CHAPTER 1

Gradings on Algebras

In this chapter we present the terminology and basic results on gradings that will be used throughout the book. In particular, we discuss the natural equivalence relations for gradings, universal grading groups, refinements and coarsenings, and the duality between gradings and actions. The latter will be our main tool for classifying gradings on simple Lie algebras. We explain it first over algebraically closed fields of characteristic zero, where gradings by abelian groups correspond to actions of quasitori, and then over arbitrary fields, where such gradings correspond to actions of diagonalizable group schemes. As a result, in Chapter 3, we will be able to reduce gradings on the classical simple Lie algebras of types A_r, B_r, C_r and D_r (except D_4), over algebraically closed fields of characteristic different from 2, to gradings on matrix algebras and certain anti-automorphisms that preserve them. In the same vein, in Chapter 4, we will reduce gradings on the simple Lie algebra of type G_2 to those on the algebra of octonions, and, in Chapter 5, gradings on the simple Lie algebra of type F_4 to those on the exceptional simple Jordan algebra (the Albert algebra). The necessary background on affine group schemes can be found in Appendix A.

1.1. General gradings and group gradings

Graded vector spaces and their maps. Let V be a vector space over a field $\mathbb{F}$ and let G be a set.

DEFINITION 1.1. A *G-grading* Γ on V is any decomposition of V into a direct sum of subspaces indexed by G,

$$\Gamma:\ V=\bigoplus_{g\in G} V_g.$$

Here we allow some of the subspaces V_g to be zero. The set

$$\operatorname{Supp}\Gamma := \{g\in G \mid V_g\neq 0\}$$

is called the *support* of Γ. The grading is *nontrivial* if the support consists of more than one element. If $v\in V_g$, then we say that v is *homogeneous of degree g* and write $\deg_\Gamma v = g$ or just $\deg v = g$ if the grading is clear from the context. The subspace V_g is called the *homogeneous component* of degree g. If a grading Γ is fixed, then V will be referred to as a *graded vector space*.

Any element $v\in V$ can be uniquely written as $\sum_{g\in G} v_g$ where $v_g\in V_g$ and all but finitely many of the elements v_g are zero. We will refer to v_g as the *homogeneous components of v*.

There are two natural ways in which a linear map $f\colon V\to W$ can respect gradings on V and W.

Definition 1.2. Let V be a G-graded vector space and let W be an H-graded vector space. A linear map $f\colon V \to W$ will be called *graded* if for any $g \in G$ there exists $h \in H$ such that $f(V_g) \subset W_h$. Clearly, if $f(V_g) \neq 0$, then h is uniquely determined.

Definition 1.3. Let V and W be G-graded vector spaces. A linear map $f\colon V \to W$ will be called a *homomorphism of G-graded spaces* if for all $g \in G$, we have $f(V_g) \subset W_g$. The set of all such maps will be denoted $\mathrm{Hom}^G(V, W)$.

The class of G-graded vector spaces with $\mathrm{Hom}^G(V, W)$ as morphisms is an $\mathbb{F}$-linear category, which will be denoted by Mod^G.

A subspace $W \subset V$ is said to be a *graded subspace* if

$$W = \bigoplus_{g \in G} (V_g \cap W).$$

It is easy to see that this happens if and only if, for any element v in W, all its homogeneous components v_g are also in W. Taking $W_g = V_g \cap W$, we turn W into a G-graded vector space so that the imbedding $W \hookrightarrow V$ is a homomorphism of G-graded spaces. In particular, if $H \subset G$, then

$$V_H := \bigoplus_{h \in H} V_h$$

is a graded subspace of V.

The image of a graded subspace under a graded linear map is again a graded subspace. Indeed, suppose $f\colon V \to W$ and $U \subset V$ are graded. Let U'_h be the sum of $f(U_g)$ over all $g \in G$ such that $f(U_g) \subset W_h$. Then $f(U) = \bigoplus_{h \in H} U'_h$.

If G is a group (or a semigroup with cancellation), then for each $g \in G$, we can define the *left shift* ${}^{[g]}V$ of a G-graded vector space V by setting ${}^{[g]}V_{gh} := V_h$, $h \in G$. In other words, we set $\deg_{[g]\Gamma} v := g \deg_\Gamma v$ for any (nonzero) homogeneous element v. We define the *right shift* $V^{[g]}$ in a similar way.

Definition 1.4. Let G be a group. A linear map $f\colon V \to W$ of G-graded vector spaces is said to be *homogeneous of degree g* if $f(V_h) \subset W_{gh}$ for all $h \in G$. In other words, $f\colon {}^{[g]}V \to W$ is a homomorphism of G-graded spaces.

Clearly, a homogeneous map of any degree is a graded map in the sense of Definition 1.2. The homomorphisms of G-graded spaces are precisely the homogeneous maps of degree e, the identity element of G.

Denote the space of all homogeneous maps of degree g by $\mathrm{Hom}_g(V, W)$ and set

$$\mathrm{Hom}^{\mathrm{gr}}(V, W) := \bigoplus_{g \in G} \mathrm{Hom}_g(V, W).$$

If V is finite-dimensional or if the supports of the gradings on V and W are finite, then $\mathrm{Hom}^{\mathrm{gr}}(V, W) = \mathrm{Hom}(V, W)$ and thus the space $\mathrm{Hom}(V, W)$ becomes G-graded.

Remark 1.5. It will be sometimes convenient for us in Chapter 2 to write maps on the right. Then the space $\mathrm{Hom}_g(V, W)$ should be defined using the right shift $W^{[g]}$ instead of the left shift ${}^{[g]}W$. This leads, in general, to a different subspace $\mathrm{Hom}^{\mathrm{gr}}(V, W)$ in $\mathrm{Hom}(V, W)$ — and in the finite case, to a different grading on $\mathrm{Hom}(V, W)$.

Finally, if U is a G-graded vector space and V is an H-graded vector space, then the tensor product $W = U \otimes V$ has a natural $G \times H$-grading given by $W_{(g,h)} = U_g \otimes V_h$. If both U and V are G-graded and G is a semigroup, then $W = U \otimes V$ can also be regarded as a G-graded space:

$$W_g = \bigoplus_{g_1, g_2 \in G \,:\, g_1 g_2 = g} U_{g_1} \otimes V_{g_2}.$$

Graded algebras. Let $\mathcal{A}$ be a nonassociative algebra. The most general concept of grading on $\mathcal{A}$ is a decomposition of $\mathcal{A}$ into a direct sum of subspaces such that the product of any two subspaces is contained in a third subspace. Using the terminology we just introduced, we can state this as follows.

DEFINITION 1.6. Let S be a set. An S-grading on $\mathcal{A}$ is a vector space grading such that the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is graded (Definition 1.2), where $\mathcal{A} \otimes \mathcal{A}$ has its natural $S \times S$ grading. If such a grading on $\mathcal{A}$ is fixed, then $\mathcal{A}$ will be referred to as a *graded algebra*.

For the following discussion, it will be convenient to discard the homogeneous components that are zero, i.e., to assume that S is the support of the grading:

$$\Gamma:\ \mathcal{A} = \bigoplus_{s \in S} \mathcal{A}_s \text{ where } \mathcal{A}_s \neq 0 \text{ for any } s \in S. \tag{1.1}$$

Then for any $s_1,\ s_2 \in S$ either $\mathcal{A}_{s_1}\mathcal{A}_{s_2} = 0$ or there is a unique $s_3 \in S$ with $\mathcal{A}_{s_1}\mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}$. Thus the support S is equipped with a partially defined (nonassociative) binary operation $s_1 \cdot s_2 := s_3$.

DEFINITION 1.7. We will say that Γ as in (1.1) is a *(semi)group grading* if $(S, \cdot)$ can be imbedded into a (semi)group G. Regarding S as a subset of G and setting $\mathcal{A}_g = 0$ for $g \in G \setminus S$, we recover Definition 0.1, which is equivalent to saying that the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a homomorphism of G-graded vector spaces (Definition 1.3), where $\mathcal{A} \otimes \mathcal{A}$ has its natural G-grading.

Replacing G with a sub(semi)group if necessary, we can assume that G is generated by S.

DEFINITION 1.8. We will say that a grading Γ as in (1.1) is *realized* as a G-grading if G is a (semi)group containing S, the subspaces

$$\mathcal{A}_g := \begin{cases} \mathcal{A}_s & \text{if } g = s \in S; \\ 0 & \text{if } g \notin S; \end{cases}$$

form a G-grading on $\mathcal{A}$, as in Definition 0.1, and S generates G. A *realization* of Γ is the G-grading determined by a (semi)group G and an imbedding $S \hookrightarrow G$ as above.

One can ask whether or not all gradings on a certain class of algebras can be realized in this way. It was asserted in [**PZ89**, Theorem 1(d)] that any grading on a Lie algebra is a semigroup grading, but later a counterexample was discovered [**Eld06a**]. In that example the grading is on a nilpotent Lie algebra of dimension 16. Here we present much easier examples of non-semigroup gradings on a metabelian Lie algebra of dimension 4 and on a semisimple Lie algebra of dimension 6.

