zero are the finite ones, and it is related to *definability of the rank*.

The four axioms for rank are called *Monotonicity, Additivity, Definability, and Uniform bounds*, and we have touched on the three nontrivial axioms already, so at this point the reader should have a fairly precise sense of the notion. As a consequence of these axioms, one gets a *Fubini principle* governing the rank of a subset of a product, or more generally of a subset of a disjoint union of sets of constant rank (a fibering). In particular, a subset S of such a disjoint union is "generic" (i.e., of full rank), if its fibers are *generically generic*; that is, the generic fiber of the ambient set meets S in a generic subset.

With rank providing a notion of dimension, there is also a notion of *multiplicity*, known as *Morley degree* in this context. The multiplicity is the number of irreducible components *of maximal dimension.* Unlike the algebraic case, these components are well defined only modulo sets of lower rank, and thus the multiplicity can only be defined in the top dimension. The fine structure of the rank notion is hard to exploit; things are always clearest in the top dimension.

The rank (and degree) provides a *descending chain condition* for definable subgroups. There is a more subtle *uniform chain condition* for uniformly definable families of subgroups, due to Baldwin and Saxl; see Lemma 2.8 for details.

From the descending chain condition on definable subgroups we derive a kind of Zariski closure for subgroups, namely the smallest definable subgroup containing a given group. A point which seems modest, but eventually plays a large role, is found in Lemma 2.16: the definable hull of a cyclic group is the direct sum of a divisible group and a finite cyclic group. A defect in the theory is the fact that the function $d(a)$ taking an element a to the definable hull of the cyclic group $\langle a \rangle$ is typically not a definable function; this defect is largely remedied in §IV 4 by introducing a definable approximation $\hat{d}$ to d with very similar properties. Another apparently innocuous result is *lifting of torsion*: the preimage of p-torsion under a definable homomorphism contains p-torsion. In view of the natural map $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ this has to be considered a particularly agreeable circumstance, and one that keeps our theory reasonably close to the finite theory; it is of course true in the algebraic context as well, when one restricts attention to Zariski closed groups and algebraic homomorphisms.

In a very similar vein, we have the *Basic Fusion Lemma* 2.20: Two involutions are either conjugate or commute with a third. This is familiar in the finite case, much less familiar but still true in the algebraic case, and true in our case as well, and essential if nontrivial parts of the theory of finite groups are to be applied in our context to classification problems.

Everything so far is essential for subsequent developments. We may then pass on (§3) to the critical notion of *connected* group and the *connected component* (of the identity) in a definable group. Obviously we are no longer following the motivation of the finite case, but the finite notions will mix well with this notion. In the algebraic category, while definable sets and Zariski closed sets are decidedly not the same thing (more's the pity), definable subgroups and Zariski closed subgroups do in fact coincide, and thus in our more general context we may expect definable groups to behave well. We call a definable group *disconnected* if it contains a proper definable subgroup of finite index, and connected otherwise. The existence of the connected component G° is then purely formal, in view of the descending chain condition on definable subgroups, but there is a useful property which lies deeper: a connected group must be *irreducible*, that is, of Morley degree one—or in other words, to put the thing more usefully: a connected group cannot contain two disjoint generic subsets. This is the foundation for a geometrical line of argument in our subject, as will be seen in Chapter IV. For the proof, see Lemma 3.6. The idea is to consider the action of G on its irreducible components of top dimension (not, admittedly, a well-defined notion, but manageable nonetheless). In carrying through this idea one makes use of the Fubini principle.

At this point we have a substantial corpus of basic ingredients to work with, and we can derive some concrete consequences, notably: Any definable action of a connected group of finite Morley rank on a finite set is trivial; any definable endomorphism with finite kernel of a group of finite Morley rank is surjective; the additive and multiplicative groups of an infinite field (or division ring) of finite Morley rank are connected; an infinite group of finite Morley rank contains an infinite definable abelian subgroup. The reader new to the subject might want to try his hand at these.

Another tool of wide applicability is the following definability result, due to Zilber: If G is a group of finite Morley rank, and H a subgroup generated by connected definable subgroups, then H is also connected. This is one of several reasons that it is useful to adapt most of the notions of group theory to *connected versions*; thus we work very often with the *connected normalizer* $N^\circ(X)$, the *connected centralizer* $C^\circ(X)$, and so on.

