Introduction

The notion of a three-dimensional Heisenberg group is an abstract algebraic formulation of a geometric phenomenon in everyday life. It occurs if we select a plane in our three-dimensional space. For example, the page you are now reading is such a plane and produces a Heisenberg group. If you take a photograph you are in the middle of Heisenberg group theory; you have transmitted information along a line and encoded it in a plane and hence you have established a Heisenberg group. These few remarks indicate already some aspects of our program encoded in the title. We will study Heisenberg groups and Heisenberg algebras as mathematical objects in detail, identify them in several physical and mathematical areas and thereby exhibit close relationships among them even though they look quite different at the beginning of our discourse. This is to say we will build bridges between fields by means of Heisenberg groups and Heisenberg algebras.

This short introductory look at our program hints that we like to invite both graduate students of mathematics and mathematicians with some interest in physics as well as graduate students of physics and physicists with some interest in mathematics on our journey through the very mathematics and physics of Heisenberg groups.

Now we go into more detail. Looking at a Euclidean oriented three-dimensional space E as a Heisenberg group or at a Heisenberg algebra amounts to a splitting of E into an oriented plane F and an oriented real $\mathbb{R} \cdot a$ line, say, orthogonal to F. Here $a \in S^2$. The orientation on E shall be made up by the orientations of the plane and the real line, respectively. A constant symplectic structure on the plane determines its orientation while the real line is oriented by a vector in it yielding a direction of the plane's oriented rotations. These geometric ingredients can be encoded in detail by a specific non-commutative group operation on the Euclidean space, yielding a Heisenberg group, a Lie group. Its center is the one-dimensional subspace. Its Lie algebra is called a Heisenberg algebra. In fact any $(2n + 1)$-dimensional Euclidean space admits a Heisenberg group structure for any integer n bigger than zero. However, with the applications we have in mind we mainly concentrate on three-dimensional Heisenberg groups and Heisenberg algebras. However, in field quantization we have to pass over to infinite dimensional ones.

Heisenberg groups have a very remarkable property: By the Stone–von Neumann theorem any Heisenberg group up to equivalence admits only one irreducible unitary representation on an infinite dimensional Hilbert space, if its action is specified on the center, i.e. on the real line introduced above. The rather simple looking Schrödinger representation of a Heisenberg group on the L_2-space of the real line (the Hilbert space) is unitary and irreducible. Therefore, up to equivalence, the Schrödinger representation is uniquely determined by its action on the center. To

define the Schrödinger representation, a coordinate system is needed turning the plane F into a phase space.

Together with the notion of time, requiring a fourth dimension, the oriented Euclidean space immediately determines the skew field of quaternions $\mathbb{H}$. This skew field provides us with a convenient mathematical structure to treat three-dimensional Heisenberg groups in a larger context. In fact, any Heisenberg algebra structure on E emanates from the multiplication of the quaternions. Moreover, the various ways of turning the oriented Euclidean space into a Heisenberg group determine the skew field $\mathbb{H}$, its three-sphere, i.e. the spin group $SU(2)$, and the Hopf bundles on the two-sphere. These Hopf bundles fibring S^3 over S are uniquely described by their respective Chern numbers. In addition, the skew field structure yields a natural Minkowski metric (which can be rescaled to meet the needs of special relativity) on $\mathbb{H}$ which intertwines Minkowski geometry with Euclidean geometry. This is nicely seen by computing the rotation angles of the inner automorphisms of $\mathbb{H}$ since any inner automorphism amounts to an oriented rotation on the Euclidean space. In fact, any Minkowski metric on an oriented four-dimensional linear space emanates from the natural Minkowski metric on $\mathbb{H}$.

Now it is conceivable that the modeling based on Heisenberg groups naturally involves the mathematical structures just described above. Let us shortly review three applications of the Heisenberg group, each of which reveals clearly this group as underlying mathematical background.

We begin these presentations by signal theory. As formulated in the seminal book of Groechenig [47], for example, the general framework of signal theory consists of three main steps, namely analysis, processing and synthesis of a signal. Since already the first step, namely signal analysis, is a rather huge field, for our purpose we need to restrict our scope. To show where Heisenberg groups and Heisenberg algebras appear in signal analysis, we focus on one of its branches, namely on time-frequency analysis, which from a mathematical point of view is a branch of harmonic analysis. Later, we describe in a rather rough fashion what time-frequency analysis focuses on by means of an analogy (cf. [47]) and consider a musical score. Time behavior is encoded horizontally whereas the frequency information is expressed vertically. The score represents an analysis of the signal in terms of time-frequency information. Playing the music is the synthesis or reconstruction of the signal. Truncation is a form of signal processing, for example. Expressed in a more abstract fashion, the time axis and the frequency axis generate the time frequency plane and a signal is given by a quadratically integrable, complex-valued function on the real line, the time axis, say. Its values give time information. Frequency information in the signal is visible in the Fourier transform of the function.

