Preface

Asymptotic convex geometry may be described as the study of convex bodies from a geometric and analytic point of view, with an emphasis on the dependence of various parameters on the dimension. This theory stands at the intersection of classical convex geometry and the local theory of Banach spaces, but it is also closely linked to many other fields, such as probability theory, partial differential equations, Riemannian geometry, harmonic analysis and combinatorics. The aim of this book is to introduce a number of basic questions regarding the distribution of volume in high-dimensional convex bodies and to provide an up to date account of the progress that has been made in the last fifteen years. It is now understood that the convexity assumption forces most of the volume of a body to be concentrated in some canonical way and the main question is whether, under some natural normalization, the answer to many fundamental questions should be independent of the dimension.

One such normalization, that in many cases facilitates the study of volume distribution, is the *isotropic position*. A convex body K in $\mathbb{R}^n$ is called isotropic if it has volume 1, barycenter at the origin, and its inertia matrix is a multiple of the identity: there exists a constant $L_K > 0$ such that

$$\int_K \langle x, \theta \rangle^2 dx = L_K^2$$

for every θ in the Euclidean unit sphere S^{n-1}. It is easily verified that the affine class of any convex body K contains a unique, up to orthogonal transformations, isotropic convex body; this is the isotropic position of K. A first example of the role and significance of the isotropic position may be given through the *hyperplane conjecture* (or *slicing problem*), which is one of the main problems in the asymptotic theory of convex bodies, and asks if there exists an absolute constant $c > 0$ such that $\max_{\theta \in S^{n-1}} |K \cap \theta^\perp| \geqslant c$ for every convex body K of volume 1 in $\mathbb{R}^n$ that has barycenter at the origin. This question was posed by Bourgain [99], who was interested in finding L_p-bounds for maximal operators defined in terms of arbitrary convex bodies. It is not so hard to check that answering his question affirmatively is equivalent to the following statement:

Isotropic constant conjecture. *There exists an absolute constant $C > 0$ such that*

$$L_n := \max\{L_K : K \text{ is isotropic in } \mathbb{R}^n\} \leqslant C.$$

This problem became well-known due to an article of V. Milman and Pajor which remains a classical reference on the subject. Around the same time, K. Ball showed in his PhD Thesis that the notion of the isotropic constant and the conjecture can be reformulated in the language of logarithmically-concave (or log-concave

for short) measures; however, without the problem becoming essentially more general. Let us note here that a finite Borel measure μ on $\mathbb{R}^n$ is called log-concave if, for any $\lambda \in (0, 1)$ and any compact subsets A, B of $\mathbb{R}^n$, we have

$$\mu(\lambda A + (1 - \lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda};$$

note also that the indicator function of a convex body is the density (with respect to the Lebesgue measure) of a compactly supported log-concave measure, but that not all log-concave measures are compactly supported. Isotropic convex bodies now form a genuine subclass of isotropic log-concave measures, but several properties and results that (may) hold for this subclass, including the boundedness or not of the isotropic constants, immediately translate in the setting of log-concave measures. Around 1990, Bourgain obtained the upper bound $L_n \leqslant c\sqrt[4]{n} \log n$ and, in 2006, this estimate was improved by Klartag to $L_n \leqslant c\sqrt[4]{n}$.

The problem remains open and has become the starting point for many other questions and challenging conjectures in high-dimensional geometry, one of those being the *central limit problem*. The latter in the asymptotic theory of convex bodies means the task to identify those high-dimensional distributions which have approximately Gaussian marginals. It is a question inspired by a general fact that has appeared more than once in the literature and states that, if μ is an isotropic probability measure on $\mathbb{R}^n$ which satisfies the thin shell condition

$$\mu\left(\left|\, \|x\|_2 - \sqrt{n}\,\right| \geqslant \varepsilon\right) \leqslant \varepsilon$$

for some $\varepsilon \in (0, 1)$, then, for all directions θ in a subset A of S^{n-1} with $\sigma(A) \geqslant 1 - \exp(-c_1\sqrt{n})$, one has

$$|\mu(\{x : \langle x, \theta \rangle \leqslant t\}) - \Phi(t)| \leqslant c_2(\varepsilon + n^{-\alpha}) \qquad \text{for all } t \in \mathbb{R},$$

where $\Phi(t)$ is the standard Gaussian distribution function and $c_1, c_2, \alpha > 0$ are absolute constants. Thus, the central limit problem is reduced to the question of identifying those high-dimensional distributions that satisfy a thin shell condition. It was the work of Anttila, Ball and Perissinaki that made this type of statement widely known in the context of isotropic convex bodies or, more generally, log-concave distributions. One of the main results in this area, first proved by Klartag in a breakthrough work, states that the assumption of log-concavity guarantees a thin shell bound, and hence an affirmative answer to the central limit problem. In fact, the following quantitative conjecture has been proposed.