EXAMPLE 1.9 ([**Eld09c**]). Consider the Lie algebra $\mathcal{L} = \operatorname{span}\{a, u, v, w\}$ of dimension 4, with multiplication given by

$$[a, u] = u, \quad [a, v] = w, \quad [a, w] = v,$$

and all other brackets of basis elements being 0. Thus $\mathcal{L}$ is the semidirect sum of the 1-dimensional subalgebra $\operatorname{span}\{a\}$ and the 3-dimensional abelian ideal $\operatorname{span}\{u, v, w\}$. Define a grading on $\mathcal{L}$ as follows:

$$\Gamma : \mathcal{L} = \mathcal{L}_{s_1} \oplus \mathcal{L}_{s_2} \oplus \mathcal{L}_{s_3}$$

where $\mathcal{L}_{s_1} = \operatorname{span}\{a, u\}$, $\mathcal{L}_{s_2} = \operatorname{span}\{v\}$, and $\mathcal{L}_{s_3} = \operatorname{span}\{w\}$. It is straightforward to check that Γ is indeed a grading on $\mathcal{L}$. But if Γ were a semigroup grading, the following equations would hold in the semigroup:

$$\begin{aligned} s_1^2 &= s_1, &\text{as}\quad [\mathcal{L}_{s_1}, \mathcal{L}_{s_1}] &= \operatorname{span}\{u\} \subset \mathcal{L}_{s_1}, \\ s_1 s_2 &= s_3, &\text{as}\quad [\mathcal{L}_{s_1}, \mathcal{L}_{s_2}] &= \operatorname{span}\{w\} = \mathcal{L}_{s_3}, \\ s_1 s_3 &= s_2, &\text{as}\quad [\mathcal{L}_{s_1}, \mathcal{L}_{s_3}] &= \operatorname{span}\{v\} = \mathcal{L}_{s_2}. \end{aligned}$$

Hence we would obtain:

$$s_3 = s_1 s_2 = s_1^2 s_2 = s_1(s_1 s_2) = s_1 s_3 = s_2,$$

a contradiction.

EXAMPLE 1.10 ([**Eld09c**]). Let $\mathcal{J} = \operatorname{span}\{x, y, h\}$ and $\mathcal{K} = \operatorname{span}\{e_1, e_2, e_3\}$ be simple Lie algebras of dimension 3, with multiplication given by

$$\begin{array}{lll} [h, x] = x, & [h, y] = -y, & [x, y] = h; \\ [e_1, e_2] = e_3, & [e_2, e_3] = e_1, & [e_3, e_1] = e_2. \end{array}$$

Let $\mathcal{L} = \mathcal{J} \oplus \mathcal{K}$. If $\operatorname{char} \mathbb{F} \neq 2$, then $\mathcal{J}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{F})$, and $\mathcal{K}$ to $\mathfrak{so}_3(\mathbb{F})$. If, in addition, $\mathbb{F}$ contains a square root of -1, then $\mathcal{J}$ and $\mathcal{K}$ are isomorphic, and $\mathcal{L}$ is isomorphic to $\mathfrak{so}_4(\mathbb{F})$. Define a grading on $\mathcal{L}$ as follows:

$$\Gamma : \mathcal{L} = \mathcal{L}_{s_1} \oplus \mathcal{L}_{s_2} \oplus \mathcal{L}_{s_3} \oplus \mathcal{L}_{s_4} \oplus \mathcal{L}_{s_5}$$

where $\mathcal{L}_{s_1} = \operatorname{span}\{h, e_1\}$, $\mathcal{L}_{s_2} = \operatorname{span}\{x\}$, $\mathcal{L}_{s_3} = \operatorname{span}\{y\}$, $\mathcal{L}_{s_4} = \operatorname{span}\{e_2\}$, $\mathcal{L}_{s_5} = \operatorname{span}\{e_3\}$. If Γ were a semigroup grading, the following equations would hold in the semigroup:

$$s_1 = s_2 s_3 = (s_1 s_2) s_3 = s_1(s_2 s_3) = s_1^2,$$

and, therefore,

$$s_5 = s_1 s_4 = s_1^2 s_4 = s_1(s_1 s_4) = s_1 s_5 = s_4,$$

a contradiction.

It is shown in [**Eld09c**] that all gradings on Lie algebras of dimension ≤ 3 are semigroup gradings, so Example 1.9 has minimal possible dimension. The direct sum of the 2-dimensional non-abelian and 1-dimensional abelian Lie algebras admits a semigroup grading that is not a group grading [**Eld09c**]; this is obviously an example of minimal possible dimension. The following still remains open:

QUESTION 1.11. Is any grading on a finite-dimensional simple Lie algebra over $\mathbb{C}$ a group grading?

If we assume from the start that the grading is a semigroup grading, then the answer is positive. In fact, we have the following result.

PROPOSITION 1.12. *Let $\mathcal{L}$ be a simple Lie algebra over any field. If G is a semigroup and $\mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ is a G-grading with support S where G is generated by S, then G is an abelian group.*

PROOF. First we prove that, for any $g \in S$, the multiplication maps,

$$l_g \colon G \to G \colon x \mapsto gx \quad \text{and} \quad r_g \colon G \to G \colon x \mapsto xg,$$

are surjective. Indeed, first we fix $s \in S$, $s \neq g$. Since $\mathcal{L}$ is simple and $\mathcal{L}_g \neq 0$, the ideal generated by $\mathcal{L}_g$ is the entire $\mathcal{L}$. It follows that there exist $s_1, \ldots, s_n \in S$ $(n \geq 1)$ such that

$$0 \neq [[\mathcal{L}_g, \mathcal{L}_{s_1}], \ldots, \mathcal{L}_{s_n}] \subset \mathcal{L}_s.$$

Hence $gs_1 \cdots s_n = s$ and $s \in l_g(G)$. We have proved that $S \setminus \{g\} \subset l_g(G)$. Now take $h \in G$, $h \neq g$. We can write $h = h_1 \cdots h_k$ with $h_1, \ldots, h_k \in S$ $(k \geq 1)$. If $h_1 = g$, then $k > 1$ and hence $h \in l_g(G)$. If $h_1 \neq g$, then $h_1 \in l_g(G)$ and hence $h \in l_g(G)$. We have proved that $G \setminus \{g\} \subset l_g(G)$. It remains to show that $g \in l_g(G)$. Since $[\mathcal{L}, \mathcal{L}] = \mathcal{L}$, we have $S \subset SS$ and hence $g = xy$ for some $x, y \in S$. If $x = g$, then $g \in l_g(G)$; otherwise $x \in l_g(G)$ and hence $g \in l_g(G)$. The proof for r_g is similar.

Since S generates G, it follows that l_g and r_g are surjective for all $g \in G$. It is easy to see that any semigroup with this property is a group.[1]

Now we can finish the proof as in [**BZ06**, Lemma 2.1] (another proof is given in [**DM06**, Proposition 1]). Namely, we show by induction on $n \geq 2$ that

$$[[\mathcal{L}_{g_1}, \mathcal{L}_{g_2}], \ldots, \mathcal{L}_{g_n}] \neq 0$$

implies that g_i commute pairwise. (This property holds for an arbitrary Lie algebra $\mathcal{L}$.) Indeed, for $n = 2$ we obtain by anticommutativity that

$$0 \neq [\mathcal{L}_{g_1}, \mathcal{L}_{g_2}] \subset \mathcal{L}_{g_1 g_2} \cap \mathcal{L}_{g_2 g_1},$$

so $g_1 g_2 = g_2 g_1$. If $n \geq 3$, then by induction $g_1, \ldots, g_{n-1}$ commute pairwise and also g_n commutes with the product $g_1 \cdots g_{n-1}$. By Jacobi identity, at least one of the subspaces $[[[\mathcal{L}_{g_1}, \mathcal{L}_{g_2}], \ldots, \mathcal{L}_{g_{n-2}}], \mathcal{L}_{g_n}]$ and $[[[\mathcal{L}_{g_1}, \mathcal{L}_{g_2}], \ldots, \mathcal{L}_{g_{n-2}}], [\mathcal{L}_{g_{n-1}}, \mathcal{L}_{g_n}]]$ is nonzero, so by induction at least one of the elements g_n and $g_{n-1} g_n$ commutes with all of $g_1, \ldots, g_{n-2}$. In either case it follows that g_n commutes with $g_1, \ldots, g_{n-1}$, as desired.

Finally, for any $g, h \in S$, using the simplicity of $\mathcal{L}$ as before, we can find $g_1, \ldots, g_n$ such that $0 \neq [[\mathcal{L}_g, \mathcal{L}_{g_1}], \ldots, \mathcal{L}_{g_n}] \subset \mathcal{L}_h$. It follows that $g g_1 \cdots g_n = h$ and hence h commutes with g. $\square$

Equivalence and isomorphism of gradings. Given a group grading Γ, there are, in general, many groups G such that Γ can be realized as a G-grading.