Methods of the analysis of a signal involve various technical tools, in particular various sorts of transformations of signals which exhibit specific properties. One of them is the ambiguity function (defined on the time-frequency plane), a fundamental tool, as for instance in radar engineering or in geometric optics in terms of its Fourier transform, the Wigner function. The ambiguity function compares two signals with each other, for example an outgoing with an incoming one. This situation is typical for radar. In the case of a plane the outgoing signal hitting the object is known in detail. The incoming signal contains information on position

and velocity. Formulated in a simplified fashion one can extract this information from the ambiguity function built up by the outgoing and the reflected signal.

As a first highlight in this discourse let us demonstrate the appearance of the Heisenberg group in time frequency analysis by means of the ambiguity function. From a technical point of view it is a simple but beautiful insight based on the Schrödinger representation first mentioned by W. Schempp in [**71**]: The first Fourier coefficient of the Schrödinger representation of the three-dimensional Heisenberg group is nothing more than the ambiguity function. Similar results on other tools can be found in [**47**]. The relation of the Heisenberg group with such a fundamental tool in time-frequency plane shows that this group is omnipotent in time-frequency analysis, as is also expressed in [**47**].

The time frequency plane is the Cartesian product of two axes, the time axis and the frequency axis. One goal of time-frequency analysis seems to be to resolve a point in this plane arbitrarily well by means of signals treated by various tools. Surprisingly, such a resolution process is not possible since time and frequency information of a signal are encoded in the signal itself, respectively in its Fourier transform. Uncertainty relations are obstructing it. These relations can be derived from a Heisenberg algebra and its infinitesimal Schrödinger representation, too.

In geometric optics in 3-space, the effect of an optical system placed in between two parallel planes can nicely be described by the Wigner function exhibiting a symplectic transformation A acting on the first plane, i.e. by a linear map of this plane which preserves a symplectic structure, as shown for example in [**24**]. The same linear map can be found by arguing in terms of wave optics as done in [**41**].

In our presentation of the appearance of the Heisenberg group in geometric optics we partly follow [**50**] since this approach naturally carries on to the quantization of homogeneous quadratic polynomials. To describe geometric optics a little more precisely, let the plane F be mapped by light rays to another plane F' parallel to F. In between these two planes an optical system is placed. A symplectic structure on F is caused by the (2+1)-splitting of the Euclidean 3-space E initiated by the choice of F and its orthogonal complement in E. The choice of a coordinate system turns the symplectic plane into a phase space. As mentioned above, the image in F' caused by the light rays passing the optical system is described by a symplectic transformation A of F. Vice versa, any map in the group $\mathrm{Sp}(F)$, consisting of all symplectic transformations of F, corresponds to an optical system. What happens with a light distribution on F? Associated with a light distribution on F is a phase distribution. The image of this phase distribution on F' caused by the optical system is computed by Fresnel integrals, a tool in wave optics (cf. [**41**]). These Fresnel integrals are very closely related to metaplectic representations, i.e. representations of the metaplectic group $\mathrm{Mp}(F)$ of F with the L_2-space of the real line as representation space (cf. [**50**]). The metaplectic group $\mathrm{Mp}(F)$ is a twofold covering of the symplectic group $\mathrm{Sp}(F)$ of the plane. Back to the Wigner function mentioned above, it detects A also and is the Fourier transform of the ambiguity function. Therefore, geometric optics are based on a Heisenberg group as well. It is determined by the (2+1)-splitting of E produced by the symplectic plane F and its orthogonal complement in E. This is to say the array naturally yields a Heisenberg group $\mathcal{H}$. The symplectic map A on F (characterizing the optical system) is extended to all of $\mathcal{H}$ by the identity on the center of $\mathcal{H}$ (still called A). It is

a Heisenberg group isomorphism and hence determines a new Heisenberg group, namely $A(\mathcal{H})$. Due to the famous Stone–von Neumann theorem, its Schrödinger representation $\rho \circ A$ is equivalent to the Schrödinger representation ρ of $\mathcal{H}$ and causes the Wigner function on F describing the optical system. This is the content of chapter 8, so you can see we are already right in the middle of our manuscript.