Thin shell conjecture. *There exists an absolute constant $C > 0$ such that, for any $n \geqslant 1$ and any isotropic log-concave measure μ on $\mathbb{R}^n$, one has*

$$\sigma_\mu^2 := \int_{\mathbb{R}^n} \left(\|x\|_2 - \sqrt{n}\right)^2 d\mu(x) \leqslant C^2.$$

A third conjecture concerns the *Cheeger constant* Is_μ of an isotropic log-concave measure μ which is defined as the best constant $\kappa \geqslant 0$ such that

$$\mu^+(A) \geqslant \kappa \min\{\mu(A), 1 - \mu(A)\}$$

for every Borel subset A of $\mathbb{R}^n$, and where

$$\mu^+(A) := \liminf_{\varepsilon \to 0^+} \frac{\mu(A_\varepsilon) - \mu(A)}{\varepsilon}$$

is the Minkowski content of A (also, $A_\varepsilon := \{x : \mathrm{dist}(x, A) < \varepsilon\}$ is the ε-extension of A).

Kannan-Lovász-Simonovits conjecture. *There exists an absolute constant $c > 0$ such that*

$$\mathrm{Is}_n := \min\{\mathrm{Is}_\mu : \mu \text{ is isotropic log-concave measure on } \mathbb{R}^n\} \geqslant c.$$

Another way to formulate this conjecture is to ask if there exists an absolute constant $c > 0$ such that, for every isotropic log-concave measure μ on $\mathbb{R}^n$ and for every smooth function φ with $\int_{\mathbb{R}^n} \varphi \, d\mu = 0$, one has

$$c \int_{\mathbb{R}^n} \varphi^2 d\mu \leqslant \int_{\mathbb{R}^n} \|\nabla\varphi\|_2^2 d\mu.$$

We then say that μ satisfies the *Poincaré inequality* with constant $c > 0$. The equivalence of the two formulations can be seen by checking that

$$\mathrm{Is}_\mu^2 \simeq \inf_\mu \inf_\varphi \frac{\int \|\nabla\varphi\|_2^2 d\mu}{\int \varphi^2 d\mu}.$$

In this book we discuss these three conjectures and what is currently known about them, as well as other problems that are related to and arise from them. We now give a brief account of the contents of every chapter; more details can be found in the introduction of each individual chapter. In Chapters 2–4, we present the hyperplane conjecture and the first attempts to an answer. This presentation is given in the more general setting of logarithmically concave probability measures, which are introduced in Chapter 2 along with their main concentration properties. Some of these properties follow immediately from the Brunn-Minkowski inequality (more precisely, from Borell's lemma) and can be expressed in the form of reverse Hölder inequalities for seminorms: if $f : \mathbb{R}^n \to \mathbb{R}$ is a seminorm, then, for every log-concave probability measure μ on $\mathbb{R}^n$, one has $\|f\|_{\psi_1(\mu)} \leqslant c\|f\|_{L_1(\mu)}$, where

$$\|f\|_{\psi_\alpha(\mu)} = \inf \left\{ t > 0 : \int \exp((|f|/t)^\alpha) \, d\mu \leqslant 2 \right\}$$

is the Orlicz ψ_α-norm of f with respect to μ, $\alpha \in [1, 2]$. Isotropic log-concave measures are the log-concave probability measures μ that have barycenter at the origin and satisfy the isotropic condition

$$\int_{\mathbb{R}^n} \langle x, \theta \rangle^2 d\mu(x) = 1$$

for every $\theta \in S^{n-1}$. The isotropic constant of a measure μ in this class is defined as

$$L_\mu := \left(\sup_{x \in \mathbb{R}^n} f(x) \right)^{1/n} \simeq (f(0))^{1/n},$$

where f is the log-concave density of μ. K. Ball introduced a family of convex bodies $K_p(\mu), p \geqslant 1$, that can be associated with a given log-concave measure μ and showed that these bodies allow us to reduce the study of log-concave measures to that of convex bodies, but also enable us to use tools from the broader class of measures to tackle problems that have naturally, or merely initially, been formulated for bodies. A first example of their use, as mentioned above, is the fact that studying the magnitude of the isotropic constant of log-concave measures is completely equivalent to the respective task inside the more restricted class of convex bodies.

The isotropic constant conjecture is discussed in detail in Chapter 3; it reads that there exists an absolute constant $C > 0$ such that

$$L_\mu \leqslant C$$

for every $n \geqslant 1$ and every log-concave measure μ on $\mathbb{R}^n$. In order to understand its equivalence to the hyperplane conjecture we formulated above, we recall that

$$\max\{L_K : K \text{ is an isotropic convex body in } \mathbb{R}^n\}$$
$$\simeq \sup\{L_\mu : \mu \text{ is an isotropic log-concave measure on } \mathbb{R}^n\},$$

and then we have to explain the relation of the moments of inertia of a centered convex body to the volume of its hyperplane sections passing through the origin. In particular, in Section 3.1.2 we show that, if K is an isotropic convex body in $\mathbb{R}^n$, then for every $\theta \in S^{n-1}$ we have