EXAMPLE 1.13 ([**DM06**]). Let $\mathcal{L} = \mathfrak{J}_1 \oplus \mathfrak{J}_2$ where $\mathfrak{J}_i$ is a copy of $\mathfrak{J}$ from Example 1.10, with basis $\{x_i, y_i, h_i\}$, $i = 1, 2$. Consider

$$\Gamma : \ \mathcal{L} = \mathcal{L}_{s_1} \oplus \mathcal{L}_{s_2} \oplus \mathcal{L}_{s_3} \oplus \mathcal{L}_{s_4}$$

where $\mathcal{L}_{s_1} = \operatorname{span}\{h_1, h_2\}$, $\mathcal{L}_{s_2} = \operatorname{span}\{x_2, y_2\}$, $\mathcal{L}_{s_3} = \operatorname{span}\{x_1\}$, $\mathcal{L}_{s_4} = \operatorname{span}\{y_1\}$. Then Γ can be realized as a grading by the cyclic group $\langle g \rangle$ of order 6 with $s_1 = e$, $s_2 = g^3$, $s_3 = g^2$, $s_4 = g^4$ and also as a grading by the symmetric group $\operatorname{Sym}(3)$ with $s_1 = e$, $s_2 = (12)$, $s_3 = (123)$, $s_4 = (132)$.

[1]The above proof was communicated to the authors by C. Draper.

We will come back to this situation in the next section (Corollary 1.19). Now it is important to note that, even if one is dealing exclusively with group gradings (as we will in this book), one should clearly indicate whether or not the grading group is considered as a part of the definition of grading. Hence there are two natural ways to define equivalence relation on group gradings. We will use the term "isomorphism" for the case when the grading group is a part of definition and "equivalence" for the case when the grading group plays a secondary role.

An *equivalence of graded vector spaces* $f\colon V \to W$ is a linear isomorphism such that both f and f^{-1} are graded maps (Definition 1.2). Let

$$\Gamma:\ \mathcal{A} = \bigoplus_{s\in S} \mathcal{A}_s \text{ and } \Gamma':\ \mathcal{B} = \bigoplus_{t\in T} \mathcal{B}_t$$

be two gradings on algebras, with supports S and T, respectively.

Definition 1.14. We say that Γ and Γ' are *equivalent* if there exists an *equivalence of graded algebras* $\varphi\colon \mathcal{A} \to \mathcal{B}$, i.e., an isomorphism of algebras that is also an equivalence of graded vector spaces. We will also say that φ is an *equivalence* of Γ and Γ'. It determines a bijection $\alpha\colon S \to T$ such that $\varphi(\mathcal{A}_s) = \mathcal{B}_{\alpha(s)}$ for all $s \in S$.

In particular, two equivalent gradings on the same algebra $\mathcal{A}$ can be obtained from one another by the action of $\mathrm{Aut}(\mathcal{A})$ and relabeling the components. In the finite-dimensional case, a simple, but important invariant of a grading is obtained by looking at the dimensions of the components: the *type* of Γ is the sequence of numbers $(n_1, n_2, \ldots)$ where n_1 is the number of 1-dimensional components, n_2 is the number of 2-dimensional components, etc.

The algebras graded by a fixed (semi)group G form a category where the morphisms are the *homomorphisms of G-graded algebras*, i.e., homomorphisms of algebras $\varphi\colon \mathcal{A} \to \mathcal{B}$ such that $\varphi(\mathcal{A}_g) \subset \mathcal{B}_g$ for all $g \in G$.

Definition 1.15. We say that two G-graded algebras, $\mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\mathcal{B} = \bigoplus_{g\in G} \mathcal{B}_g$, are *isomorphic* if there exists an isomorphism of algebras $\varphi\colon \mathcal{A} \to \mathcal{B}$ such that $\varphi(\mathcal{A}_g) = \mathcal{B}_g$ for all $g \in G$. We will also say that φ is an *isomorphism* of the G-gradings on $\mathcal{A}$ and $\mathcal{B}$.

In particular, two isomorphic gradings on the same algebra $\mathcal{A}$ can be obtained from one another by the action of $\mathrm{Aut}(\mathcal{A})$ (without relabeling) and hence have the same support.

Following [**PZ89**], we can associate three subgroups of $\mathrm{Aut}(\mathcal{A})$ to a grading Γ on an algebra $\mathcal{A}$.

Definition 1.16. The *automorphism group of* Γ, denoted $\mathrm{Aut}(\Gamma)$, consists of all self-equivalences of Γ, i.e., automorphisms of $\mathcal{A}$ that permute the components of Γ. Each $\varphi \in \mathrm{Aut}(\Gamma)$ determines a self-bijection $\alpha = \alpha(\varphi)$ of the support S such that $\varphi(\mathcal{A}_s) = \mathcal{A}_{\alpha(s)}$ for all $s \in S$. The *stabilizer of* Γ, denoted $\mathrm{Stab}(\Gamma)$, is the kernel of the homomorphism $\mathrm{Aut}(\Gamma) \to \mathrm{Sym}(S)$ given by $\varphi \mapsto \alpha(\varphi)$. (In the case of a G-graded algebra, this is the same as the group of automorphisms, $\mathrm{Aut}_G(\mathcal{A})$, in the category of G-graded algebras.) Finally, the *diagonal group of* Γ, denoted $\mathrm{Diag}(\Gamma)$, is the (abelian) subgroup of the stabilizer consisting of all automorphisms φ such that the restriction of φ to any homogeneous component of Γ is the multiplication by a (nonzero) scalar.

The quotient group $\mathrm{Aut}(\Gamma)/\,\mathrm{Stab}(\Gamma)$, which is a subgroup of $\mathrm{Sym}(S)$, will be called the *Weyl group of* Γ and denoted by $W(\Gamma)$. If Γ is the Cartan grading on a semisimple Lie algebra $\mathfrak{g}$ (see Example 0.6), then $W(\Gamma)$ is isomorphic to the so-called extended Weyl group of $\mathfrak{g}$, i.e., the automorphism group of the root system of $\mathfrak{g}$ (see Chapter 3). The importance of $\mathrm{Diag}(\Gamma)$ will become clear later, when we discuss duality between gradings and actions.

1.2. The universal group of a grading

As was pointed out earlier, a group grading Γ, in general, can be realized as a G-grading for many groups G. It turns out [**PZ89**] that there is one distinguished group among them.

Definition 1.17. Let Γ be a grading on an algebra $\mathcal{A}$. Suppose that Γ admits a realization as a G_0-grading for some group G_0. We will say that G_0 is a *universal group of* Γ if for any other realization of Γ as a G-grading, there exists a unique homomorphism $G_0 \to G$ that restricts to identity on $\mathrm{Supp}\,\Gamma$.

Note that, by definition, G_0 is a group with a distinguished generating set, $\mathrm{Supp}\,\Gamma$. A standard argument shows that, if a universal group exists, it is unique up to an isomorphism over $\mathrm{Supp}\,\Gamma$. We will denote it by $U(\Gamma)$. The following proposition shows that $U(\Gamma)$ exists and depends only on the equivalence class of Γ.

Proposition 1.18. *Let* Γ *be a group grading on an algebra* $\mathcal{A}$. *Then there exists a universal group* $U(\Gamma)$. *Two group gradings,* Γ *on* $\mathcal{A}$ *and* Γ' *on* $\mathcal{B}$, *are equivalent if and only if there exist an algebra isomorphism* $\varphi\colon \mathcal{A} \to \mathcal{B}$ *and a group isomorphism* $\alpha\colon U(\Gamma) \to U(\Gamma')$ *such that* $\varphi(\mathcal{A}_g) = \mathcal{B}_{\alpha(g)}$ *for all* $g \in U(\Gamma)$.

Proof. We define $U(\Gamma)$ to be the group with generating set $S = \mathrm{Supp}\,\Gamma$ and relations $s_1 s_2 = s_3$ for all $0 \neq \mathcal{A}_{s_1}\mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}$. Then, for any realization of Γ as a G-grading, we have a unique homomorphism $U(\Gamma) \to G$ induced by the identity map on S. Since S is imbedded in G, the canonical map $S \to U(\Gamma)$ is also an imbedding. The second assertion of the proposition follows from the universal property of $U(\Gamma)$. $\square$

Corollary 1.19. *For a given group grading* Γ *and a group* G, *the realizations of* Γ *as a* G*-grading are in one-to-one correspondence with the epimorphisms* $U(\Gamma) \to G$ *that are injective on* $\mathrm{Supp}\,\Gamma$. $\square$

It follows from Proposition 1.18 that any $\varphi \in \mathrm{Aut}(\Gamma)$ gives rise to a unique automorphism $u(\varphi)$ of $U(\Gamma)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Supp}\,\Gamma \arrow[r] \arrow[d, "\alpha(\varphi)"'] & U(\Gamma) \arrow[d, "u(\varphi)"] \\ \mathrm{Supp}\,\Gamma \arrow[r] & U(\Gamma) \end{tikzcd}$$

where the horizontal arrows are the canonical imbeddings. This gives an action of $\mathrm{Aut}(\Gamma)$ by automorphisms of the group $U(\Gamma)$. The kernel of this action is $\mathrm{Stab}(\Gamma)$, so we may regard $W(\Gamma) := \mathrm{Aut}(\Gamma)/\,\mathrm{Stab}(\Gamma)$ as a subgroup of $\mathrm{Aut}(U(\Gamma))$.