Before proceeding, we extend the notion of connected component to subgroups which are not necessarily definable: $H^\circ = H \cap [d(H)^\circ]$ when H is not necessarily definable. The most interesting case of this arises in conjunction with the Sylow theory, as Sylow subgroups are typically not definable. We note also that definable hulls of abelian, nilpotent, or solvable groups are again abelian, nilpotent, or solvable, respectively.

For definable groups H it follows from what we have said already that the Morley degree of H is the index $[H : H^\circ]$. A considerably more delicate point is the following: if $\mathcal{F}$ is a uniformly definable family of definable groups, then these indices $[H : H^\circ]$ are uniformly bounded—this would be obvious if the family

$$\{H^\circ : H \in \mathcal{F}\}$$

were itself known to be uniformly definable. For the proof, it seems to be necessary to go back into the foundations of the subject.

Next we take up the theory of *fields* of finite Morley rank (§4), which contains both the earliest results in this area, and some of the most recent, all of them very useful in applications. This consists of the following ingredients.

- Macintyre's theorem: An infinite field of finite Morley rank is algebraically closed.
- Zilber's Linearization Theorem: Let G be a connected group of finite Morley rank acting definably, faithfully, and irreducibly on an abelian group V, and let $T \lhd G$ be infinite abelian. Then the

subring of $\mathrm{End}(A)$ generated by T is a an algebraically closed field, with respect to which V becomes a finite dimensional vector space on which G acts linearly.
- The Newelski-Wagner Genericity Lemma: A definable subset of a field of finite Morley rank which contains an infinite field is generic.
- The Good Torus Principle: Any definable subgroup of the multiplicative group of a field of finite Morley rank, in *positive characteristic* is the definable hull of its torsion subgroup.

Most of this can be extracted reasonably directly based on the foundations established in the previous two sections, with the exception of the final point, which is a reformulation of results of Wagner which require some further foundational work, but in the specific context of fields. As will be seen in Chapter VI (and to some extent already in Chapter IV) this last result provides the basis for our geometrical lines of argument which fill the gap which would otherwise result from our inability to handle nonsolvable groups without involutions—not by casting any light on the class of groups, but by allowing us to work around it entirely.

For Macintyre's theorem, we have already laid sufficient foundations. We know that the additive and multiplicative groups of our field are connected, and in Galois theoretic terms it follows that the field has no Kummer or (in characteristic p) Artin-Schreier extensions, and is perfect. While this by itself will not yield algebraic closure, these properties are inherited by finite algebraic extensions, which "live inside" the universe of definable sets. And indeed, a Galois theoretic argument then shows that a field with these properties holding *hereditarily* is algebraically closed.

Zilber's theorem (found in this form in [**52**]) is a distant cousin of Schur's lemma, coupled with the following point (Lemma 4.5) any definable group of automorphisms of a field of finite Morley rank is trivial. *"But what of the Frobenius automorphism?"* The alert reader will object—observe the distinction between a definable group of automorphisms and a group of definable automorphisms. The triviality of such definable automorphism groups is largely a consequence of Macintyre's theorem (the fixed field must be either finite or algebraically closed, and we need only concern ourselves with the former case, which indeed requires a little attention).

The Newelski-Wagner Genericity Lemma would be obvious if the "infinite field" referred to were itself definable, as the structure consisting of a pair of algebraically closed fields, nested, has *infinite rank*. So this becomes largely a matter of trading a not necessarily definable field in for a definable one.

Finally, the Good Torus Principle is both essential and subtle, and not in the direct line of thought we have followed to this point. The underlying model theoretic result is the following: if F is a field of finite Morley rank, then the subfield F_{alg} of *model theoretically algebraic* elements is an elementary substructure. In positive characteristic one may show (via the Frobenius automorphism, which respects both the field structure and whatever multiplicative subgroups may be definable) that these algebraic elements are algebraic in the conventional sense, and thus our field has a *locally finite* elementary substructure. This then decodes into the Good Torus Principle. The point of this principle will be seen considerably later, starting with §IV 1 (Proposition IV 1.15).