$$\frac{c_1}{L_K} \leqslant |K \cap \theta^\perp| \leqslant \frac{c_2}{L_K},$$

where $c_1, c_2 > 0$ are absolute constants, and, thus, all hyperplane sections through the barycenter of K have approximately the same volume, this volume being large enough if and only if L_K is small enough. The hyperplane conjecture is also equivalent to the asymptotic versions of several classical problems in convex geometry. We discuss two of them: Sylvester's problem on the expected volume of a random simplex contained in a convex body and the Busemann-Petty problem. In Sections 3.3 and 3.4 we discuss Bourgain's upper bound $L_K \leqslant C\sqrt[4]{n}\log n$ for the isotropic constant of convex bodies K in $\mathbb{R}^n$. We describe two proofs of Bourgain's result. A key observation is that, if K is an isotropic convex body in $\mathbb{R}^n$, then, as we saw above for every log-concave probability measure μ, one has $\|\langle \cdot, \theta\rangle\|_{\psi_1(K)} \leqslant C\|\langle \cdot, \theta\rangle\|_{L_1(K)} \leqslant CL_K$ for all $\theta \in S^{n-1}$, where $C > 0$ is an absolute constant. In fact, Alesker's theorem shows that one has a stronger ψ_2-estimate for the function $f(x) = \|x\|_2$: one has $\|f\|_{\psi_2(K)} \leqslant C\|f\|_{L_2(K)} \leqslant C\sqrt{n}L_K$. Markov's inequality then implies exponential concentration of the mass of K in a strip of width CL_K and normal concentration in a ball of radius $C\sqrt{n}L_K$.

Chapter 4 is devoted to some partial affirmative answers to the hyperplane conjecture that were obtained soon after the problem became known. In order to make this statement more precise, we say that a class $\mathcal{C}$ of centered convex bodies satisfies the hyperplane conjecture uniformly if there exists a positive constant C such that $L_K \leqslant C$ for all $K \in \mathcal{C}$. The hyperplane conjecture has been verified for several important classes of convex bodies. A first example is the class of unconditional convex bodies; these are the centrally symmetric convex bodies K in $\mathbb{R}^n$ that have a position that is symmetric with respect to the standard coordinate subspaces, namely they have a position $\tilde{K}$ such that, if $(x_1, \ldots, x_n)$ belongs to $\tilde{K}$, then $(\epsilon_1 x_1, \ldots, \epsilon_n x_n)$ also belongs to $\tilde{K}$ for every $(\epsilon_1, \ldots, \epsilon_n) \in \{-1, 1\}^n$. The class of unconditional convex bodies will appear often in this book, mainly as a model for results or conjectures regarding the general cases. In this chapter, we also describe uniform bounds for the isotropic constants of some other classes of convex bodies and we give simple geometric proofs of the best known estimates for the isotropic constants of polytopes with N vertices or polyhedra with N facets, estimates that are logarithmic in N.

In Chapters 5–7, we discuss more recent approaches to the slicing problem and some very useful tools that have been developed for these approaches as well as for

related problems in the theory. Bourgain's approach exploited the ψ_1-information we have for the behavior of the linear functionals $x \mapsto \langle x, \theta \rangle$ on an isotropic convex body. The aim to understand the distribution of linear functionals in an isotropic convex body or, more precisely, the behavior of their L_q-norms with respect to the uniform measure on the body, has been furthered by the introduction of the family of L_q-centroid bodies of a convex body K of volume 1 or, more generally, of a log-concave probability measure μ. For every $q \geqslant 1$, the L_q-centroid body $Z_q(K)$ of K or, respectively, the L_q-centroid body $Z_q(\mu)$ of μ is defined through its support function, which is given by

$$h_{Z_q(K)}(y) := \|\langle \cdot, y \rangle\|_{L_q(K)} = \left(\int_K |\langle x, y \rangle|^q dx \right)^{1/q},$$

$$\text{or by} \quad h_{Z_q(\mu)}(y) := \|\langle \cdot, y \rangle\|_{L_q(\mu)} = \left(\int |\langle x, y \rangle|^q d\mu(x) \right)^{1/q},$$

respectively, for every vector y. Note that, according to our normalization, a convex body K of volume 1 in $\mathbb{R}^n$ is isotropic if and only if it is centered and $Z_2(K) = L_K B_2^n$ and, respectively, a log-concave probability measure μ on $\mathbb{R}^n$ is isotropic if and only if it is centered and $Z_2(\mu) = B_2^n$. The development of an asymptotic theory for this family of bodies, and for their behavior as q increases from 2 up to the dimension n, was initiated by Paouris and has proved to be a very fruitful idea.