COROLLARY 1.20. *For a given group grading* Γ *and a group* G*, the isomorphism classes of the realizations of* Γ *as a* G*-grading are in one-to-one correspondence with the* $W(\Gamma)$*-orbits in the set of all epimorphisms* $U(\Gamma) \to G$ *that are injective on* $\operatorname{Supp}\Gamma$*.* $\square$

We note that one can construct the group $U(\Gamma)$ as in the proof of Proposition 1.18 for any grading Γ. Then Γ is a group grading, i.e., can be realized as a grading by a group, if and only if the canonical map $S \to U(\Gamma)$ is an imbedding.

From Proposition 1.12, we immediately obtain the following result.

COROLLARY 1.21. *Let* Γ *be a group grading on a simple Lie algebra. Then* $U(\Gamma)$ *is an abelian group.* $\square$

REMARK 1.22. For any group grading Γ, we can define the universal *abelian* group $U_{ab}(\Gamma)$ by the same generators and relations as in the proof of Proposition 1.18. The canonical map $S \to U_{ab}(\Gamma)$ is an imbedding if and only if Γ can be realized as a grading by an abelian group.

1.3. Fine gradings

Change-of-group functor and weak isomorphism of gradings. Let G and H be (semi)groups. Then any homomorphism $\alpha\colon G \to H$ induces a functor from the category Mod^G of G-graded spaces to the category Mod^H of H-graded spaces as follows. If $\Gamma:\ V = \bigoplus_{g\in G} V_g$ is a G-grading on V, then the decomposition ${}^\alpha\Gamma:\ V = \bigoplus_{h\in H} V'_h$ defined by

$$V'_h = \bigoplus_{g\in G\,:\,\alpha(g)=h} V_g$$

is an H-grading on A. The functor sends V with grading Γ to V with grading ${}^\alpha\Gamma$; it is the identity map on morphisms. We will say that ${}^\alpha\Gamma$ is the grading *induced from* Γ *by the homomorphism* α.

It is sometimes desirable to extend the notion of isomorphism of graded spaces V and W to include the situation when they are graded by different groups, say, G and H, respectively. Then V and W are said to be isomorphic if there exist an isomorphism of vector spaces $\varphi\colon V \to W$ and a group isomorphism $\alpha\colon G \to H$ such that $\varphi(V_g) = W_{\alpha(g)}$ for all $g \in G$. Note that, for $G = H$, this definition gives a *weaker* notion of isomorphism than that in the category Mod^G. The two notions are related as follows. A grading $\Gamma:\ V = \bigoplus_{g\in G} V_g$ is isomorphic to $\Gamma':\ W = \bigoplus_{h\in H} W_h$ in the weaker sense if and only if Γ' is isomorphic in the stronger sense to ${}^\alpha\Gamma$ for some isomorphism $\alpha\colon G \to H$. We will refer to the stronger notion as *isomorphism* and to the weaker notion as *weak isomorphism*.

We can also speak of weak isomorphism of graded algebras. Then the second assertion of Proposition 1.18 can be restated as follows: if G and H are the universal groups of gradings $\Gamma:\ \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ and $\Gamma':\ \mathcal{B} = \bigoplus_{h\in H} \mathcal{B}_h$, respectively, then Γ and Γ' are equivalent if and only if they are weakly isomorphic. We also have the following

COROLLARY 1.23. *For a given group grading* Γ*, the weak isomorphism classes of realizations of* Γ *are in one-to-one correspondence with the* $W(\Gamma)$*-orbits in the set of all normal subgroups* N *of* $U(\Gamma)$ *such that the quotient map* $U(\Gamma) \to U(\Gamma)/N$ *is injective on* $\operatorname{Supp}\Gamma$*.*

PROOF. Suppose we have two realizations of Γ, one as a G-grading and the other as an H-grading. By Corollary 1.19, they are determined by some epimorphisms $\pi_1\colon U(\Gamma) \to G$ and $\pi_2\colon U(\Gamma) \to H$, both injective on $\operatorname{Supp}\Gamma$. These realizations are weakly isomorphic if and only if there exists $\varphi \in \operatorname{Aut}(\Gamma)$ and an isomorphism $\alpha\colon G \to H$ such that the following diagram commutes:

$$\begin{tikzcd} U(\Gamma) \arrow[r, "\pi_1"] \arrow[d, "u(\varphi)"'] & G \arrow[d, "\alpha"] \\ U(\Gamma) \arrow[r, "\pi_2"] & H \end{tikzcd}$$

Hence the realizations are weakly isomorphic if and only if there exists $\varphi \in \operatorname{Aut}(\Gamma)$ such that $u(\varphi)$ maps $\ker \pi_1$ onto $\ker \pi_2$. $\square$

We will now see that the functor $\operatorname{Mod}^G \to \operatorname{Mod}^H$ induced by a homomorphism of groups $\alpha\colon G \to H$ is analogous to the base change functor for categories of modules.

Gradings and comodules. Let G be a set. Let $\mathbb{F}G$ be the vector space that has G as a basis. Then $\mathbb{F}G$ is a coalgebra with comultiplication and counit defined, respectively, by $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$, for all $g \in G$. It is well-known that a G-grading is equivalent to the structure of an $\mathbb{F}G$-comodule. (For the background on coalgebras, comodules, etc. the reader may refer to Appendix A.) The equivalence is set up as follows. If $\Gamma : \ V = \bigoplus_{g\in G} V_g$ is a G-grading on V, then the corresponding structure of a right $\mathbb{F}G$-comodule is given by the coaction $\rho_\Gamma\colon V \to V \otimes \mathbb{F}G$ where

$$\rho_\Gamma(v) := v \otimes g \quad \text{for all} \quad v \in V_g,\ g \in G. \tag{1.2}$$

Conversely, given a coaction $\rho\colon V \to V \otimes \mathbb{F}G$, we can define a grading Γ on V by setting

$$V_g := \{v \in V \mid \rho(v) = v \otimes g\} \quad \text{for all} \quad g \in G.$$

It follows from the axioms of comodule that V is the direct sum of V_g, $g \in G$. Clearly, $\rho_\Gamma = \rho$. Also, a linear map is a homomorphism of G-graded spaces if and only if it is a homomorphism of $\mathbb{F}G$-comodules.

If G is a semigroup with identity element, then the semigroup algebra $\mathbb{F}G$, with the above comultiplication and counit, is a bialgebra. If G is a group, then the group algebra $\mathbb{F}G$ is a Hopf algebra, with antipode given by $S(g) = g^{-1}$, for all $g \in G$.

Recall that if U and V are comodules over some bialgebra $\mathcal{B}$, then their tensor product $W = U \otimes V$ is itself a comodule. A G-graded space $\mathcal{A}$ with multiplication $\mu\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a G-graded algebra if and only if the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \otimes \mathcal{A} \arrow[r, "\rho_{\mathcal{A}\otimes\mathcal{A}}"] \arrow[d, "\mu"'] & \mathcal{A} \otimes \mathcal{A} \otimes \mathbb{F}G \arrow[d, "\mu \otimes \mathrm{id}"] \\ \mathcal{A} \arrow[r, "\rho_{\mathcal{A}}"] & \mathcal{A} \otimes \mathbb{F}G \end{tikzcd}$$

This can be expressed by saying that $\rho_{\mathcal{A}}$ is a homomorphism of algebras or, equivalently, that μ is a homomorphism of comodules.

If $\alpha\colon G \to H$ is a group homomorphism, then we can extend it to a Hopf algebra homomorphism $\mathbb{F}G \to \mathbb{F}H$, which we also denote by α. One immediately verifies that

$$\rho_{{}^\alpha\Gamma} = (\mathrm{id} \otimes \alpha) \circ \rho_\Gamma. \tag{1.3}$$

Refinements and coarsenings. If we apply an arbitrary homomorphism $\alpha\colon G \to H$ to a G-grading Γ, then some components of Γ may coalesce in ${}^\alpha\Gamma$.

Definition 1.24. Let Γ and Γ' be two gradings on V with supports S and T, respectively. We will say that Γ is a *refinement* of Γ', or that Γ' is a *coarsening* of Γ, and write $\Gamma' \leq \Gamma$, if for any $s \in S$ there exists $t \in T$ such that $V_s \subset V'_t$. If, for some $s \in S$, this inclusion is strict, then we will speak of a *proper* refinement or coarsening.

Clearly, $\leq$ is a partial order on the set of all gradings on V (if we regard all relabelings as one grading). The trivial grading is the unique minimal element. If V is finite-dimensional, then there also exist maximal elements, which are called *fine gradings*. It should be pointed out that the notion of fine grading depends on the class of gradings one is working with. For example, grading (0.1) is fine in the class of group gradings if $(g_i g_j^{-1})^2 \neq e$ for all $i \neq j$ (see Proposition 2.31), but, for $n \geq 2$, it admits a proper refinement in the class of semigroup gradings: namely, take the 1-dimensional subspaces span$\{E_{ij}\}$ as the components. It is remarkable that, by virtue of Proposition 1.12, the notions of fine semigroup grading, fine group grading and fine abelian group grading are all equivalent for simple Lie algebras.

The element $t \in T$ in Definition 1.24 is uniquely determined by $s \in S$, so $s \mapsto t$ defines a mapping $\pi\colon S \to T$. Clearly, this mapping is surjective, and we have $W_t = \bigoplus_{s \in S\,:\,\pi(s)=t} V_s$.