Now let us pass on to ordinary quantum mechanics. There are several different looking approaches to it. For example, one way of formulating quantum mechanics is by concentrating more on analytic aspects such as Wigner functions and operator theory (cf. [**4**], [**37**], [**47**] and [**32**]). Another one is based directly on symplectic geometry (cf. [**50**] and [**75**]); operator theory enters here via representation theory. In the very beautiful book [**32**] it becomes clear how these approaches are intertwined. Since we concentrate on the appearance of the Heisenberg group and other geometric structures based on them, it is natural to focus on a geometric basis of Quantum mechanics. In doing so, in our investigations we adopt the view point taken in [**50**].

Keeping the role of time and frequency in time-frequency analysis in mind, in classical mechanics the analogous object of the time-frequency plane is the phase space of a line, which is a plane F, say. F is equipped with a coordinate system in which one coordinate axis is identified with the line, on which a point is thought to move. Let us call it the q-axis. At any instant this point has a position q and a momentum p, say, visualized on the second coordinate axis, the p-axis. Hence the pair (q, p) of coordinates characterizes a point in phase space F. As in time-frequency analysis, F is equipped with a symplectic structure, hence determines a Heisenberg group structure on E.

Quantization of position and momentum in classical mechanics is achieved by means of the infinitesimal Schrödinger representation $d\rho$ multiplied by $-i$ where i is the imaginary unit of the complex plane. Hence $-i \cdot d\rho$ converts each element of the Heisenberg algebra into a self-adjoint operator acting on the L_2-space of the real line, a Hilbert space. It consists of all quadratically integrable, complex-valued functions of the real line. The quantization of q and p yields two non-commuting operators, obeying Heisenberg's uncertainty relations.

Kinetic energy (a classical observable) of the moving point is a homogeneous quadratic polynomial. The quantization of this type of polynomials (called classical observables) defined on the phase space F is in a sense an infinitesimal version of geometric optics. Here is why: The infinitesimal metaplectic representation multiplied by $-i$ represents the Lie-algebra of the metaplectic group $\mathrm{Mp}(F)$ in the space of the self-adjoint operators of the L_2-space of the real line. This Lie algebra is identical with the Lie algebra $\mathrm{sp}(F)$ of $\mathrm{Sp}(F)$ and the Lie algebra (F) made up of all trace-free linear maps of F. Now the Lie algebra $sl(F)$ is naturally isomorphic to the Poisson algebra of homogeneous quadratic polynomials on the plane F, an algebra of classical observables of the moving point. Thus any homogeneous quadratic polynomial on F is converted to a self-adjoint operator acting on the L_2-space of the real line. Of course the definition of a homogeneous quadratic polynomial requires the coordinate system on F. This resembles the situation of the Schrödinger representation, and in fact, the metaplectic representation can be constructed out of the Schrödinger representation. Representing all quadratic polynomials of the plane requires a representation of the semidirect product of the Heisenberg group

with the metaplectic group and the Lie algebra of this product. At this stage we point out that the collection of quantized homogeneous quadratic polynomials together with the identity allows the reconstruction of the field of quaternions and hence of a Minkowski space. This is found in chapter 9.

Of a quite different nature is the quantization associated with a vector field in 3-space, elaborated in chapter 10. The goal here is to specify a collection of classical observables of the vector field and to associate field operators (on some infinite dimensional Hilbert space) to them. But first let us analyze the vector field in order to single out a collection of classical observables.

Given a vector field on a possibly bounded three-dimensional submanifold in the oriented Euclidean space E we may cut out all its singularities and obtain a singularity free vector field X on a smaller topological space of which we assume that it is a manifold M in E with or without boundary, say. A complex line bundle $\mathbb{F}$ on M is obtained by taking the orthogonal complement in E of each field vector as fibres. The points in $\mathbb{F}$ are called internal variables of the vector field. Each fibre of $\mathbb{F}$ admits a constant symplectic form determined by inserting the respective field vector into the volume form of E. Thus at each point in M, the fibre together with the line containing the field vector yields a three-dimensional Heisenberg group and hence in total a Heisenberg group bundle $\mathbf{H^a}$ as well as a Heisenberg algebra bundle on M. The bundle $\mathbf{H^a} \subset M \times E$ determines the vector field and vice versa. Passing on to the collection $\Gamma\mathbf{H^a}$ of all Schwartz sections and integrating up the fibrewisely given symplectic forms yields an infinite dimensional commutative Weyl algebra, a C^*-algebra whose involution sends a section into its negative and which in addition contains a natural Poisson algebra. The elements of this Poisson algebra $P^{\mathbf{a}}$ are called the classical observables of the vector field. This natural Poisson algebra determines the vector field and vice versa, as shown in chapter 6.