In Chapter 5 we present the basic properties of the family $\{L_q(\mu) : q \geqslant 2\}$ of the centroid bodies of a centered log-concave probability measure μ on $\mathbb{R}^n$ and prove some fundamental formulas. The first main application of this theory is a striking and very useful deviation inequality of Paouris: for every isotropic log-concave probability measure μ on $\mathbb{R}^n$ one has

$$\mu(\{x \in \mathbb{R}^n : \|x\|_2 \geqslant ct\sqrt{n}\}) \leqslant \exp\left(-t\sqrt{n}\right)$$

for every $t \geqslant 1$, where $c > 0$ is an absolute constant. This is a consequence of the following statement: there exists an absolute constant $C_1 > 0$ such that, if μ is an isotropic log-concave measure on $\mathbb{R}^n$, then

(1) $$I_q(\mu) \leqslant C_1 I_2(\mu)$$

for every $q \leqslant \sqrt{n}$, where $I_q(\mu)$ is defined by

$$I_q(\mu) = \left(\int_{\mathbb{R}^n} \|x\|_2^q d\mu(x) \right)^{1/q}$$

for all $0 \neq q > -n$. Paouris has, moreover, proved an extension to this theorem which we also present: there exists an absolute constant c_2 such that, if μ is an isotropic log-concave measure on $\mathbb{R}^n$, then for any $1 \leqslant q \leqslant c_2\sqrt{n}$ one has

$$I_{-q}(\mu) \simeq I_q(\mu).$$

In particular, this shows that, for all $1 \leqslant q \leqslant c_2\sqrt{n}$, one has $I_q(\mu) \leqslant CI_2(\mu)$, where $C > 0$ is an absolute constant. Using the extended result one can derive a small ball probability estimate: for every isotropic log-concave measure μ on $\mathbb{R}^n$ and for any $0 < \varepsilon < \varepsilon_0$, one has

$$\mu(\{x \in \mathbb{R}^n : \|x\|_2 < \varepsilon\sqrt{n}\}) \leqslant \varepsilon^{c_3\sqrt{n}},$$

where $\varepsilon_0, c_3 > 0$ are absolute constants. In a few words, the main results of Paouris imply that for any isotropic log-concave measure one has

$$\mu(\{x : c\sqrt{n} \leqslant \|x\|_2 \leqslant C\sqrt{n}\}) \geqslant 1 - \exp(-\sqrt{n}).$$

This is a rough version of the thin shell estimate, that is often enough for the applications. In fact, as we will explain in Chapter 13, a way to obtain a thin shell estimate is to prove a more precise version of (1), with the constant C_1 being, for example, of the form $1 + cq/\sqrt{n}$ for some absolute constant $c > 0$ and for as large $q \in [1, \sqrt{n}]$ as possible.

In Chapters 6 and 7 we discuss some recent approaches to and reductions of the hyperplane conjecture. Chapter 6 deals with properties that bodies with maximal isotropic constant have, namely bodies whose isotropic constant is equal to or very close to L_n. It turns out that the isotropic position of such bodies is closely related to their M-position and this enables one to establish several interesting facts: for example, a reduction of the hyperplane conjecture, due to Bourgain, Klartag and V. Milman, to the question of boundedness of the isotropic constant of a restricted class of convex bodies, those that have volume ratio bounded by an absolute constant. Next, we give two more reductions of the conjecture to the study of parameters that can be associated with any isotropic convex body. The proofs of these reductions rely heavily on the existence of convex bodies with maximal isotropic constant whose isotropic position is not only closely related to their M-position, but is also compatible with regular covering estimates. The first of these reductions is a continuation of the work of Paouris on the behavior of the negative moments of the Euclidean norm with respect to an isotropic measure μ on $\mathbb{R}^n$. As we mentioned above, we already know that $I_{-q}(\mu) \simeq I_2(\mu) = \sqrt{n}$ for $0 < q \leqslant \sqrt{n}$, however, the behavior of the negative moments $I_{-q}(\mu)$ for $q > \sqrt{n}$ is not known at all and, in fact, our current knowledge does not exclude the possibility that the moments stay constant for all positive q up to $n - 1$. Dafnis and Paouris have actually proved that this question is equivalent to the hyperplane conjecture: they introduce a parameter that, for each $\delta \geqslant 1$, is given by

$$q_{-c}(\mu, \delta) := \max\{1 \leqslant q \leqslant n - 1 : I_{-q}(\mu) \geqslant \delta^{-1} I_2(\mu) = \delta^{-1}\sqrt{n}\},$$

and they establish that

$$L_n \leqslant C\delta \sup_\mu \sqrt{\frac{n}{q_{-c}(\mu, \delta)}} \, \log^2\left(\frac{en}{q_{-c}(\mu, \delta)}\right)$$

for every $\delta \geqslant 1$; additionally, they show that, if the hyperplane conjecture is correct, then we must have $q_{-c}(\mu, \delta_0) = n - 1$ for some $\delta_0 \simeq 1$, for every isotropic log-concave measure μ on $\mathbb{R}^n$. The other reduction is a work of Giannopoulos, Paouris and Vritsiou, based on the study of the parameter

$$I_1(K, Z_q^\circ(K)) = \int_K h_{Z_q(K)}(x)dx = \int_K \|\langle \cdot, x \rangle\|_{L_q(K)}dx,$$

and can be viewed as a continuation of Bourgain's initial approach that led to the upper bound $L_K \leqslant c\sqrt[4]{n} \log n$. Roughly speaking, this last reduction can be formulated as follows: given $q \geqslant 2$ and $\frac{1}{2} \leqslant s \leqslant 1$, an upper bound of the form

$$I_1(K, Z_q^\circ(K)) \leqslant C_1 q^s \sqrt{n} L_K^2 \quad \text{for all bodies } K \text{ in isotropic position}$$

leads to the estimate

$$L_n \leqslant \frac{C_2 \sqrt[4]{n} \log^2 n}{q^{\frac{1-s}{2}}}.$$

Bourgain's estimate is (almost) recovered by choosing $q = 2$, however, the behavior of $I_1(K, Z_q^\circ(K))$ may allow one to use $s < 1$ along with large values of q to obtain improved bounds if possible.