If G is a group, $\Gamma : \mathcal{A} = \bigoplus_{g \in G} \mathcal{A}_g$ is a G-grading on an algebra $\mathcal{A}$, and $\alpha\colon G \to H$ is a homomorphism of groups, then the H-grading ${}^\alpha\Gamma$ is a coarsening of Γ (not necessarily proper). However, it is not true in general that all coarsenings of Γ arise in this way. In fact, the example of a non-group grading on $\mathfrak{sl}_2(\mathbb{F}) \times \mathfrak{sl}_2(\mathbb{F})$ in [**Eld09c**] shows that a coarsening of a group grading is not necessarily a group grading. The following result shows what can still be salvaged in this situation.

Proposition 1.25. *Let Γ be a grading on an algebra $\mathcal{A}$. Assume that Γ is a group grading and $G = U(\Gamma)$ is its universal group. If Γ' is a coarsening of Γ which is itself a group grading, then, for any realization of Γ' as an H-grading for some group H, there exists a unique epimorphism $\alpha\colon G \to H$ such that $\Gamma' = {}^\alpha\Gamma$. Moreover, if $S = \mathrm{Supp}\,\Gamma$, $T = \mathrm{Supp}\,\Gamma'$ and $\pi\colon S \to T$ is the map associated to the coarsening, then $U(\Gamma')$ is the quotient of G by the normal subgroup generated by the elements $s_1 s_2^{-1}$ for all $s_1, s_2 \in S$ with $\pi(s_1) = \pi(s_2)$.*

Proof. Since $0 \neq \mathcal{A}_{s_1}\mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}$ implies $\mathcal{A}'_{\pi(s_1)}\mathcal{A}'_{\pi(s_2)} \cap \mathcal{A}'_{\pi(s_3)} \neq 0$, we conclude that $\pi(s_1)\pi(s_2) = \pi(s_3)$ in any realization of Γ' as an H-grading. It follows that π induces an epimorphism $U(\Gamma) \to H$. The uniqueness is obvious.

Now let N be the normal subgroup of G stated above. Then for any realization of Γ' as an H-grading, the epimorphism $G \to H$ factors through $G/N \to H$. Hence T is imbedded into G/N, and G/N satisfies the universal property of $U(\Gamma')$. $\square$

In the finite-dimensional case, Proposition 1.25 allows us, in principle, to construct all group gradings once the fine group gradings and their universal groups are known.

COROLLARY 1.26. *Let $\mathcal{A}$ be a finite-dimensional algebra. Then all group gradings on $\mathcal{A}$, up to equivalence, are obtained by taking, for each fine group grading Γ on $\mathcal{A}$, the coarsenings Γ_N induced by all quotient maps $U(\Gamma) \to U(\Gamma)/N$ where N is the normal subgroup generated by some elements of the form $s_1 s_2^{-1}$, $s_1, s_2 \in \operatorname{Supp}\Gamma$. Moreover, $U(\Gamma)/N$ is the universal group of Γ_N. If N_1 and N_2 belong to one $W(\Gamma)$-orbit, then Γ_{N_1} and Γ_{N_2} are equivalent (in fact, weakly isomorphic).*

PROOF. Since $\mathcal{A}$ is finite-dimensional, any group grading is a coarsening of some fine group grading, so Proposition 1.25 applies. If, for some $\varphi \in \operatorname{Aut}(\Gamma)$, $u(\varphi)$ maps N_1 onto N_2, then we have a commutative diagram

$$\begin{array}{ccc} U(\Gamma) & \longrightarrow & U(\Gamma)/N_1 \\ {\scriptstyle u(\varphi)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ U(\Gamma) & \longrightarrow & U(\Gamma)/N_2 \end{array}$$

where α is an isomorphism. Hence the induced $U(\Gamma)/N_1$-grading is weakly isomorphic to the induced $U(\Gamma)/N_2$-grading. □

COROLLARY 1.27. *Let $\mathcal{A}$ be a finite-dimensional algebra and let G be a group. Then all G-gradings on $\mathcal{A}$, up to isomorphism, are obtained by taking, for each fine group grading Γ on $\mathcal{A}$, the G-gradings induced by all homomorphisms $U(\Gamma) \to G$. The homomorphisms belonging to one $W(\Gamma)$-orbit result in isomorphic G-gradings.* □

Note that, in general, a given grading can be induced from many fine gradings, so the descriptions given in Corollaries 1.26 and 1.27 do not yet give classifications of gradings up to equivalence and up to isomorphism, respectively.

1.4. Duality between gradings and actions

Throughout this section we assume that all grading groups are *abelian* (which can be done without loss of generality for simple Lie algebras in view of Proposition 1.12). We want to reformulate G-gradings on a given (nonassociative) algebra $\mathcal{A}$ in the language of actions of a suitable object on $\mathcal{A}$. We will assume that $\mathcal{A}$ is *finite-dimensional* unless stated otherwise. Replacing G with a smaller group if necessary, we may also assume that the support of a G-grading generates G and hence G is finitely generated.

Algebraically closed field of characteristic zero. The situation is easier if $\mathbb{F}$ is algebraically closed and $\operatorname{char}\mathbb{F} = 0$, so we will consider this case in some detail before handling the general case. For a finitely generated abelian group G, let $\widehat{G}$ be the group of characters, i.e., homomorphisms $G \to \mathbb{F}^\times$. Given a G-grading $\Gamma :\ \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$, we can define a $\widehat{G}$-action on $\mathcal{A}$ by setting

$$\chi * x = \chi(g)x \quad \text{for all} \quad x \in \mathcal{A}_g,\ g \in G \text{ and } \chi \in \widehat{G}. \tag{1.4}$$

Thus we obtain a homomorphism $\eta_\Gamma : \widehat{G} \to \operatorname{Aut}(\mathcal{A})$. This homomorphism is injective if and only if the support of Γ generates G. Furthermore, both $\widehat{G}$ and $\operatorname{Aut}(\mathcal{A})$

are algebraic groups, and η_Γ is a homomorphism of algebraic groups. (For the background on algebraic groups the reader may refer to Appendix A.) Writing $G \cong \mathbb{Z}^n \times G_f$ where G_f is the torsion subgroup of G, we see that $\widehat{G} \cong (\mathbb{F}^\times)^n \times \widehat{G_f}$. Thus $\widehat{G}$ is isomorphic (as an algebraic group) to the direct product of a torus, $(\mathbb{F}^\times)^n$, and a finite abelian group, $\widehat{G_f}$. Such algebraic groups are called *quasitori*. Conversely, if Q is a quasitorus, then $Q = \widehat{G}$ where $G = \mathfrak{X}(Q)$, the group of characters of Q, i.e., homomorphisms of algebraic groups $Q \to \mathbb{F}^\times$. Since any subgroup or quotient group of a finitely generated abelian group is again finitely generated and abelian, we see that any quotient group or subgroup of a quasitorus is again a quasitorus. Among algebraic groups, the quasitori are characterized by the property that all their representations are diagonalizable. Hence any quasitorus $Q \subset \mathrm{Aut}(\mathcal{A})$ gives rise to a grading on $\mathcal{A}$ by a finitely generated abelian group — namely, the eigenspace decomposition of $\mathcal{A}$ relative to Q, where the eigenspaces are labeled by elements of $\mathfrak{X}(Q)$. This discussion implies the following result.

PROPOSITION 1.28. *The G-gradings on $\mathcal{A}$ are in one-to-one correspondence with the homomorphisms of algebraic groups $\widehat{G} \to \mathrm{Aut}(\mathcal{A})$. Two G-gradings are isomorphic if and only if the corresponding homomorphisms are conjugate by an element of* $\mathrm{Aut}(\mathcal{A})$. *The weak isomorphism classes of gradings on $\mathcal{A}$ with the property that the support generates the grading group are in one-to-one correspondence with the conjugacy classes of quasitori in* $\mathrm{Aut}(\mathcal{A})$. □

Note that everything boils down to the structure of the algebraic group $\mathrm{Aut}(\mathcal{A})$. So if two algebras, $\mathcal{A}$ and $\mathcal{B}$, have isomorphic automorphism groups, then $\mathcal{A}$ and $\mathcal{B}$ have the same classification of gradings up to isomorphism or weak isomorphism.

