Introduction

The aim of this book is to give a self-contained presentation of an asymptotic theory for eigenvalue problems using layer potential techniques. This theory and its application in the field of inverse problems have been developed over the last number of years by the authors.

Since the early part of the twentieth century, the use of integral equations has developed into a range of tools for the study of partial differential equations. This includes the use of single- and double-layer potentials to treat classical boundary value problems.

Our main objective in this book is to show how powerful the layer potential techniques are for solving not only boundary value problems but also eigenvalue problems if they are combined with the elegant theory of Gohberg and Sigal on meromorphic operator-valued functions.

There are two prominent eigenvalue perturbation problems: one under variation of domains or boundary conditions and the other due to the presence of small-volume inclusions.

There have been several interesting works on the problem of eigenvalue changes under variation of domains since the seminal formula of Hadamard [**119**]: the works by Garabedian and Schiffer [**113**], Kato [**149**], Fujiwara and Ozawa [**102**], Sanchez Hubert and Sanchez Palencia [**221**], Ward and Keller [**248**], Gadyl'shin and Il'in [**112**], Gadyl'shin [**110, 111**], Daners [**80**], McGillivray [**181**], Noll [**192**], Planida [**212**], Bruno and Reitich [**61**], Burenkov and Lamberti [**62**], and Kozlov [**153**].

For the second problem, Rauch and Taylor [**216**] have shown that the spectrum of a bounded domain does not change after imposing Dirichlet conditions on compact subsets of capacity zero. Subsequently, many people have studied the asymptotic expansions of the eigenvalues for the case of small holes with a Dirichlet or a Neumann boundary condition. In particular, Ozawa provided in a series of papers [**202**]–[**207**] leading-order terms (A_0, A_1, A_2 in (0.3)) in eigenvalue expansions; see also [**248**] and [**177**]. Besson [**54**] has proved the existence of a complete expansion (0.3) of the eigenvalue perturbation in the two-dimensional case. Courtois [**76**] has established a perturbation theory for the Dirichlet spectrum in a compactly perturbed domain in terms of the capacity of the compact perturbation. We also refer to the book by Maz'ya, Nazarov, and Plamenevskii [**178**], where the method of matched expansions [**133, 134**] has been used to construct asymptotic representations of eigenvalues of problems of conduction and elasticity theory for bodies with small holes.

In this book, we shall consider both the first and the second eigenvalue problems. For the first problem, we consider an inclusion inside a bounded domain and derive high-order asymptotic expansions of the perturbations of the eigenvalues that are due to shape variations of the inclusion. We also study the effect of a change

in the boundary condition on a small part of the boundary on the eigenvalues. For the second problem, we provide a complete asymptotic expansion of the eigenvalue perturbation with respect to the size of the inclusion for a domain containing a small inclusion. Our exposition is accompanied by many new applications of our asymptotic theory, especially to imaging and optimal design. It is worth emphasizing that the asymptotic results derived in this book have numerous other important applications in practice.

To be more precise, let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, with a connected Lipschitz boundary $\partial\Omega$. Let ν denote the unit outward normal to $\partial\Omega$. Suppose that Ω contains a small inclusion D, of the form $D = z + \epsilon B$, where B is a bounded Lipschitz domain in $\mathbb{R}^d$ containing the origin. We also assume that the "background" is homogeneous with conductivity 1. Even though we will deal with other cases as well, for the moment let us assume that the inclusion is grounded, meaning that zero Dirichlet conditions are imposed on ∂D. Then the eigenvalue problem for the domain with the inclusion is given by

$$(0.1) \qquad \begin{cases} \Delta u^\epsilon + \mu^\epsilon u^\epsilon = 0 & \text{in } \Omega_\epsilon, \\ \dfrac{\partial u^\epsilon}{\partial \nu} = 0 & \text{on } \partial\Omega, \\ u^\epsilon = 0 & \text{on } \partial D, \end{cases}$$

where $\Omega_\epsilon := \Omega \setminus \overline{D}$.

Let $0 = \mu_1 < \mu_2 \leq \ldots$ be the eigenvalues of $-\Delta$ in Ω with Neumann conditions, namely, those of the problem

$$(0.2) \qquad \begin{cases} \Delta u + \mu u = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

The eigenvalues are arranged in an increasing sequence and counted according to multiplicity. Fix j and suppose that the eigenvalue μ_j is simple. Note that this assumption is not essential in what follows and, moreover, it is proved in [**3, 4, 243**] that the eigenvalues are generically simple. By generic, we mean the existence of arbitrary small deformations of $\partial\Omega$ such that in the deformed domain the eigenvalue is simple. Throughout this book, the assumption of the simplicity is made for ease of exposition. Then there exists a simple eigenvalue μ_j^ϵ near μ_j associated to the normalized eigenfunction u_j^ϵ; that is, u_j^ϵ satisfies (0.1).

One of our goals in this book is to find *complete asymptotic expansions* for the eigenvalues μ_j^ϵ as ϵ tends to 0. In other words, we seek to find a series expansion of the form

$$(0.3) \qquad \mu_j^\epsilon = A_0 + A_1 \epsilon^n + A_2 \epsilon^{n+1} + \ldots.$$

Existence of such a series is a part of what we are going to prove.