Now we begin to describe the quantization procedure of the vector field X as done in chapter 10. Here this procedure is split up into two steps, namely into a prequantization and into the specification of the physical observables reached by representations. The Poisson algebra $P^{\mathbf{a}}$ is the domain of the quantization map $Q_\hbar$, called here the prequantization. The quantization map $Q_\hbar$ represents the $*$-algebra $P^{\mathbf{a}}$ on the C^*-algebra $\mathcal{W}^\hbar\Gamma\mathbf{H^a}$, involving a parameter $\hbar$ varying on the real line. It may in particular assume the value of Planck's constant. If this parameter differs from zero, the multiplication (an $\hbar$ dependent deformed convolution) of the Weyl algebra $W^\hbar\Gamma\mathbf{H^a}$ is non-commutative. This construction yields a real parameterized family of Weyl algebras $W^\hbar\Gamma\mathbf{H^a}$ with parameter $\hbar$ and the Poisson algebra $P^{\mathbf{a}}$ for vanishing $\hbar$. In fact, $Q_\hbar$ is a strict and continuous deformation quantization in the sense of Rieffel (cf. [**55**]). The range of the map $Q_\hbar$ can be reproduced from a C^*-group algebra $C^*\mathcal{H}^\infty$ of the infinite dimensional Heisenberg group $\mathcal{H}^\infty := \Gamma\mathbf{H^a} + \mathbb{R}\cdot e$. This group is characteristic for the vector field, too. On $\mathbb{R}\cdot e$ in $\mathcal{H}^\infty$ varies the deformation parameter $\hbar$ mentioned above. If this parameter approaches 0 the Heisenberg group deforms to $W^0\Gamma\mathbf{H^a}$ containing the Poisson algebra $P^{\mathbf{a}}$. This family of Weyl algebras mentioned determines a C^*-algebra of so-called Weyl fields, $*$-isomorphic to $C^\infty\mathcal{H}^\infty$.

A representation of $W^\hbar\Gamma\mathbf{H^a}$ or in some cases also of $C^*\mathcal{H}^\infty$ represents these C^*-algebras on the C^*-algebra $\mathbf{B}\mathcal{H}$, the C^*-algebra of all bounded operators of $\mathcal{H}$. The respective images are called the collection of physical quantum observables.

From here we construct the field operators and derive the canonical commutation relations (CCR). This construction is called the Weyl quantization of the vector field.

Thus the infinite dimensional Heisenberg group $\mathcal{H}^\infty$ governs the Weyl quantization and allows a classical limit as $\hbar$ tends to 0 in a continuous rigorous fashion.

We close chapter 10 by studying the influence of the topology of the three-manifold M to the quantization of the vector field X defined on M.

Now we have already alluded to the content of the later chapters of these notes. But let us start from the beginning. The first two chapters collect and prepare the mathematical material for the later ones. We intend to show that the quaternions $\mathbb{H}$ are a convenient tool to describe the geometry in three- and four-space naturally hidden in $SU(2)$. In particular, we investigate the automorphisms of $\mathbb{H}$; these automorphisms provide a link between the natural Minkowski geometry on $\mathbb{H}$ and the Euclidean one on E. These studies open the doors to the Hopf bundles on S^2. We hence pay a little more attention to this skew field than the mere application of Heisenberg groups and Heisenberg algebras would require.

The C^*-quantization associated with a singularity free vector field in 3-space requires the notion of a Heisenberg algebra bundle associated with it. These bundles naturally contain the complex line bundles of such vector fields. The geometry of these line bundles is treated in chapter 3. A classification of them, in terms of homotopy theory and Chern classes, is the goal of chapter 4. It prepares the effect of the topology to the field quantization done in chapter 10.

In chapter 5 Heisenberg groups and their Lie algebras are introduced. We need them in the quantization of homogeneous polynomials in two variables. We observe that the skew field of quaternions is determined by only one Heisenberg group or one Heisenberg algebra inside of $\mathbb{H}$. These groups and algebras link Euclidean and Minkowski geometry. The close ties of $\mathbb{H}$ and Heisenberg algebras with Minkowski geometry are exhibited and group theoretically formulated. Here the symplectic group and $\mathrm{SL}(2, \mathbb{C})$ reproduce isometry groups of three- and four-dimensional Minkowski spaces.
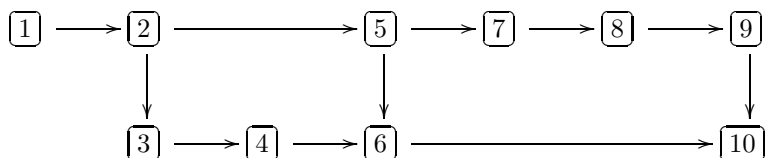
The main tools of the quantization of vector fields in 3-space are their Heisenberg group bundles and the C^*-algebras of sections of them. The infinite dimensional C^*-Heisenberg group as well as a natural C^*-Weyl algebra emanate from the vector fields. Both are characteristic for the field. To show this is the topic of chapter 6.