In Chapter 7 we first discuss Klartag's solution to the isomorphic slicing problem, an isomorphic variation of the hyperplane conjecture that asks whether, given any convex body, we can find another convex body, with absolutely bounded isotropic constant, that is geometrically close to the first body. Klartag's method relies on properties of the logarithmic Laplace transform of the uniform measure on a convex body. In general, given a finite Borel measure μ on $\mathbb{R}^n$, the logarithmic Laplace transform of μ is given by

$$\Lambda_\mu(\xi) := \log\left(\frac{1}{\mu(\mathbb{R}^n)} \int_{\mathbb{R}^n} e^{\langle \xi, x \rangle} d\mu(x)\right).$$

Klartag proved that, if K is a convex body in $\mathbb{R}^n$, then, for every $\varepsilon \in (0, 1)$, we can find a centered convex body $T \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$ such that $\frac{1}{1+\varepsilon}T \subseteq K + x \subseteq (1 + \varepsilon)T$ and

$$L_T \leqslant C/\sqrt{\varepsilon},$$

where $C > 0$ is an absolute constant. Most remarkably, by combining this fact with the deviation inequality of Paouris, one may also deduce the currently best known upper bound for the isotropic constant, which is that

$$L_\mu \leqslant C' n^{1/4}$$

for every isotropic log-concave measure μ on $\mathbb{R}^n$. The logarithmic Laplace transform is another important tool of the theory that, since it was first employed in the setting of isotropic convex bodies and log-concave measures, has proved to be extremely useful given its various and interesting applications; these include Klartag's solution to the isomorphic slicing problem, that we already mentioned, as well as an alternative approach of Klartag and E. Milman that combines the advantages of both the logarithmic Laplace transform and the extensive theory of the L_q-centroid bodies, and occupies the second part of Chapter 7. Klartag and E. Milman looked for lower bounds for the volume radius of the L_q-centroid bodies of an isotropic log-concave measure μ. Through a delicate analysis of the logarithmic Laplace transform of μ, they showed that

$$(2) \qquad\qquad |Z_q(\mu)|^{1/n} \geqslant c_1 \sqrt{q/n}$$

for all $q \leqslant \sqrt{n}$, where $c_1 > 0$ is an absolute constant. Apart from being interesting on its own, this result leads again to the estimate $L_\mu \leqslant c_2 \sqrt[4]{n}$. It is also plausible that (2) can hold for larger values of $q \in [1, n]$ as well; this is the content of a recent work of Vritsiou that is also discussed in the chapter. She showed that (2) holds for every q up to a variant of the parameter $q_{-c}(\mu, \delta)$ of Dafnis and Paouris, which, as we previously mentioned, could be of the order of n (in fact, recall that the hyperplane conjecture is correct if and only if $q_{-c}(\mu, \delta_0)$ is of the order of n for some $\delta_0 \simeq 1$ and every isotropic log-concave measure μ on $\mathbb{R}^n$). However, even a small improvement to the estimates we currently have for $q_{-c}(\mu, \delta)$ and its variant could permit one to extend the range of q with which the method of Klartag and Milman can be applied, and also improve on the currently known bounds for the isotropic

constant problem. Other applications of the logarithmic Laplace transform are discussed in some of the following chapters, the most important of these appearing in Chapters 12 and 15.

In Chapters 8–11, we deviate a little from those lines of results that are directly related to the hyperplane conjecture and the other two main conjectures of the theory so as to look at different applications of the tools that were developed in the previous part. Chapters 8 and 9 are devoted to some open questions, whose study so far has already shed more light on various geometric properties of convex bodies and log-concave measures. The first question was originally posed by V. Milman in the framework of convex bodies: it asks if there exists an absolute constant $C > 0$ such that every centered convex body K of volume 1 has at least one sub-Gaussian direction with constant C. Following some positive results for special classes of convex bodies, Klartag was the first to prove the existence of "almost sub-Gaussian" directions for any isotropic convex body. More precisely, using again properties of the logarithmic Laplace transform, he proved that for every log-concave probability measure μ on $\mathbb{R}^n$ there exists $\theta \in S^{n-1}$ such that

$$\mu\left(\{x : |\langle x, \theta \rangle| \geqslant ct\|\langle \cdot, \theta \rangle\|_2\}\right) \leqslant e^{-\frac{t^2}{(\log(t+1))^{2\alpha}}},$$