The question when two gradings on $\mathcal{A}$ are equivalent can also be answered in the language of actions. However, the answer depends not only on $\mathrm{Aut}(\mathcal{A})$, but also on $\mathcal{A}$ itself. Let Γ be an abelian group grading on $\mathcal{A}$. Let $U = U_{ab}(\Gamma)$, the universal abelian group of Γ as in Remark 1.22. Then, by Proposition 1.28, we have an imbedding $\widehat{U} \to \mathrm{Aut}(\mathcal{A})$. Denote the image of this imbedding by Q. Recalling Definition 1.16, we see that Q is contained in $\mathrm{Diag}(\Gamma)$. Conversely, if $\varphi \in \mathrm{Diag}(\Gamma)$, then, for any $s \in \mathrm{Supp}\,\Gamma$, there is a scalar $\lambda(s) \in \mathbb{F}^\times$ such that $\varphi|_{\mathcal{A}_s} = \lambda(s)\mathrm{id}$, and, whenever $0 \neq \mathcal{A}_{s_1}\mathcal{A}_{s_2} \subset \mathcal{A}_{s_3}$, we have $\lambda(s_1)\lambda(s_2) = \lambda(s_3)$. Looking at the defining relations of U, we see that the map $\lambda\colon \mathrm{Supp}\,\Gamma \to \mathbb{F}^\times$ extends uniquely to a homomorphism $U \to \mathbb{F}^\times$, which we also denote by λ. By construction, λ acts as φ on $\mathcal{A}$. We have proved that $Q = \mathrm{Diag}(\Gamma)$. Therefore, we have the following:

$$U_{ab}(\Gamma) \cong \mathfrak{X}(\mathrm{Diag}(\Gamma)), \tag{1.5}$$

$$\mathrm{Aut}(\Gamma) = N_{\mathrm{Aut}(\mathcal{A})}(\mathrm{Diag}(\Gamma)), \tag{1.6}$$

$$\mathrm{Stab}(\Gamma) = C_{\mathrm{Aut}(\mathcal{A})}(\mathrm{Diag}(\Gamma)), \tag{1.7}$$

$$W(\Gamma) = N_{\mathrm{Aut}(\mathcal{A})}(\mathrm{Diag}(\Gamma))/C_{\mathrm{Aut}(\mathcal{A})}(\mathrm{Diag}(\Gamma)). \tag{1.8}$$

DEFINITION 1.29. Let $Q \subset \mathrm{Aut}(\mathcal{A})$ be a quasitorus. Let Γ be the eigenspace decomposition of $\mathcal{A}$ relative to Q. Then the quasitorus $\mathrm{Diag}(\Gamma)$ in $\mathrm{Aut}(\mathcal{A})$ will be called the *saturation* of Q. We always have $Q \subset \mathrm{Diag}(\Gamma)$, and we will say that Q is *saturated* if $Q = \mathrm{Diag}(\Gamma)$.

Note that Q is saturated if and only if $\mathfrak{X}(Q)$ is the universal group of Γ. Combining Proposition 1.18 (modified to the case of abelian groups) and Proposition 1.28, we obtain the following result.

PROPOSITION 1.30. *The equivalence classes of gradings on $\mathcal{A}$ are in one-to-one correspondence with the conjugacy classes of saturated quasitori in* $\mathrm{Aut}(\mathcal{A})$. $\square$

Given a grading Γ on $\mathcal{A}$, Corollary 1.23 describes all possible realizations of Γ as a grading by an abelian group. It translates to the dual language as follows:

COROLLARY 1.31. *For a given grading Γ on $\mathcal{A}$, the weak isomorphism classes of the realizations of Γ are in one-to-one correspondence with the $W(\Gamma)$-orbits in the set of all quasitori $Q \subset \mathrm{Diag}(\Gamma)$ whose saturation equals* $\mathrm{Diag}(\Gamma)$. $\square$

Note that Definition 1.29 is *not* intrinsic to the algebraic group $\mathrm{Aut}(\mathcal{A})$. Hence, if two algebras have isomorphic automorphism groups, they need not have the same classification of gradings up to equivalence. We will see examples of this in Chapters 4 and 5.

Let Γ and Γ' be two abelian group gradings on $\mathcal{A}$ and let Q and Q' be the corresponding saturated quasitori in $\mathrm{Aut}(\mathcal{A})$. The abelian version of Proposition 1.25 implies that Γ is a refinement of Γ' if and only if $\Gamma' = {}^{\alpha}\Gamma$ for some epimorphism $\alpha\colon U_{ab}(\Gamma) \to U_{ab}(\Gamma')$. The latter is equivalent to saying that $\eta_{\Gamma'} = \eta_{\Gamma} \circ \widehat{\alpha}$. Hence we obtain

$$\Gamma' \leq \Gamma \ \Leftrightarrow\ Q' \subset Q.$$

In particular, fine gradings correspond to maximal saturated quasitori. Clearly, any maximal quasitorus in $\mathrm{Aut}(\mathcal{A})$ is automatically saturated. Therefore, the notion of maximal saturated quasitorus coincides with the notion of maximal quasitorus, which is intrinsic to the algebraic group $\mathrm{Aut}(\mathcal{A})$. Maximal quasitori were called "MAD subgroups" in [**PZ89**] (for "Maximal Abelian Diagonalizable"). To summarize, we have the following

PROPOSITION 1.32. *The equivalence classes of fine gradings on $\mathcal{A}$ are in one-to-one correspondence with the conjugacy classes of maximal quasitori in* $\mathrm{Aut}(\mathcal{A})$. $\square$

It follows that, if two algebras have isomorphic automorphism groups, then they have the same classification of fine gradings up to equivalence.

A maximal torus T in $\mathrm{Aut}(\mathcal{A})$ gives rise to a grading Γ_0. Since all maximal tori are conjugate (see e.g. [**Hum75**, §21.3]), there is only one such grading up to equivalence. If $\mathrm{Aut}(\mathcal{A})$ is connected, then the centralizer C of T is connected [**Hum75**, §22.3] and nilpotent [**Hum75**, §21.4], so T coincides with the set of all semisimple elements in C [**Hum75**, §19.2]. Therefore, T is maximal as a quasitorus and hence Γ_0 is a fine grading with universal group $\mathfrak{X}(T)$.

For a semisimple Lie algebra $\mathcal{L}$, Γ_0 is the Cartan decomposition (Example 0.6) where $\mathfrak{h}$ is the tangent algebra of T. The grading group $\mathfrak{X}(T)$ can be identified with the root lattice. Although $\mathrm{Aut}(\mathcal{L})$ is not always connected (see e.g. [**Hum78**, §16.5]), still Γ_0 is a fine grading and $\mathfrak{X}(T)$ is its universal group (see the proof of the proposition below).

DEFINITION 1.33. A grading Γ on $\mathcal{A}$ is said to be *toral* if it can be realized as a G-grading such that the image of $\widehat{G}$ in $\mathrm{Aut}(\mathcal{A})$ is contained in a torus.

Since any torus is contained in a maximal one, the toral gradings on $\mathcal{A}$ are, up to equivalence, the coarsenings of Γ_0.

PROPOSITION 1.34 ([**DM06**]). *Let $\mathcal{L}$ be a semisimple Lie algebra and let G be an abelian group. Let Γ_0 be the Cartan decomposition of $\mathcal{L}$, relative to some* *maximal torus $T \subset \mathrm{Aut}(\mathcal{L})$, regarded as an $\mathfrak{X}(T)$-grading. Then, for a G-grading Γ on $\mathcal{L}$, the following conditions are equivalent:*

1) *Γ is toral;*
2) *Γ is isomorphic to ${}^{\alpha}\Gamma_0$ for some homomorphism $\alpha\colon \mathfrak{X}(T) \to G$;*
3) *the identity component of Γ contains a Cartan subalgebra of $\mathcal{L}$.*

PROOF. 1) $\Rightarrow$ 2) Since all root spaces are 1-dimensional, any automorphism belonging to $\mathrm{Stab}(\Gamma_0)$ must act as a scalar on each of them. Also, since Γ_0 is the eigenspace decomposition of $\mathcal{L}$ relative to $\mathfrak{h}$, any such automorphism must act as identity on $\mathfrak{h}$. Thus $\mathrm{Diag}(\Gamma_0) = \mathrm{Stab}(\Gamma_0) = C_{\mathrm{Aut}(\mathcal{L})}(T)$. It can be shown [**Jac79**, p. 278] that $\mathrm{Diag}(\Gamma_0)$ consists of inner automorphisms, so it is contained in the connected component of $\mathrm{Aut}(\mathcal{L})$. But the centralizer of T in the connected component is connected, so $C_{\mathrm{Aut}(\mathcal{L})}(T)$ is a torus containing T. We conclude that $\mathrm{Diag}(\Gamma_0) = C_{\mathrm{Aut}(\mathcal{L})}(T) = T$. Hence T is maximal as a quasitorus, and Γ_0 is a fine grading with universal group $\mathfrak{X}(T)$. Since Γ is equivalent to a coarsening of Γ_0, it is isomorphic to ${}^{\alpha}\Gamma_0$ by Proposition 1.25.

2) $\Rightarrow$ 3) The identity component of Γ_0 is a Cartan subalgebra, and it is contained in the identity component of ${}^{\alpha}\Gamma_0$.

3) $\Rightarrow$ 1) Since all Cartan subalgebras are conjugate [**Hum78**, §16.4], we may assume that the identity component of Γ contains $\mathfrak{h}$. Let $Q = \eta_\Gamma(\widehat{G})$. Then any element of Q restricts to identity on $\mathfrak{h}$ and hence is contained in $\mathrm{Stab}(\Gamma_0)$. As shown above, $\mathrm{Stab}(\Gamma_0) = T$. $\square$

COROLLARY 1.35. *Let $\mathcal{L}$ be a semisimple Lie algebra. Then any grading on $\mathcal{L}$ by a torsion-free abelian group is induced from a Cartan decomposition.* $\square$

Toral gradings and methods of refining them to obtain non-toral gradings play an important role in [**DM06, DM09, DMV10, DV12**].