At this point we have the foundations well in hand, and we can develop various standard group theoretic topics on that basis (§§5–9): the structure of nilpotent groups, the Fitting subgroup (and later, the generalized Fitting subgroup), and the theory of solvable groups, which includes the theory of Hall and Carter subgroups, the solvable radical, and the important Schur-Zassenhaus lemma. The latter requires a considerable development. The solvable theory has some special features which are reminiscent of the algebraic theory. It is much simpler than the theory in the finite case, because of our attention to connected groups. The most striking parallel to algebraic group theory is a version of the Lie-Kolchin theorem: if H is a connected solvable group of finite Morley rank, then the quotient $H/F^\circ(H)$ modulo the connected Fitting subgroup is a divisible abelian group (Lemma 8.3). The idea of the proof is to use the Zilber Linearization Theorem on a composition series for H to get H acting as a subgroup of a product of fields, where the kernel of the action has a nilpotent action on H and hence lies in the Fitting subgroup (after which one may pass to the connected Fitting subgroup with a little more argument).

One can also define a reasonable notion of p-unipotent subgroup, and show in the solvable case that these groups necessarily lie in the Fitting subgroup. We would have more trouble introducing a notion of 0-unipotence; this has been done, for use in groups of odd type, but for us the critical case is 2-unipotence, and we may be spared these interesting refinements.

In the Introduction we took special note of the Sylow theory, for good reason. On the one hand it presents several subtleties: the groups in question are typically not definable, which makes their management a delicate point; furthermore, while we have a good Hall theory, in general, inside solvable groups, we have no real Sylow theory at all for any prime other than 2. And indeed the treatment of Sylow theory in the finite case is resolutely arithmetical, whichever of the variety of approaches one adopts. We must adopt an entirely different approach, working inductively, and relying ultimately on the Basic Fusion Lemma when all else fails.

One may deal with either Sylow subgroups or their connected analog, Sylow° subgroups; the latter is the workhorse of the subject, but the former intervenes on occasion. Sylow 2-subgroups are maximal 2-subgroups, and 2-subgroups are those whose elements have order a power of 2. Existence is therefore not an issue; conjugacy is, but can be proved by an inductive argument. Structure is also an issue. An essential point is the *local finiteness* of the Sylow 2-subgroups. If one wishes to extend the theory to p-subgroups, one should probably build local finiteness into the definition; but even so, problems will remain.

Considerably more can be said about the structure of a Sylow° 2-subgroup S of a group of finite Morley rank, and indeed the situation runs closely parallel to what one sees in the algebraic case. The structure is as follows:

$$S = U * T$$

where:

(1) The $*$ represents a central product, with a finite intersection.
(2) U is 2-*unipotent*: definable, connected, solvable of bounded exponent (and hence nilpotent, by Lemma 5.5.
(3) T is a 2-*torus*: divisible abelian but presumably not definable, as the definable hull should contain elements of infinite order.

This structural result gives us a way of distinguishing groups of "characteristic 2" type from the others, at the outset. Either or both of U and T may be trivial; in a simple group one expects exactly one of the two to be absent. There are all told four possibilities, and hence four possible "types": *mixed type*, with both factors present; *degenerate type*, with $S = 1$; and *even* or *odd* type, with, respectively, $S = U$ or $S = T$. Each of these types is approached differently, though as we shall see the two types with $U > 1$ are approached similarly, and in the end the case of mixed type reduces to the case of even type.

The chapter contains four further sections, dealing with automorphism, modules, the Thompson $A \times B$ theorem, and complex reflection groups (§§10–13). This is a motley collection, ending up with topics that lie somewhere between general tools and the sort of more specialized developments treated in Chapter III.

To begin with, a number of general principles involving automorphisms should be considered part of the basic tools of the trade, and have been collected together. More specialized topics relating to automorphisms of Chevalley groups will be found under that more specialized heading. But we have buried one topic of the first importance in this section, namely Proposition 10.13: *If a* 2-*unipotent group acts definably on a definable subgroup without* 2-*unipotent subgroups (for example, on a group of degenerate type), then the action is trivial.* This result furnishes one of the key mechanisms for *neutralizing* degenerate sections of a group. And in the course of the analysis we make our first acquaintance with the important notion of strong embedding, to be investigated in detail in Chapter VI.

The topics considered under "modules" do not form a particularly coherent whole. Logically, this section could equally well include the Zilber Linearization Lemma, some at least of the Schur-Zassenhaus analysis, and other topics. But we collect here a few points for which the language of modules is particularly convenient, and which do not belong to any more notable category.

The last two topics ($A \times B$ theorem, complex reflection groups) are of a very specific character. They are close in spirit to the specialized topics which are treated on their own in Chapter III, but seem to us to have more of the character of general group theory.