for all $1 \leqslant t \leqslant \sqrt{n}\log^\alpha n$, where $\alpha = 3$. We describe the best known estimate, due to Giannopoulos, Paouris and Valettas, according to which one can always have $\alpha = 1/2$. The main idea is to define the symmetric convex set $\Psi_2(\mu)$ whose support function is $h_{\Psi_2(\mu)}(\theta) = \|\langle \cdot, \theta \rangle\|_{\psi_2}$ and to estimate its volume. One can show that for every centered log-concave probability measure μ in $\mathbb{R}^n$ one has

$$c_1 \leqslant \left(\frac{|\Psi_2(\mu)|}{|Z_2(\mu)|}\right)^{1/n} \leqslant c_2\sqrt{\log n},$$

where $c_1, c_2 > 0$ are absolute constants. An immediate consequence is the existence of at least one sub-Gaussian direction for μ with constant $b = O(\sqrt{\log n})$. The main tool in the proof of this result is estimates for the covering numbers $N(Z_q(K), sB_2^n)$. An even more interesting question is to determine the distribution of the function $\theta \mapsto \|\langle \cdot, \theta \rangle\|_{\psi_2}$ on the unit sphere; that is, to understand whether most of the directions have ψ_2-norm that is, say, logarithmic in the dimension.

In Chapter 9 we discuss the questions of obtaining an upper bound for the mean width

$$w(K) := \int_{S^{n-1}} h_K(x) \, d\sigma(x),$$

that is, the L_1-norm of the support function of K with respect to the Haar measure on the sphere, as well as the respective L_1-norm of the Minkowski functional of K,

$$M(K) := \int_{S^{n-1}} \|x\|_K \, d\sigma(x),$$

when K is an isotropic convex body. We present some non-trivial but non-optimal estimates. We also discuss the same questions for the L_q-centroid bodies of an isotropic log-concave measure. Answering these questions requires a deeper understanding of the behavior of linear functionals and of the local structure of the centroid bodies; this would bring new insights to the reductions of the hyperplane conjecture that were discussed in the previous chapters.

Chapters 10 and 11 contain applications of the theory of L_q-centroid bodies and of the main inequalities of Paouris to random matrices and random polytopes.

In Chapter 10 we discuss a question of Kannan, Lovász and Simonovits on the approximation of the covariance matrix of a log-concave measure. If K is an isotropic convex body in $\mathbb{R}^n$, then one has

$$I = \frac{1}{L_K^2} \int_K x \otimes x dx,$$

where I is the identity operator. Given $\varepsilon \in (0,1)$, the question is to find N_0, as small as possible, for which the following holds true: if $N \geqslant N_0$, then N independent random points $x_1, \ldots, x_N$ that are uniformly distributed in K must have, with probability greater than $1 - \varepsilon$, the property that

$$(1 - \varepsilon)L_K^2 \leqslant \frac{1}{N} \sum_{i=1}^{N} \langle x_i, \theta \rangle^2 \leqslant (1 + \varepsilon)L_K^2$$

for every $\theta \in S^{n-1}$. The question had its origin in the problem of finding a fast algorithm for the computation of the volume of a given convex body, and Kannan, Lovász and Simonovits proved that one can take $N_0 = C(\varepsilon)n^2$ for some constant $C(\varepsilon) > 0$ depending only on ε. This was improved to $N_0 = C(\varepsilon)n(\log n)^3$ by Bourgain and to $N_0 = C(\varepsilon)n(\log n)^2$ by Rudelson. It was finally proved by Adamczak, Litvak, Pajor and Tomczak-Jaegermann that the best estimate for N_0 is $C(\varepsilon)n$. We describe the history and the solution of the problem.

In Chapter 11 we discuss the asymptotic shape of the random polytope $K_N :=$ conv$\{\pm x_1, \ldots, \pm x_N\}$ that is spanned by N independent random points $x_1, \ldots, x_N$ uniformly distributed in an isotropic convex body K in $\mathbb{R}^n$. The literature on the approximation of convex bodies by random polytopes is very rich, but the main point here is that N is fixed in the range $[n, e^n]$ and we are interested in estimates which do not depend on the affine class of a convex body K. Some basic tasks in this spirit are: to determine the asymptotic behavior of the volume radius $|K|^{1/n}$, to understand the typical "asymptotic shape" of K_N and to estimate the isotropic constant of K_N. The same questions can be formulated and studied more generally if we assume that we have N independent copies $X_1, \ldots, X_N$ of an isotropic log-concave random vector X. A general, and rather precise, description was obtained by Dafnis, Giannopoulos and Tsolomitis: given any isotropic log-concave measure μ on $\mathbb{R}^n$ and any $n \leqslant N \leqslant \exp(n)$, the random polytope K_N defined by N i.i.d. random points $X_1, \ldots, X_N$ which are distributed according to μ satisfies, with high probability, the next two conditions: (i) $K_N \supseteq c\, Z_{\log(N/n)}(\mu)$ and (ii) for every $\alpha > 1$ and $q \geqslant 1$,

$$\mathbb{E}\left[\sigma(\{\theta : h_{K_N}(\theta) \geqslant \alpha h_{Z_q(\mu)}(\theta)\})\right] \leqslant N\alpha^{-q}.$$