Arbitrary field. We want to extend Propositions 1.28 and 1.32 to algebras over an arbitrary field $\mathbb{F}$ so we will have a way of transfering the classification of G-gradings up to isomorphism and the classification of fine gradings up to equivalence from one algebra to another if these algebras have isomorphic *automorphism group schemes*. The reason we have to use automorphism group schemes rather than automorphism groups is that the action of the group of characters $\widehat{G}$ does not capture the information about G-gradings if $\mathbb{F}$ is too small or if G has p-torsion and $\mathrm{char}\,\mathbb{F} = p$. In the case of an algebraically closed field of characteristic zero, the theory of affine algebraic group schemes is equivalent to the theory of affine algebraic groups, so, for this case, we will obtain just a reformulation of previous results. For the convenience of the reader, the necessary background on affine group schemes is briefly presented in Appendix A.

Recall that a G-grading Γ on a finite-dimensional vector space V is equivalent to a comodule structure $\rho_\Gamma\colon V \to V \otimes \mathbb{F}G$, as defined by (1.2). Since G is abelian, the Hopf algebra $\mathbb{F}G$ is commutative and thus represents an affine group scheme, which we denote by G^D. (Recall that affine group schemes of this form are called *diagonalizable* — see Proposition A.31; G can be identified with the *group of characters* $\mathfrak{X}(G^D)$ — see Example A.21.) Then ρ_Γ is equivalent to a morphism $\eta_\Gamma\colon G^D \to \mathbf{GL}(V)$, i.e., a linear representation of G^D on V. (This may go back to Grothendieck — see [**DG80**, II §2, no. 2.5].) If we pick a homogeneous basis $\{v_1, \ldots, v_n\}$ in V, $\deg_\Gamma(v_i) = g_i$, then the comorphism of representing objects $\eta_\Gamma^*\colon \mathbb{F}[X_{ij}, D^{-1}] \to \mathbb{F}G$ can be written explicitly as follows: $X_{ij} \mapsto \delta_{ij} g_i$, $i, j = 1, \ldots, n$. In particular, η_Γ is a closed imbedding if and only if η_Γ^* is onto if and only if $\mathrm{Supp}\,\Gamma$ generates G.

If Γ is a G-grading on an algebra $\mathcal{A}$, then the multiplication map $\mu\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ is a morphism of G^D-representations, which is equivalent to saying that G^D stabilizes μ as an element of $\mathrm{Hom}(\mathcal{A}\otimes\mathcal{A}, \mathcal{A})$, or that the image of $\eta_\Gamma\colon G^D \to \mathbf{GL}(\mathcal{A})$ is a subgroupscheme of $\mathbf{Aut}(\mathcal{A})$ (see Examples A.6 and A.29). Conversely, a morphism $\eta\colon G^D \to \mathbf{Aut}(\mathcal{A})$ gives rise to a G-grading Γ on the algebra $\mathcal{A}$ such that $\eta_\Gamma = \eta$. For any unital commutative associative $\mathbb{F}$-algebra $\mathcal{R}$, the action of $\mathcal{R}$-points of G^D by automorphisms of the $\mathcal{R}$-algebra $\mathcal{A}\otimes\mathcal{R}$ can be written explicitly:
(1.9)
$$(\eta_\Gamma)_{\mathcal{R}}(f)(x\otimes r) = x\otimes f(g)r \quad \text{for all} \quad x\in\mathcal{A}_g,\ r\in\mathcal{R},\ g\in G,\ f\in\mathrm{Alg}(\mathbb{F}G,\mathcal{R}),$$
which reduces to (1.4) when $\mathcal{R} = \mathbb{F}$, because the algebra homomorphisms $\mathbb{F}G \to \mathbb{F}$ can be identified with the group homomorphisms $G \to \mathbb{F}^\times$.

A group homomorphism $\alpha\colon G \to H$ gives rise to a morphism $\alpha^D\colon H^D \to G^D$. Then (1.3) implies that $\eta_{{}^\alpha\Gamma} = \eta_\Gamma \circ \alpha^D$.

Now if $\mathcal{B}$ is another algebra and we have a morphism $\theta\colon \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$, then any G-grading Γ on $\mathcal{A}$ induces a G-grading on $\mathcal{B}$ via the morphism $\theta\circ\eta_\Gamma\colon G^D \to \mathbf{Aut}(\mathcal{B})$. We will denote the induced grading by $\theta(\Gamma)$. Clearly, $\theta({}^\alpha\Gamma) = {}^\alpha(\theta(\Gamma))$.

The group $\mathrm{Aut}(\mathcal{A})$ of the $\mathbb{F}$-points of $\mathbf{Aut}(\mathcal{A})$ acts by automorphisms of $\mathbf{Aut}(\mathcal{A})$ via conjugation. Namely, $\varphi \in \mathrm{Aut}(\mathcal{A})$ defines a morphism $\mathrm{Ad}_\varphi\colon \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{A})$ as follows:

$$(1.10) \qquad (\mathrm{Ad}_\varphi)_{\mathcal{R}}(f) := (\varphi\otimes\mathrm{id})\circ f\circ(\varphi^{-1}\otimes\mathrm{id}) \quad \text{for all} \quad f\in\mathrm{Aut}_{\mathcal{R}}(\mathcal{A}\otimes\mathcal{R}).$$

Comparing (1.9) and (1.10), we see that $\mathrm{Ad}_\varphi(\Gamma)$ is the grading $\mathcal{A} = \bigoplus_{g\in G}\varphi(\mathcal{A}_g)$. We have obtained the following generalization of Proposition 1.28:

PROPOSITION 1.36. *The G-gradings on $\mathcal{A}$ are in one-to-one correspondence with the morphisms of affine group schemes $G^D \to \mathbf{Aut}(\mathcal{A})$. Two G-gradings are isomorphic if and only if the corresponding morphisms are conjugate by an element of* $\mathrm{Aut}(\mathcal{A})$. *The weak isomorphism classes of gradings on $\mathcal{A}$ with the property that the support generates the grading group are in one-to-one correspondence with the* $\mathrm{Aut}(\mathcal{A})$*-orbits of diagonalizable subgroupschemes in* $\mathbf{Aut}(\mathcal{A})$. ☐

Let Γ be an abelian group grading on $\mathcal{A}$. Define the subgroupscheme $\mathbf{Diag}(\Gamma)$ of $\mathbf{Aut}(\mathcal{A})$ as follows:

$$\mathbf{Diag}(\Gamma)(\mathcal{R}) := \{f\in\mathrm{Aut}_{\mathcal{R}}(\mathcal{A}\otimes\mathcal{R}) \mid f|_{\mathcal{A}_g\otimes\mathcal{R}} \in \mathcal{R}^\times\mathrm{id}_{\mathcal{A}_g\otimes\mathcal{R}} \text{ for all } g\in G\}.$$

Clearly, $\mathrm{Diag}(\Gamma) = \mathbf{Diag}(\Gamma)(\mathbb{F})$. Since $\mathbf{Diag}(\Gamma)$ is a subgroupscheme of a torus in $\mathbf{GL}(\mathcal{A})$, it is diagonalizable, so $\mathbf{Diag}(\Gamma) = U^D$ for some finitely generated abelian group U. If Γ is realized as a G-grading, then (1.9) shows that the image of the imbedding $\eta_\Gamma\colon G^D \to \mathbf{Aut}(\mathcal{A})$ is a subgroupscheme of $\mathbf{Diag}(\Gamma)$. The imbedding $G^D \to \mathbf{Diag}(\Gamma)$ corresponds to an epimorphism $U \to G$. We conclude that U satisfies the definition of the universal abelian group of Γ and hence equations (1.5) through (1.8) remain valid if we replace $\mathrm{Diag}(\Gamma)$ by $\mathbf{Diag}(\Gamma)$.

Let Γ and Γ' be two abelian group gradings on $\mathcal{A}$ and let $\mathbf{Q} = \mathbf{Diag}(\Gamma)$ and $\mathbf{Q}' = \mathbf{Diag}(\Gamma')$. Now Γ is a refinement of Γ' if and only if $\Gamma' = {}^\alpha\Gamma$ for some epimorphism $\alpha\colon U_{ab}(\Gamma) \to U_{ab}(\Gamma')$ (the abelian version of Proposition 1.25) if and only if $\eta_{\Gamma'} = \eta_\Gamma\circ\alpha^D$. Hence we obtain

$$\Gamma' \le \Gamma \ \Leftrightarrow\ \mathbf{Q}' \text{ is a subgroupscheme of } \mathbf{Q}.$$

It follows that fine gradings correspond to maximal diagonalizable subgroupschemes of $\mathbf{Aut}(\mathcal{A})$. We have obtained the following generalization of Proposition 1.32:

PROPOSITION 1.37. *The equivalence classes of fine gradings on $\mathcal{A}$ are in one-to-one correspondence with the* $\mathrm{Aut}(\mathcal{A})$*-orbits of maximal diagonalizable subgroupschemes in* $\mathbf{Aut}(\mathcal{A})$. □

Transfer theorems. The following results will be crucial in Chapters 3, 4, 5 and 7.