The Thompson $A \times B$ Lemma goes over smoothly from finite group theory to our context, and it plays much the same role for us that it does classically; and it could easily be hidden in the section on nilpotent groups, or modules, or automorphisms, which are themselves three faces of a single thing. But it plays a distinguished role in the subject, and we let it stand alone. The theory of complex reflection groups, on the other hand, is a topic which clearly belongs, as far as the content is concerned, in our first section, which was devoted to topics in general group theory which we take over and use—we use this one rarely, but to great effect.

In other words, we deal at the end with four afterthoughts (excepting the fundamental Proposition 10.13) perhaps out of their proper places, but all playing an important role in the sequel.

If the reader is still with us, he has been here long enough and should either browse the chapter or move on to points of greater interest (probably Chapters III or IV, as taste may dictate).

1. General group theory

In this section we record some general group theoretic facts which are occasionally useful, and establish our group theoretic notations, particularly in areas in which conventions are not entirely standardized. We note at the outset that in model theoretic contexts, the notation X^n stands for the Cartesian power $X \times \cdots \times X$ of the set X, but the notation G^n, for G a group, will also refer to a term in the upper central series; this usage should not result in any substantial ambiguity. But we will avoid the use of the same notation for the set of n-th powers. In abelian contexts, that set may be denoted nG, and in nonabelian contexts we have no special notation for this set.

When we work with a group H having a normal subgroup K, we will sometimes avoid passing to the quotient $\bar{H} = H/K$ by a standard notational device: We write for example $Z(H \bmod K)$ for the pullback to H of $Z(\bar{H})$, $C_H(X \bmod K)$ for the pullback to H of $C_{\bar{H}}(\bar{X})$, and $N_H(L \bmod K)$ for the pullback to H of $N_{\bar{H}}(\bar{L})$.

1.1. Notations.

NOTATION 1.1. *Let G be a group, and π a set of primes.*

(1) *For $a, b \in G$, we set $a^b = b^{-1}ab$ and $[a, b] = a^{-1}a^b$.*
(2) *For $X, Y \subseteq G$, $[X, Y]$ denotes the* subgroup *generated by commutators $[x, y]$ with $x \in X$, $y \in Y$; but for $x \in X$, $[x, Y]$ denotes the* set *of commutators $\{[x, y] : y \in Y\}$.*

(3) *G^i and $G^{(i)}$ are defined inductively by:*

$$\begin{aligned} G^0 = G^{(0)} &= G, \\ G^{i+1} &= [G, G^i], \\ G^{(i+1)} &= [G^{(i)}, G^{(i)}]. \end{aligned}$$

The series G^i is called the descending central series*; the series $G^{(i)}$ is called the* commutator series*.*

(4) *The* ascending central series *$Z_i(G)$ is defined inductively by*

$$\begin{aligned} Z_0(G) &= 1, \\ Z_{i+1}(G)/Z_i(G) &= Z(G/Z_i(G)). \end{aligned}$$

(5) *A π-number is a positive integer all of whose factors belong to π.*
(6) *For $a \in G$, $H \leq G$, a^H denotes the set $\{a^h : h \in H\}$.*
(7) *A π-element of G is an element whose order is a π-number.*
(8) *A π-group is a group all of whose elements are π-elements.*
(9) *A $\pi^\perp$-group is a group none of whose elements other than 1 is a π-element.*
(10) *π' is the complement of π in the set of prime numbers.*
(11) *We write $G^\times$ for $G \setminus \{1\}$.*
(12) *G is* π-radicable *if for every $g \in G$ and every π-number n, g has an n-th root in G, and if G is abelian, the term* π-divisible *means the same thing (but may be expressed additively).*
(13) *If P is a $\{p\}$-group, then $\Omega_i(P)$ is the subgroup generated by elements of order at most p^i.*

Note that a π'-group is a $\pi^\perp$ group, but that π'-groups are necessarily *periodic* (all elements are of finite order) whereas $\pi^\perp$-groups may contain elements of infinite order.

When $\pi = \{p\}$ for a single prime p, we lighten the notation accordingly: p-group, p'-group, $p^\perp$-group, and so on.

1.2. Commutator laws.

LEMMA 1.2. [**108**, ...] *Let G be a group, and $a, b, c, g \in G$.*

(L) $[ab, g] = [a, g]^b [b, g]$,

(R) $[g, ab] = [g, b][g, a]^b$,

(J) $[[a, b^{-1}], c]^b [[b, c^{-1}], a]^c [[c, a^{-1}], b]^a = 1$.