Using this description of the shape of K_N and the theory of centroid bodies which was developed in the previous chapters, one can determine the volume radius and the quermassintegrals of a random K_N, at least in the range $n \leqslant N \leqslant \exp(\sqrt{n})$. A question concerning the isotropic constant of K_N can be made precise in the following way: one would like to show that, with probability tending to 1 as $n \to \infty$, the isotropic constant of the random polytope $K_N := \text{conv}\{\pm x_1, \ldots, \pm x_N\}$ is bounded by CL_K where $C > 0$ is a constant independent of K, n and N. We describe a method that was initiated by Klartag and Kozma when dealing with the class of Gaussian random polytopes. Variants of the method also work in the cases that the vertices x_j of K_N are distributed according to the uniform measure on an

isotropic convex body which is either ψ_2 (with constant b) or unconditional. The general case remains open.

Chapters 12–14 provide an exposition of our state of knowledge on the thin shell and Kannan-Lovász-Simonovits (or KLS for short) conjectures. Historical and other information about the thin shell conjecture and its connections with the central limit problem is given in Chapter 12. We present the work of Anttila, Ball and Perissinaki and various central limit theorems for isotropic convex bodies which would follow from thin shell estimates. This question has been studied by many authors and has been verified in some special cases. Klartag was the first to give a positive answer in full generality. In fact, aside from the immediate consequence of a general thin shell estimate that, as we mentioned again earlier in the Introduction, is that most one-dimensional marginals are close to Gaussian distributions, Klartag also established normal approximation for multidimensional marginal distributions. In Section 12.4 we give an account of Klartag's positive answer to the thin shell conjecture for the class of unconditional isotropic log-concave random vectors, which is one of the special cases for which this question was fully verified. Klartag proved that if K is an unconditional isotropic convex body in $\mathbb{R}^n$, then

$$\sigma_K^2 := \mathbb{E}_{\mu_K} \left(\|x\|_2 - \sqrt{n} \right)^2 \leqslant C^2,$$

where $C \leqslant 4$ is an absolute positive constant. We also describe a result of Eldan and Klartag which shows that the thin shell conjecture is stronger than the hyperplane conjecture and implies it; more precisely, they prove that $L_n \leqslant C\sigma_n$ where

$$\sigma_n := \max\{\sigma_\mu : \mu \text{ is isotropic log-concave measure on } \mathbb{R}^n\},$$

and, hence, any estimate one establishes for the former conjecture immediately holds for the latter too. Chapter 13 is then devoted to a complete proof of the currently best known estimate for the thin shell conjecture, $\sigma_n \leqslant Cn^{1/3}$, which is due to Guédon and E. Milman.

Chapter 14 is devoted to the Kannan-Lovász-Simonovits conjecture. We first introduce various isoperimetric constants which provide information on the interplay between a log-concave probability measure μ and the underlying Euclidean metric (the Cheeger constant Is_μ, the Poincaré constant Poin_μ, the exponential concentration constant Exp_μ and the first moment concentration constant FM_μ) and we discuss their relation. Complementing classical results of Maz'ya, Cheeger, Gromov, V. Milman, Buser, Ledoux and others, E. Milman established the equivalence of all four constants in the log-concave setting: one has

$$\mathrm{Is}_\mu \simeq \sqrt{\mathrm{Poin}_\mu} \simeq \mathrm{Exp}_\mu \simeq \mathrm{FM}_\mu$$

for every log-concave probability measure, where $a \simeq b$ means that $c_1 a \leqslant b \leqslant c_2 b$ for some absolute constants $c_1, c_2 > 0$. As an application, E. Milman obtained stability results for the Cheeger constant of convex bodies. Loosely speaking, if K and T are two convex bodies in $\mathbb{R}^n$ and if $|K| \simeq |T| \simeq |K \cap T|$, then $\mathrm{Is}_K \simeq \mathrm{Is}_T$. We introduce the KLS-conjecture in Section 14.4 and we present the first general lower bounds for Is_μ in the isotropic log-concave case. From the work of Kannan, Lovász and Simonovits and Bobkov one has that $\sqrt{n}\mathrm{Is}_\mu \geqslant c$, where $c > 0$ is an absolute constant. Actually, Bobkov proved that

$$\sqrt[4]{n}\sqrt{\sigma_\mu}\mathrm{Is}_\mu \geqslant c;$$

this provides a direct link between the KLS-conjecture and the thin shell conjecture: combined with the thin shell estimate of Guédon and E. Milman his result leads to the bound $n^{5/12} \mathrm{Is}_\mu \geqslant c$. In Section 14.5 we describe Klartag's logarithmic in the dimension lower bound for the Poincaré constant Poin_K of an unconditional isotropic convex body K in $\mathbb{R}^n$; one has $\mathrm{Is}_K \simeq \sqrt{\mathrm{Poin}_K} \geqslant \frac{c}{\log n}$, where $c > 0$ is an absolute positive constant. We close this discussion with a result of Eldan which, again, connects the thin shell conjecture with the KLS-conjecture: there exists an absolute constant $C > 0$ such that

$$\frac{1}{\mathrm{Is}_n^2} \leqslant C \log n \sum_{k=1}^n \frac{\sigma_k^2}{k}.$$

Taking into account the result of Guédon and E. Milman, one gets the currently best known bound for Is_n: $\mathrm{Is}_n^{-1} \leqslant C n^{1/3} \log n$.

