

# Contents

Preface	ix
History	x
Related works	xi
Acknowledgments	xii
Differences from the preliminary version	xiii
Introduction	xv
Notation	xxix
Chapter 1. The Berkovich unit disc	1
1.1. Definition of $\mathcal{D}(0, 1)$	1
1.2. Berkovich's classification of points in $\mathcal{D}(0, 1)$	2
1.3. The topology on $\mathcal{D}(0, 1)$	7
1.4. The tree structure on $\mathcal{D}(0, 1)$	9
1.5. Metrizable	17
1.6. Notes and further references	18
Chapter 2. The Berkovich projective line	19
2.1. The Berkovich affine line $\mathbb{A}_{\text{Berk}}^1$	19
2.2. The Berkovich "Proj" construction	23
2.3. The action of a rational map $\varphi$ on $\mathbb{P}_{\text{Berk}}^1$	30
2.4. Points of $\mathbb{P}_{\text{Berk}}^1$ revisited	35
2.5. The tree structure on $\mathbb{H}_{\text{Berk}}$ and $\mathbb{P}_{\text{Berk}}^1$	38
2.6. Discs, annuli, and simple domains	40
2.7. The strong topology	42
2.8. Notes and further references	47
Chapter 3. Metrized graphs	49
3.1. Definitions	49
3.2. The space $\text{CPA}(\Gamma)$	50
3.3. The potential kernel $j_z(x, y)$	52
3.4. The Zhang space $\text{Zh}(\Gamma)$	54

3.5.	The space $BDV(\Gamma)$	56
3.6.	The Laplacian on a metrized graph	61
3.7.	Properties of the Laplacian on $BDV(\Gamma)$	68
Chapter 4.	The Hsia kernel	73
4.1.	Definition of the Hsia kernel	73
4.2.	The extension of $j_z(x, y)$ to $\mathbb{P}_{\text{Berk}}^1$	76
4.3.	The spherical distance and the spherical kernel	79
4.4.	The generalized Hsia kernel	81
4.5.	Notes and further references	85
Chapter 5.	The Laplacian on the Berkovich projective line	87
5.1.	Continuous functions	87
5.2.	Measures on $\mathbb{P}_{\text{Berk}}^1$	91
5.3.	Coherent systems of measures	95
5.4.	The Laplacian on a subdomain of $\mathbb{P}_{\text{Berk}}^1$	97
5.5.	Properties of the Laplacian	101
5.6.	The Dirichlet pairing	104
5.7.	Favre–Rivera-Letelier smoothing	113
5.8.	The Laplacians of Favre, Jonsson, and Rivera-Letelier, and of Thuillier	116
5.9.	Notes and further references	119
Chapter 6.	Capacity theory	121
6.1.	Logarithmic capacities	121
6.2.	The equilibrium distribution	123
6.3.	Potential functions attached to probability measures	128
6.4.	The transfinite diameter and the Chebyshev constant	136
6.5.	The Fekete-Szegő theorem	141
6.6.	Notes and further references	144
Chapter 7.	Harmonic functions	145
7.1.	Harmonic functions	145
7.2.	The Maximum Principle	150
7.3.	The Poisson formula	155
7.4.	Uniform convergence	161
7.5.	Harnack’s principle	162
7.6.	Green’s functions	163
7.7.	Pullbacks	174
7.8.	The multi-center Fekete-Szegő theorem	177

7.9. A Bilu-type equidistribution theorem	184
7.10. Notes and further references	191
Chapter 8. Subharmonic functions	193
8.1. Subharmonic and strongly subharmonic functions	193
8.2. Domination subharmonicity	199
8.3. Stability properties	207
8.4. The Domination Theorem	213
8.5. The Riesz Decomposition Theorem	214
8.6. The topological short exact sequence	219
8.7. Convergence of Laplacians	227
8.8. Hartogs's lemma	230
8.9. Smoothing	234
8.10. The Energy Minimization Principle	240
8.11. Notes and further references	248
Chapter 9. Multiplicities	249
9.1. An analytic construction of multiplicities	249
9.2. Images of segments and finite graphs	271
9.3. Images of discs and annuli	278
9.4. The pushforward and pullback measures	285
9.5. The pullback formula for subharmonic functions	287
9.6. Notes and further references	290
Chapter 10. Applications to the dynamics of rational maps	291
10.1. Construction of the canonical measure	293
10.2. The Arakelov-Green's function $g_{\mu_\varphi}(x, y)$	299
10.3. Adelic equidistribution of dynamically small points	306
10.4. Equidistribution of preimages	318
10.5. The Berkovich Fatou and Julia sets	328
10.6. Equicontinuity	333
10.7. Fixed point theorems and their applications	340
10.8. Dynamics of polynomial maps	354
10.9. Rational dynamics over $\mathbb{C}_p$	357
10.10. Examples	370
10.11. Notes and further references	374
Appendix A. Some results from analysis and topology	377
A.1. Convex functions	377
A.2. Upper and lower semicontinuous functions	378

A.3. Nets	379
A.4. Measure-theoretic terminology	381
A.5. Radon measures	381
A.6. Baire measures	382
A.7. The Portmanteau theorem	383
A.8. The one-point compactification	384
A.9. Uniform spaces	385
A.10. Newton polygons	387
Appendix B. $\mathbb{R}$ -trees and Gromov hyperbolicity	393
B.1. Definitions	393
B.2. An equivalent definition of $\mathbb{R}$ -tree	394
B.3. Geodesic triangles	395
B.4. The Gromov product	397
B.5. $\mathbb{R}$ -trees and partial orders	401
B.6. The weak and strong topologies	402
Appendix C. A Brief overview of Berkovich's theory	405
C.1. Motivation	405
C.2. Seminorms and norms	406
C.3. The spectrum of a normed ring	406
C.4. Affinoid algebras and affinoid spaces	409
C.5. Global $k$ -analytic spaces	412
C.6. Properties of $k$ -analytic spaces	414
Bibliography	417
Index	423