The Schrödinger representation of Heisenberg groups and the metaplectic representation are the basic topics of chapter 7. These representations are essential for the quantization of inhomogeneous quadratic polynomials. Both representations influence signal analysis and geometric optics fundamentally.

The notes end with a remarkable appendix by Serge Preston. The deep relations between information theory and thermodynamics are well recognized and utilized as documented in the references of the appendix. Therefore, in the spirit of the approach of this monograph, one might expect the Heisenberg group to play some prominent role in geometrical structures of thermodynamics. This is beautifully presented in this appendix.

Hence after the applications in Chapters 8 to 10 described above, the appendix "Thermodynamics, Geometry and Heisenberg group" provides an answer to the following question: The energy-phase space (P, θ, G) of a homogeneous thermodynamical system, together with its contact structure θ and natural indefinite metric G introduced by R. Mrugala, is isomorphic to the Heisenberg group $\mathcal{H}_n$ endowed with a right-invariant contact structure and the right-invariant indefinite metric G. Different properties of these structures are studied in terms of curvature and isometries of the metric G. Geodesics of the metric G are closely related with the three-dimensional Heisenberg subgroups $\mathcal{H}_1$ of the group $\mathcal{H}_n$. A natural compactification $(\hat{P}, \hat{\theta}, \hat{G})$ of the triple (P, θ, G) with its stratification by the subgroups of the type $R_k \times \mathcal{H}_{n-k}, k = 0, \ldots, n$ is investigated.

The above outlines of the chapters shall be complemented by a short description of the main interdependencies which are graphically visualized in the following diagram:

$$
\boxed{1} \longrightarrow \boxed{2} \longrightarrow \boxed{5} \longrightarrow \boxed{7} \longrightarrow \boxed{8} \longrightarrow \boxed{9}
$$
$$
\boxed{3} \longrightarrow \boxed{4} \longrightarrow \boxed{6} \longrightarrow \boxed{10} .
$$

The first chapter provides the basis for the second one and presents techniques used throughout the book. Results of both of them are applied to smooth vector fields in 3-space by means of complex line bundles in the third chapter. The classification of these vector fields in terms of complex line bundles in chapter 4 is based on the earlier chapters, however, new technical means are introduced and interrelated with earlier ones in order to understand and formulate the classifications mentioned. The techniques provided by the first two chapters are used in chapter 5 to interrelate the concept of a Heisenberg group and a Heisenberg algebra with Euclidean and Minkowski as well as with symplectic geometry. The Schrödinger and the metaplectic representations introduced in chapter 7 form the basis for all the later ones. Chapter 6 lays the foundation for chapter 10 and uses among newly introduced techniques the ones provided in the first three chapters. The Chapters 8 and 9 use the representation theory presented in chapter 7 and require material from chapter 5. The last chapter on field quantization is based on chapter 6 and part of the study of the topological influence to this quantization relies on chapter 4.

Finally, a word on the prerequisites: The first three chapters as well as chapter 5 can easily be read with a background in linear algebra and elementary differential geometry as provided by senior undergraduate or low level graduate courses. Technically more involved are chapters 4, 6 and 7. With the prerequisites mentioned for chapter 4, the reader has to invest some time reading the topological part involving the classification of complex line bundles and the section concerning the mapping degree. In chapter 6 different concepts are introduced and studied; it is partly self-contained. However, it is technically more advanced than earlier ones. In chapter 7 some knowledge in representation theory would be helpful. The literature referenced contains all the technicalities which are used. Technically less involved are the self-contained parts of chapters 8 and 9. Other, more advanced parts are complemented by references to standard literature. Some basic knowledge in functional

analysis would make the reading easier. Both chapters, however, require knowledge from earlier ones, in particular from chapter 7 on representation theory. More knowledge from the earlier chapters and from the literature referenced is needed to follow the last chapter. Together with the appendix on thermodynamics it is the most complex one.

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