In the last two chapters of the book we are concerned with two more approaches to the main questions in this theory. Chapter 15 is devoted to a probabilistic approach and related conjectures of Latała and Wojtaszczyk on the geometry of log-concave measures. The starting point is an *infimum convolution inequality* which was first introduced by Maurey when he gave a simple proof of Talagrand's two level concentration inequality for the product exponential measure. In general, if μ is a probability measure and φ is a non-negative measurable function on $\mathbb{R}^n$, one says that the pair (μ, φ) has *property* (τ) if, for every bounded measurable function f on $\mathbb{R}^n$,

$$\left(\int_{\mathbb{R}^n} e^{f \Box \varphi} d\mu \right) \left(\int_{\mathbb{R}^n} e^{-f} d\mu \right) \leqslant 1,$$

where $f \Box \varphi$ is the infimum convolution of f and φ, defined by

$$(f \Box \varphi)(x) = \inf \left\{ f(x - y) + \varphi(y) : \ y \in \mathbb{R}^n \right\}.$$

That the property (τ) is satisfied by a pair (μ, φ) is directly related to concentration properties of the measure μ since the former property implies that, for every measurable $A \subseteq \mathbb{R}^n$ and every $t > 0$, we have

$$\mu \left(x \notin A + B_\varphi(t) \right) \leqslant (\mu(A))^{-1} e^{-t},$$

where $B_\varphi(t) = \{\varphi \leqslant t\}$. Therefore, given a measure μ it makes sense to ask for the optimal cost function φ for which we have that (μ, φ) has property (τ). The first main observation is that, if we restrict ourselves to even probability measures μ and convex cost functions φ, then the (pointwise) largest candidate for a cost function is the Cramer transform Λ_μ^* of μ; this is the Legendre transform of the logarithmic Laplace transform of μ. In the setting of log-concave probability measures, the conjecture Latała and Wojtaszczyk formulate is that the pair (μ, Λ_μ^*) always has property (τ). A detailed analysis shows that this conjecture would imply an affirmative answer to most of the conjectures addressed in this book: among them, the thin shell conjecture as well as the hyperplane conjecture. The problems that are raised through this approach are very interesting and challenging. An affirmative answer has been given for some rather restricted classes of measures: even log-concave product measures, uniform distributions on ℓ_p^n-balls and rotationally invariant log-concave measures.

In the last chapter we give an account of K. Ball's information theoretic approach, which is based on the study of the Shannon entropy $\mathrm{Ent}(X) = -\int_{\mathbb{R}^n} f \log f$ of an isotropic random vector X with density f. It is known that, among all

isotropic random vectors, the standard Gaussian random vector G has the largest entropy, and the main observation is that comparing the entropy gap $\text{Ent}\left(\frac{X+Y}{\sqrt{2}}\right) - \text{Ent}(X)$ (with Y being an independent copy of X) to $\text{Ent}(G) - \text{Ent}(X)$ provides a link between the KLS-conjecture and the hyperplane conjecture. A first result of this type was obtained by Ball, Barthe and Naor for one-dimensional distributions. The main result of this chapter is a recent high-dimensional analogue for isotropic log-concave random vectors, which is due to Ball and Nguyen: if X is an isotropic log-concave random vector in $\mathbb{R}^n$ and its density f satisfies the Poincaré inequality with constant $\kappa > 0$, then

$$\text{Ent}\left(\frac{X+Y}{\sqrt{2}}\right) - \text{Ent}(X) \geqslant \frac{\kappa}{4(1+\kappa)}\left(\text{Ent}(G) - \text{Ent}(X)\right),$$

where G is a standard Gaussian random vector in $\mathbb{R}^n$. In addition, Ball and Nguyen show that this implies $L_X \leqslant e^{17/\kappa}$. Thus, for each individual isotropic log-concave distribution X, a lower bound for the Poincaré constant implies a bound for the isotropic constant.

The book is primarily addressed to readers who are familiar with the basic theory of convex bodies and the asymptotic theory of finite dimensional normed spaces as these are developed in the books of Milman and Schechtman and of Pisier. Nevertheless, we have included an introductory chapter where all the prerequisites are described; short proofs are also provided for the most important results that are used in the sequel.

This book grew out of our working seminar in the last fifteen years. Among the main topics that were discussed in our meetings were the developments on the basic questions addressed in the text. A large part of the material forms the basis of PhD and MSc theses that were written at the University of Athens and the University of Crete. We are grateful to Nikos Dafnis, Dimitris Gatzouras, Marianna Hartzoulaki, Labrini Hioni, Lefteris Markessinis, Nikos Markoulakis, Grigoris Paouris, Eirini Perissinaki, Pantelis Stavrakakis and Antonis Tsolomitis for their active participation in our seminar, for collaborating with us at various stages, for numerous discussions around the subject of this book and for their friendship over the years.

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