THEOREM 1.38. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite-dimensional (nonassociative) algebras. Assume we have a morphism $\theta\colon \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$. Then, for any abelian group G, we have a mapping, $\Gamma \to \theta(\Gamma)$, from G-gradings on $\mathcal{A}$ to G-gradings on $\mathcal{B}$. If Γ and Γ' are isomorphic (respectively, weakly isomorphic), then $\theta(\Gamma)$ and $\theta(\Gamma')$ are isomorphic (respectively, weakly isomorphic).*

PROOF. We have already defined $\theta(\Gamma)$. Let $\varphi \in \mathrm{Aut}(\mathcal{A})$ and $\psi = \theta_{\mathbb{F}}(\varphi)$. Then the following diagram commutes:

$$\begin{array}{ccc}
\mathbf{Aut}(\mathcal{A}) & \xrightarrow{\theta} & \mathbf{Aut}(\mathcal{B}) \\
{\scriptstyle \mathrm{Ad}_\varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_\psi} \\
\mathbf{Aut}(\mathcal{A}) & \xrightarrow{\theta} & \mathbf{Aut}(\mathcal{B})
\end{array}$$

This follows immediately from (1.10) and the equation $\theta_{\mathcal{R}}(\varphi \otimes \mathrm{id}) = \psi \otimes \mathrm{id}$, which is a consequence of the naturality of θ.

Now if φ sends Γ to Γ' (respectively, ${}^{\alpha}\Gamma$ to Γ'), then ψ sends $\theta(\Gamma)$ to $\theta(\Gamma')$ (respectively, $\theta({}^{\alpha}\Gamma) = {}^{\alpha}(\theta(\Gamma))$ to $\theta(\Gamma')$). □

THEOREM 1.39. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite-dimensional (nonassociative) algebras. Assume we have an isomorphism $\theta\colon \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$. Let Γ be a G-grading on $\mathcal{A}$ such that G is its universal abelian group. Then Γ is a fine abelian group grading if and only if so is $\theta(\Gamma)$. If this is the case, G is the universal abelian group of $\theta(\Gamma)$. Moreover, two fine abelian group gradings, Γ and Γ', are equivalent if and only if $\theta(\Gamma)$ and $\theta(\Gamma')$ are equivalent.*

PROOF. If Γ is fine, then the image of $\eta_\Gamma\colon G^D \to \mathbf{Aut}(\mathcal{A})$ is a maximal diagonalizable subgroupscheme of $\mathbf{Aut}(\mathcal{A})$. Hence the image of $\eta_{\theta(\Gamma)} = \theta \circ \eta_\Gamma$ is a maximal diagonalizable subgroupscheme of $\mathbf{Aut}(\mathcal{B})$, so $\theta(\Gamma)$ is fine and G is its universal abelian group. It remains to recall that, if universal groups are used, two fine gradings are equivalent if and only if they are weakly isomorphic, so we can apply Theorem 1.38. □

REMARK 1.40. Theorem 1.38 has obvious analogs for algebras carrying some additional structure. In Chapter 3, we will encounter the following situation: there is an isomorphism $\theta\colon \mathbf{Aut}(\mathcal{A}, \varphi) \to \mathbf{Aut}(\mathcal{B})$ where φ is an involution on $\mathcal{A}$. Then we obtain a one-to-one correspondence $\Gamma \mapsto \theta(\Gamma)$ between φ-gradings on $\mathcal{A}$ and gradings on $\mathcal{B}$. Here a grading $\Gamma :\ \mathcal{A} = \bigoplus_{g\in G} \mathcal{A}_g$ is said to be a *φ-grading* if $\varphi(\mathcal{A}_g) = \mathcal{A}_g$ for all $g \in G$, which is equivalent to saying that $\mathbf{Diag}(\Gamma)$ is contained in $\mathbf{Aut}(\mathcal{A}, \varphi)$. In this situation, the concepts of isomorphism, equivalence, refinement, etc. should be understood in the class of φ-gradings when referring to $\mathcal{A}$. Then an analog of Theorem 1.39 also holds.

The action of the finite dual Hopf algebra. There is another approach to duality between gradings and actions that works in arbitrary characteristic and does not require $\mathcal{A}$ to be finite-dimensional. It allows us to translate a G-grading on $\mathcal{A}$ to an action of certain operators on $\mathcal{A}$ as follows. For any group G, consider the Hopf algebra $\mathcal{K} = (\mathbb{F}G)^\circ$, the finite dual of $\mathbb{F}G$ (see Appendix A). It consists of the so-called representative functions $f\colon G \to \mathbb{F}$, i.e., the functions satisfying $\dim \operatorname{span}\{Gf\} < \infty$, where G acts on functions by translation, and functions are multiplied point-wise — see [**Mon93**, 1.3.6]. $\mathcal{K}$ acts on $\mathcal{A}$ by extension of (1.4):

$$f * x = f(g)x \quad \text{for all} \quad x \in \mathcal{A}_g,\ g \in G \text{ and } f \in \mathcal{K}.$$

With respect to this action, $\mathcal{A}$ becomes a $\mathcal{K}$-module algebra, i.e.,

$$f * (xy) = \sum_i (f_i' * x)(f_i'' * y) \quad \text{for all} \quad f \in \mathcal{K},\ x, y \in \mathcal{A},$$

where $\Delta(f) = \sum_i f_i' \otimes f_i''$. If G is a finitely generated abelian group (or, more generally, a residually linear group), then $\mathcal{K}$ separates points of G and hence the G-grading can be recovered from the $\mathcal{K}$-action.

If $f \in \mathcal{K}$ is group-like, then f acts on $\mathcal{A}$ as an automorphism:

$$f * (xy) = (f * x)(f * y) \quad \text{for all} \quad x, y \in \mathcal{A}.$$

The group $\mathrm{G}(\mathcal{K})$ of group-like elements can be identified with $\widehat{G}$, the group of multiplicative characters of G. If $\mathbb{F}$ is algebraically closed and either $\operatorname{char}\mathbb{F} = 0$ or $\operatorname{char}\mathbb{F} = p$ and G has no p-torsion, then the G-grading on $\mathcal{A}$ can be recovered from the $\widehat{G}$-action.

If $f \in \mathcal{K}$ is primitive, then f acts on $\mathcal{A}$ as a derivation:

$$f * (xy) = (f * x)y + x(f * y) \quad \text{for all} \quad x, y \in \mathcal{A}.$$

The space $\operatorname{Prim}(\mathcal{K})$ of primitive elements can be identified with the space of additive characters of G, i.e, homomorphisms $G \to \mathbb{F}$. Unless $\mathbb{F}$ has enough roots of unity and the p-torsion subgroup of G has period p, $\mathcal{K}$ will not be generated by group-like and primitive elements, so one has to study elements $f \in \mathcal{K}$ with more complicated expansion formulas for $f * (xy)$. This approach was taken in [**BK09**]. The action of $\mathcal{K}$ on $\mathcal{A}$ can be put into the context of *formal groups* [**BKM09**]. In this book we avoid formal groups and use affine group schemes instead. However, in our study of simple Lie algebras of Cartan type in Chapter 7, we will encounter non-smooth automorphism group schemes and will use the action of their distribution algebras (see Appendix A) to understand their structure.

1.5. Exercises

(1) Consider the $\mathbb{Z}$-grading on $M_n(\mathbb{F})$, $n \geq 2$, defined by setting $\deg E_{ij} = i - j$. This is the special case of (0.1) where the n-tuple is $(1, 2, \dots, n)$. Prove that $\mathbb{Z}$ is the universal group of this grading.

(2) What is the universal group of the $\mathbb{Z}_n^2$-grading (0.4) on $M_n(\mathbb{F})$ constructed using the generalized Pauli matrices?

(3) The grading Γ in Example 1.9 cannot be realized as a group grading. Find the universal group $U(\Gamma)$ and the proper coarsening of Γ determined by the canonical map $\{s_1, s_2, s_3\} \to U(\Gamma)$.

(4) For the grading Γ in Example 1.13, find $U(\Gamma)$ and $U_{ab}(\Gamma)$.

(5) Find all nonequivalent coarsenings of the $\mathbb{Z}$-grading on $M_n(\mathbb{F})$ in Exercise 1 (in the class of group gradings).
(6) Give an example of two G-gradings that are weakly isomorphic, but not isomorphic.
(7) Describe the quasitori in $\mathrm{Aut}(M_n(\mathbb{F}))$ corresponding to the $\mathbb{Z}$-grading in Exercise 1 and the $\mathbb{Z}_n^2$-grading in Exercise 2 (where $\mathbb{F}$ is an algebraically closed field of characteristic zero).
(8) Let G be an abelian group and let $\Gamma: V = \bigoplus_{g\in G} V_g$ be a G-grading on a vector space V over an algebraically closed field of characteristic zero. Prove that the group G is generated by the support of Γ if and only if the corresponding homomorphism $\eta_\Gamma\colon \widehat{G} \to \mathrm{GL}(V)$ is injective.
(9) Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras over $\mathbb{F}$, and let $\theta\colon \mathbf{Aut}(\mathcal{A}) \to \mathbf{Aut}(\mathcal{B})$ be an isomorphism of affine group schemes. Let Γ be a fine abelian group grading on $\mathcal{A}$ with universal abelian group G, and let $\Gamma' = \theta(\Gamma)$ be the induced G-grading on $\mathcal{B}$. Prove that $\theta_{\mathbb{F}}$ induces isomorphisms $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(\Gamma')$, $\mathrm{Stab}(\Gamma) \to \mathrm{Stab}(\Gamma')$ and $W(\Gamma) \to W(\Gamma')$.