The book consists of three parts. The first part is devoted to the theory developed by Gohberg and Sigal. In the second part, we provide rigorous derivations of complete asymptotic expansions of eigenvalue perturbations such as (0.3). A key feature of our work is the approach we develop: a *general and unified boundary integral approach* with rigorous justification based on the Gohberg-Sigal theory explained in Part 1. By using layer potential techniques, we show that the square roots of the eigenvalues are exactly the real characteristic values of meromorphic

operator-valued functions that are of Fredholm type with index 0. We then proceed from the *generalized argument principle* to construct their complete asymptotic expressions with respect to the perturbations. Our main idea is to reduce the eigenvalue problem to the study of characteristic values of systems of certain integral operators. A similar approach is extended in Part 3 to investigate the band gap structure of the frequency spectrum for waves in a high contrast, two-component periodic medium. This provides a new tool for investigating photonic and phononic crystals and solving difficult mathematical problems arising in these fields. Photonic and phononic crystals have attracted enormous interest in the last decade because of their unique optical and acoustic properties. Such structures have been found to exhibit interesting spectral properties with respect to classical wave propagation, including the appearance of band gaps. An important example of these crystals consists of a background medium which is perforated by a periodic array of arbitrary-shaped holes with different material parameters.

As we said, the method to derive the asymptotic expansion of the eigenvalues of problem (0.1) can be applied to other types of eigenvalue perturbation problems. As a first example, instead of being grounded, suppose that the inclusion may have a different conductivity, say $0 < k \neq 1 < +\infty$. Then the eigenvalue problem to be considered is

$$(0.4) \qquad \begin{cases} \nabla \cdot (1 + (k-1)\chi(D))\nabla u^\epsilon + \mu^\epsilon u^\epsilon = 0 & \text{in } \Omega, \\ \dfrac{\partial u^\epsilon}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\chi(D)$ denotes the indicator function of D which is of the form $D = z + \epsilon B$, as before. Another example is the derivation of high-order terms in the asymptotic expansions of the eigenvalue perturbations resulting from small perturbations of the shape of the conductivity inclusion D. A third example is concerned with the effect of internal corrosion on eigenvalues. Suppose that ∂D contains a corroded part I of small Hausdorff measure $|I| = \epsilon$ and let a positive constant γ denote the surface impedance (the corrosion coefficient) of I. The eigenvalue problem in the presence of corrosion consists of finding $\mu^\epsilon > 0$ such that there exists a nontrivial solution u^ϵ to

$$\begin{cases} \Delta u^\epsilon + \mu^\epsilon u^\epsilon = 0 & \text{in } \Omega \setminus \overline{D}, \\ -\dfrac{\partial u^\epsilon}{\partial \nu} + \gamma\chi(I)u^\epsilon = 0 & \text{on } \partial D, \\ u^\epsilon = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\chi(I)$ denotes the characteristic function on I.

We will also derive an asymptotic expansion of the eigenvalues for the elasticity equations with Neumann boundary conditions in the presence of a small elastic inclusion.

As will be shown in this book, these asymptotic expansions can be used for identifying the inclusions. We provide a general method for determining the locations and/or shape of small inclusions by taking eigenvalue and eigenfunction measurements. It should be emphasized that in its most general form the inverse spectral problem is severely ill-posed and nonlinear. This has been the main obstacle to finding noniterative reconstruction algorithms with limited modal data.

Our method of asymptotic expansions of small-volume inclusions provides a useful framework to accurately and efficiently reconstruct the location and geometric features of the inclusions in a stable way, even for moderately noisy modal data.

Indeed, the asymptotic expansions of the eigenvalue perturbations resulting from small perturbations of the shape of a conductivity inclusion, which extend those for small-volume conductivity inclusions, lead to very effective algorithms, aimed at determining certain properties of the shape of the conductivity inclusion based on eigenvalue measurements. We propose an original and promising optimization approach for reconstructing interface changes of a conductivity inclusion from measurements of eigenvalues and eigenfunctions associated with the transmission problem for the Laplacian or the Lamé system. A key identity, dual to the asymptotic expansion for the perturbations in the modal measurements that are due to small changes in the interface of the inclusion, is established. It naturally leads to the formulation of the proposed optimization problem. The viability of our reconstruction algorithms is documented by a variety of numerical results. Their resolution limit is discussed. The case of multiple eigenvalues is rigorously handled as well.

Our general approach is also applied for defect classification and sizing by vibration testing. Following the asymptotic formalism developed in this book, we derive asymptotic formulas for the effects of corrosion on resonance frequencies and mode shapes and use them to design a simple method for localizing the corrosion and estimating its extent.

Our asymptotic theory for eigenvalue problems also leads to efficient algorithms for solving shape optimization problems. Shape optimization arises in many different fields, such as mechanical design and shape reconstruction. It can be generally described as a problem of finding the optimal shapes in a certain sense under certain constraints. We incorporate the asymptotic expansions derived in this book into a level set method to investigate optimal design of photonic and phononic crystals. The level set is used to represent the interface between two materials with different physical parameters. We present efficient algorithms for finding the optimal shapes for maximal band gaps and acoustic drum problems.