We will refer to (J) as the Jacobi identity.

COROLLARY 1.3. *Let $i \in I(G)$ (that is, an involution), $x \in G$, $\gamma = [i, x]$. Then $\gamma^i = \gamma^{-1}$.*

COROLLARY 1.4. *Let G be a group, A an abelian subgroup, and $g \in N_G(A)$. Then the commutator map $\gamma_g : A \to A$ defined by $\gamma_g(a) = [g, a]$ is an endomorphism of A.*

LEMMA 1.5. *Let G be a group, and X, Y subgroups. Then X and Y normalize $[X, Y]$.*

PROOF. Let $x_1, x \in X$, $y_1 \in Y$, $\gamma = [x_1, y_1]$. Then $[x_1 x, y_1] = [x_1, y_1]^x [x, y_1]$, hence $\gamma^x \in [X, Y]$. □

LEMMA 1.6 (Three subgroups lemma). *Let G be a group, H, K, L three subgroups, and $N \triangleleft G$. If two of the three subgroups*

$$[[H, K], L], [[K, H], L], [[L, H], K]$$

are contained in N, then so is the third.

PROOF. Suppose the first two are contained in N. By the Jacobi identity, for $l \in L$, $h \in H$, $k \in K$, we find

$$[[l, h], k]^{h^{-1}} \in [[H, K], L]^K [[K, L], H]^L \subseteq N$$

and thus $[[l, h], k] \in N$. □

DEFINITION 1.7. *Let G be a group.*

(1) *G is* quasisimple *if $G' = G$ and $G/Z(G)$ is simple.*
(2) *G is* quasisemisimple *if G is a central product of quasisimple groups.*
(3) *For any group G, $E(G)$ is the subgroup of G generated by its subnormal quasisimple subgroups.*

LEMMA 1.8. *Let G be a quasisemisimple group. If $H \triangleleft G$ then H' is quasisemisimple, $H = H'Z(H)$, $Z(H) = H \cap Z(G)$, and the quasisimple normal subgroups of H are normal in G.*

PROOF. Let $\bar{G} = G/Z(G)$. Then $\bar{G}$ is a direct product of simple groups and $\bar{H}$ is normal in $\bar{G}$, so the same applies to $\bar{H}$. Thus $H = H'Z(H)$ and $Z(H) = H \cap Z(G)$, and it follows that $H' = H''$. It remains to check the last claim.

If H_1 is a quasisimple normal subgroup of H, then $\bar{H}_1 \triangleleft \bar{G}$ and hence $H_1 Z(G) \triangleleft G$. Then $H_1 = H_1' = (H_1 Z(G))'$ is normal in G. □

LEMMA 1.9. *Let G be a group.*

(1) *If H, K are subnormal in G and quasisimple, then either $[H, K] = 1$ or $H = K$.*
(2) *$E(G)$ is the central product of the subnormal quasisimple subgroups of G.*

PROOF. As the second claim follows from the first, we concern ourselves with the first.

Let $H = H_0 \triangleleft H_1 \triangleleft \dots H_m = G$ and $K = K_0 \triangleleft K_1 \triangleleft \dots K_n = G$, and proceed by induction on $\max(m, n)$. Then the conjugates of H in G lie inside H_{m-1}, so by induction distinct conjugates of H commute with one another, and generate a subgroup $\hat{H}$ which is their central product; similarly the conjugates of $\hat{K}$ generate a subgroup $\hat{K}$ which is their central product.

Suppose first that $L = \hat{H} \cap \hat{K}$ is abelian. Then $L \leq Z(\hat{K})$ so $[[H,K],K]=1$ and by the three subgroups lemma, $[H,K] = [H,[K,K]] = 1$.

Now suppose that $L = \hat{H} \cap \hat{K}$ is nonabelian, and let L_1 be a quasisimple normal factor of L. Then L_1 is also quasisimple normal in $\hat{H}$ and in $\hat{K}$. Accordingly there are conjugates H_1 of H and K_1 of K such that $L_1 Z(\hat{H}) = H_1 Z(\hat{H})$ and similarly for K_1. Then $L_1 = L_1' = (L_1 Z(\hat{H}))' = (H_1 Z(\hat{H}))' = H_1$ and similarly $L_1 = K_1$, so H and K are conjugate, and the claim follows as already remarked. □

Lemma 1.10. *Let G be a group, and H a solvable normal subgroup. Then $[E(G),H] = 1$.*

Proof. The commutator $[E(G),H]$ is a solvable normal subgroup of $E(G)$, hence is contained in $Z(E(G))$, or in other words $[E(G),[E(G),H]]=1$ and hence $[E(G),H] = 1$ since $E(G)' = E(G)$. □

Lemma 1.11. *Let G be a group, and H a normal subgroup. Then $E(H) = (E(G) \cap H)'$.*

Proof. By the definitions, $E(H) \leq E(G)$ and hence $E(H) = E(H)' \leq (E(G) \cap H)'$. Conversely $E(G) \cap E(H) \triangleleft E(G)$, hence $(E(G) \cap H)'$ is quasisemisimple. □

1.3. Commutator subgroup.

Lemma 1.12 ([**52**, Ex. 21, p. 7]). *Let G be a group with $G/Z(G)$ finite. Then G' is finite.*

Proof. Let $X = \{[a,b] : a, b \in G\}$. Then $|X|$ is finite. Let $X = \{x_1, \dots, x_N\}$ in some definite order. We claim

(∗) Any element of G' may be written in the form $x_1^* \cdot \cdots \cdot x_N^*$ with x_i^* a positive power of x_i

For this, let $g \in G'$ have the representation $g = x_{i_1} \cdot \cdots \cdot x_{i_l}$. We proceed by induction on l.

If $i_1 \leq \cdots \leq i_l$, we have our claim. Assuming the contrary, let j be minimal so that $i_j > i_k$ for some $k > j$, and with l fixed, choose the representation of g to maximize j. Choose $k > j$ to minimize i_k. Let $g_1 = x_{i_1} \cdot \cdots \cdot x_{i_{j-1}}$, $g_2 = x_{i_j} \cdot \cdots \cdot x_{i_{k-1}}$, and $g_3 = x_{i_{k+1}} \cdot \cdots \cdot x_{i_l}$. Note that $g_2^{x_{i_k}}$ can be expressed as a product of length $k-j$ since X is invariant under conjugation. Consider the representation of g as $g_1 x_{i_k} g_2^{x_{i_k}} g_3$. This again has length l, but has i_k in place of i_j in the j-th position. Thus relative to this second representation, the value of j has increased, a contradiction. This proves (∗).

Note also that the transformation of an expression for g into the standard form of (∗) does not increase length. Let $n = |G/Z(G)|$. We claim now that the representation can always be taken with length at most $|X| \cdot n$. To see this, it suffices to show that any expression of the form x^{n+1} (considered as length $n+1$), with $x = [a,b] \in G'$, can be shortened by rewriting. Since $x^n \in Z(G)$, this may be done as follows:

$$x^{n+1} = a^{-1}x^n b^a = [(x^{n-1})^a x^a a^{-1}]b^a = (x^{n-1})^a [a,b]^a [a,b] = (x^{n-1})^a [a^2,b]$$

where the last expression can be construed as length n, since $[a^2,b] \in X$. □

LEMMA 1.13 ([**52**, Ex. 22, p. 8]). *Let G be a group, and H_1, H_2 subgroups of G which normalize each other. If the set of commutators $X = \{[h_1,h_2] : h_1 \in H_1, h_2 \in H_2\}$ is finite, then the commutator subgroup $H = [H_1,H_2]$ is finite.*

PROOF. We may take $G = H_1H_2$ and thus $C_G(X) = C_G(H)$ is a normal subgroup of finite index. Hence $Z(H)$ has finite index in H, and it follows that H' is finite. Accordingly we may factor out H' without loss of generality, and assume that H is abelian.

Therefore the commutator maps $\gamma_{h_1} : H_1 \to H$ defined by $\gamma_{h_1}(h_2) = [h_1,h_2]$ are homomorphisms, so X is closed under taking powers. Thus the elements of X are of finite order, and H is generated by a finite set of elements of finite order; H is finite. □

1.4. Abelian and nilpotent groups.

NOTATION 1.14. *Let A be an abelian group:*

(1) *For $n \geq 1$, $A[n] = \{a \in A : na = 0\}$.*
(2) *A subgroup B of A is* pure *in A if $nA \cap B = nB$ for all $n \geq 1$.*

LEMMA 1.15 ([**96**, 28.2, Kulikov]). *If A is an abelian group, and B a pure subgroup with A/B of bounded exponent, then A splits as $B \oplus C$ for some complement C.*

PROOF. By Zorn's Lemma one can find a maximal pure subgroup $\hat{B}$ of A containing B, such that $\hat{B}/B$ splits over B. We claim $\hat{B} = A$. If not, take $a \in A$ of maximal prime power order $q = p^n$ modulo $\hat{B}$, and use the purity of $\hat{B}$ to write $\langle \hat{B}, a\rangle$ as $\langle \hat{B}\rangle \oplus C$ with C cyclic. To reach a contradiction it suffices to check that $\hat{B} \oplus C$ is again pure in A. This reduces easily to checking that $pA \cap (B \oplus C) = pB \oplus pC$ and follows from the maximization of q. □

LEMMA 1.16 ([**96**, Theorem 17.2]). *Let A be an abelian group of bounded exponent. Then A is a direct sum of cyclic groups.*

PROOF. Take a maximal direct sum of cyclic subgroups which is pure in A, and apply the preceding lemma. This reduces the problem to one of finding a single cyclic direct factor of A, and for this one takes a cyclic subgroup of maximal prime power order, which is again pure in A and hence is a direct factor. □

LEMMA 1.17. *If H is a nilpotent group and π is a set of primes, then the set H_π of π-elements of H is a subgroup, and is the direct sum of the* *subgroups H_p (i.e., $H_{\{p\}}$) for $p \in \pi$. In particular the set H_{tor} of elements of finite order in H is a subgroup, and is the direct sum of all the subgroups H_p.*

LEMMA 1.18 ([**71, 72**]). *Let H be a π-radicable nilpotent group. Then:*

(1) *$H_\pi \le Z(H)$.*
(2) *$Z_i(H)/Z_{i-1}(H)$ is π-torsion free and π-divisible for $i > 1$.*

PROOF.
Ad (1) We show for each prime $p \in \pi$ that

$$(*) \qquad \text{If } h^p \in Z(H) \text{ with } h \in H, \text{ then } h \in Z(H).$$

We may suppose that the corresponding statement holds in $H/Z(H)$, and accordingly if $h^p \in Z(H)$, then $h \in Z_2(H)$.

Fix such an h. Then for $x \in H$ we have $1 = [h^p, x] = [h, x]^p = [h, x^p]$, and as H is p-radicable we find $h \in Z(H)$ as required.

Ad (2) In $\bar{H} = H/Z_{i-1}(H)$, our claim is that $Z(\bar{H})$ is π-torsion free and π-divisible. It is π-torsion free by part (1) applied to $H/Z_{i-2}(H)$, and it is π-divisible by $(*)$ applied to $\bar{H}$. □

LEMMA 1.19. *Let G be a group and H a normal nilpotent subgroup such that G/H' is nilpotent. Then G is nilpotent.*

PROOF. Let $H_0 = H$, $H_{i+1} = [G, H_i]$, and take n minimal so that $H_n \le H'$. We proceed by induction on n. It suffices therefore to show that G/H_1' is nilpotent.

Let $K_i = [H_i, H]$. As G/H' is nilpotent, it suffices to show that $[G, K_i] \le K_{i+1}H_1'$ for all i. We apply the three subgroups lemma: $[[G, H_i], H] = K_{i+1}$ and $[[G, H], H_i] = [H_1, H_i]$, which is contained in H_1' for $i \ge 1$ and is equal to K_{i+1} for $i = 0$. Thus $[G, K_i] \le K_{i+1}H_1'$ for all i. □

LEMMA 1.20. *If H is a nilpotent group, π a set of primes, and H has a central series H_i such that every section H_i/H_{i+1} is π-radicable, then H is π-radicable.*

LEMMA 1.21. *If H, K are normal nilpotent subgroups of the group G, then HK is nilpotent.*

LEMMA 1.22. *Let G be a nilpotent by finite p-group for some prime p.*

(1) *$Z(G) \ne 1$.*
(2) *For $H < G$, we have $N_G(H) > H$.*

PROOF.
Ad (1) Let $G_0 \triangleleft G$ be nilpotent and normal, with $\bar{G} = G/G_0$ finite. Let $A = \Omega_1(Z(G_0))$. Then the group $A \rtimes \bar{G}$ is an elementary abelian by finite p-group. Take $a \in A^\times$ and consider the subgroup $A_0 = \langle a^{\bar{G}} \rangle$ of A; this is a finite $\bar{G}$-invariant group.